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VARIETIES OF SKEW BOOLEAN ALGEBRAS WITH INTERSECTIONS

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Abstract

Skew Boolean algebras for which pairs of elements have natural meets, called intersections, are studied from a universal algebraic perspective. Their lattice of varieties is described and shown to coincide with the lattice of quasi-varieties. Some connections of relevance to arbitrary skew Boolean algebras are also established.

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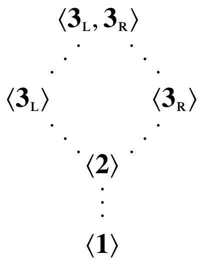
1. Introduction

Noncommutative variations of (generalized) Boolean algebras go back at least to Robert Bignall's 1976 dissertation [3] and the 1980 paper by his advisor, William Cornish [12]. If one includes variations of substructures of Boolean algebras, then the path no doubt goes back further. Jonathan Leech published his 1990 paper 'Skew Boolean algebras' [22] in an initial sequence of papers on skew lattices. Aspects of Bignall and Cornish's earlier work were then integrated into a developing theory of skew lattices in the joint paper by Bignall and Leech [4] in 1995. These papers, along with Leech's 1996 survey article on skew lattices [24], provide ample background for reading this paper.

For skew Boolean algebras (or SBAs), the lattice of varieties is rather simple.



Generator(s)	Variety
$\mathbf{3}_{\mathrm{L}}, \mathbf{3}_{\mathrm{R}}$	Skew Boolean algebras
$\mathbf{3}_{\mathrm{L}}$	Left-handed SBAs
$\mathbf{3}_{\mathrm{R}}$	Right-handed SBAs
$\mathbf{2}$	Generalized Boolean algebras
$\mathbf{1}$	Trivial algebras

The generators $\mathbf{3}_{\mathrm{L}}$ and $\mathbf{3}_{\mathrm{R}}$ give dual ways of placing an SBA structure on $\{0, 1, 2\}$ that generalizes the Boolean algebra $\mathbf{2}$ on $\{0, 1\}$. On both, 0 behaves as one would expect a bottom element: $x \wedge 0 = 0 = 0 \wedge x$ and $x \vee 0 = x = 0 \vee x$. On $\{1, 2\}$ in the case of $\mathbf{3}_{\mathrm{L}}$, $x \wedge_{\mathrm{L}} y = x$ on the left and $x \vee_{\mathrm{L}} y = y$ on the right; for $\mathbf{3}_{\mathrm{R}}$, it is the opposite. Also, $x \vee y = y \wedge x$ on $\{1, 2\}$ for $\mathbf{3}_{\mathrm{L}}$ and $\mathbf{3}_{\mathrm{R}}$.

Besides a (skew) join $\vee$, a (skew) meet $\wedge$, and a constant 0, skew Boolean algebras have a difference (or relative complement) operation. For both $\mathbf{3}_{\mathrm{L}}$ and $\mathbf{3}_{\mathrm{R}}$, it is given by

$$x \backslash y = \begin{cases} x & \text{if } y = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Another operation can at times be defined on skew lattices and on skew Boolean algebras in particular. The intersection $x \cap y$ is the infimum of x and y in the natural partial order (see details below), if it exists. In the case of both $\mathbf{3}_{\mathrm{L}}$ and $\mathbf{3}_{\mathrm{R}}$,

$$x \cap y = \begin{cases} x & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

The intersection $\cap$, when it exists for all pairs x and y, is of course there already and not a new operation forced upon the system. But, brought into the signature, some significant consequences occur, not the least of which is that SBAs with intersections are congruence distributive and congruence permutable. Indeed, the lattice of congruences on an SBA with intersections (often written 'skew Boolean $\cap$-algebra' or SBIA for short) is isomorphic to the lattice of ideals of its SBA reduct. Of special interest, left- or right-handed SBIAs are term-equivalent to pointed discriminator algebras. (See Bignall and Leech [4].) On the other hand, the lattice of (quasi-)varieties is more complex. It is the purpose of this paper to explore this lattice and related universal algebraic features.

In Section 2, some general background about skew lattices, skew Boolean algebras, and intersections is presented. In the next section, SBIAs are studied from the perspective of their SBA reducts. Its main result is Theorem 3.3, which states that a given SBA $\mathbf{S}$ has intersections if and only if both of its algebraic images $\mathbf{S}/\mathcal{L}$ and $\mathbf{S}/\mathcal{R}$ (where $\mathcal{L}$ and $\mathcal{R}$ are the canonical Green's congruences) have intersections (even though neither the induced map $\mathbf{S} \to \mathbf{S}/\mathcal{L}$ nor $\mathbf{S} \to \mathbf{S}/\mathcal{R}$ need preserve intersections). In Section 4, nontrivial subdirectly irreducible SBIAs are characterized in Theorem 4.3 as

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primitive algebras, the (generally) noncommutative variants of the Boolean algebra **2**. In Section 5, we describe the lattices of (quasi-)varieties of skew Boolean ∩-algebras. Following a result of Brian Davey, Theorem 5.5 states that this lattice is completely distributive and isomorphic to the lattice of order ideals of a prior lattice Ω_0 of finite primitive algebras (including the trivial case **1**). We also show in Theorem 5.3 that the question of a given identity holding for all [left-handed/right-handed] skew Boolean ∩-algebras is decidable. We conclude with a corresponding discussion for the case of quasi-varieties, showing that all quasi-varieties of SBIAs are varieties and thus collectively possess the same lattice structure.

Skew Boolean algebras, possibly with intersections, have received a fair amount of attention. Recently, Bauer and Cvetko-Vah [1] developed a Stone duality theory for SBIAs, while Kudryavtseva [19, 20] developed a similar theory for arbitrary SBAs. Cvetko-Vah and Leech [15, 16] have studied rings whose idempotents are closed under multiplication, and thus form SBAs that often have intersections. Both of the present authors with Cvetko-Vah [17] have studied applications to theoretical computer science. Spinks and his former advisor, Bignall, have studied connections with other types of algebras. See, for example, [5, 6, 30, 31]. These references are but few of even more, as will be evident from a search of publications by the individuals just mentioned.

2. Some background

A skew lattice is an algebra $\mathbf{S} = \langle S; \vee, \wedge\rangle$, where $\vee$ and $\wedge$ are associative binary operations on a set S that satisfy the absorption identities:

$$x \wedge (x \vee y) = x = (y \vee x) \wedge x \quad \text{and} \quad x \vee (x \wedge y) = x = (y \wedge x) \vee x.$$

Both operations are necessarily idempotent and the following dualities hold:

$$u \wedge v = u \iff u \vee v = v \quad \text{and} \quad u \wedge v = v \iff u \vee v = u.$$

The reducts $\langle S; \vee\rangle$ and $\langle S; \wedge\rangle$ are *regular* bands, that is, semigroups of idempotents that satisfy $xyxzx = xyzx$. All skew lattices possess a coherent *natural partial order*: $x \geq y$ if $x \wedge y = y = y \wedge x$ or dually $x \vee y = x = y \vee x$. This refines the *natural preorder*: $x \succeq y$ if $y \wedge x \wedge y = y$ or dually $x \vee y \vee x = x$.

A skew lattice $\mathbf{S}$ is *symmetric* if $a \vee b = b \vee a$ if and only if $a \wedge b = b \wedge a$ for all $a, b \in S$. Symmetry thus makes instances of commutation unambiguous. A skew lattice $\mathbf{S}$ is *distributive* if, for all $x, y, z \in S$,

$$x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$$

and

$$x \vee (y \wedge z) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x).$$

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EXAMPLE 2.1. Let $\mathcal{P}(A, B)$ denote the set of all partial functions from a given set A to a second set B. Given supports $F, G \subseteq A$ and functions $f : F \to B$ and $g : G \to B$, we define functions $f \vee g$ and $f \wedge g$ with supports $F \cup G$ and $F \cap G$ as follows: $f \vee g = f \cup (g \mid G\backslash F)$ and $f \wedge g = g \mid (G \cap F)$. The function $f \vee g$ is often called the *override* since f overrides g on the common part of their support. Clearly, $f \vee g$ favors f while the restriction $f \wedge g$ favors g. The algebra $\mathbf{P} = \langle \mathcal{P}(A, B); \vee, \wedge \rangle$ is a skew lattice that is easily seen to be distributive and symmetric. In fact, $\mathbf{P}$ is *strongly distributive* in that it satisfies the identities

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \text{and} \quad (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z).$$

This leads us to recall that a skew lattice is *normal* if it satisfies the identity $x \wedge y \wedge z \wedge w = x \wedge z \wedge y \wedge w$.

THEOREM 2.2 [23]. *A skew lattice* $\mathbf{S}$ *is strongly distributive if and only if it is symmetric, distributive, and normal.* □

Being normal is equivalent to requiring that for each $e \in S$, $\lceil e \rceil = \{x \in S \mid e \geq x\}$ is commutative, *thus forming a sublattice of* $\mathbf{S}$. (Normality was studied by Leech in [23].)

Two other operations can be defined on $\mathcal{P}(A, B)$: the *difference*, $f \backslash g = f \mid (F \backslash G)$, an operation favoring f, and the nullary operation given by the constant $\emptyset$. The latter is the *zero* of $\mathcal{P}(A, B)$. In general, a zero element of a skew lattice, if it exists, is characterized by the identities $0 \vee x = x = x \vee 0$ and $0 \wedge x = 0 = x \wedge 0$. Zero elements, when they exist, are always unique. The difference and $\emptyset$ turn $\mathbf{P}$ into a variant of the Boolean algebra on the power set $\mathbb{P}(A)$.

A *skew Boolean algebra* is an algebra $\mathbf{S} = \langle S; \vee, \wedge, \backslash, 0 \rangle$ such that the $\langle \vee, \wedge, 0 \rangle$-reduct is a strongly distributive skew lattice with zero element 0 and $\backslash$ is a binary operation on S satisfying

$$(x \wedge y \wedge x) \vee (x \backslash y) = x \quad \text{and} \quad (x \wedge y \wedge x) \wedge (x \backslash y) = 0 = (x \backslash y) \wedge (x \wedge y \wedge x).$$

By symmetry, one has $(x \backslash f) \vee (x \wedge f \wedge x) = x$ also, so that $x \wedge y \wedge x$ commutes with $x \backslash y$. This all implies that each $\lceil x \rceil = \{u \in S \mid x \geq u\}$ is a Boolean sublattice of $\mathbf{S}$ with $x \backslash y$ being the unique complement of $x \wedge y \wedge x$ in $\lceil x \rceil$. Clearly, we have the following result.

THEOREM 2.3 [22, Theorem 1.8]. *Skew Boolean algebras form a variety.* □

The algebra $\langle \mathcal{P}(A, B); \vee, \wedge, \backslash, \emptyset \rangle$ is a skew Boolean algebra. In fact, it is *right-handed* in that $x \wedge y \wedge x = y \wedge x$ holds and, dually, $x \vee y \vee x = x \vee y$ also. (Likewise, a skew lattice is *left-handed* if $x \wedge y \wedge x = x \wedge y$ and $x \vee y \vee x = y \vee x$.) As with Boolean algebras and their power set exemplars, every right-handed skew Boolean algebra can be embedded in some partial function algebra $\langle \mathcal{P}(A, B); \vee, \wedge, \backslash, \emptyset \rangle$. (See Leech [22].)

The four operations $\vee$, $\wedge$, $\backslash$, and $\emptyset$ are not the only ones that can be defined on $\mathcal{P}(A, B)$. Given f, g in $\mathcal{P}(A, B)$, their *intersection* $f \cap g$ is exactly that given upon viewing f and g as subsets of the Cartesian product $A \times B$. As such, $f \cap g \leq$ both f and g and, given any $h \leq$ both f and g, then $h \leq f \cap g$ follows.

In general, a skew lattice $\langle S; \vee, \wedge\rangle$ *has intersections* (or is *with intersections*) if every pair $x, y \in S$ possesses a *natural* meet with respect to the natural partial order $\geq$ on S. We denote the natural meet of x and y, when it exists, by $x \cap y$ and call it their *intersection*. For any pair x and y, $x \wedge y$ coincides with $x \cap y$ if and only if $x \wedge y = y \wedge x$. The following results are from Bignall and Leech [4, Proposition 2.6 and Theorem 2.8].

Lemma 2.4. *A skew lattice with intersections is an algebra $\langle S; \vee, \wedge, \cap\rangle$ such that $\langle S; \cap\rangle$ is a meet semilattice, $\langle S; \vee, \wedge\rangle$ is a skew lattice, and the following identities hold:*

$$e \cap (e \wedge f \wedge e) = e \wedge f \wedge e \quad \text{and} \quad e \wedge (e \cap f) = e \cap f = (e \cap f) \wedge e.$$

Skew lattices with intersections, and in particular SBAs with intersections, thus form varieties of algebras.

Proof. The identities state in essence that the two partial orders on S induced by $\wedge$ and $\cap$ must contain each other and thus coincide. □

Theorem 2.5. *Skew lattices with intersections form a congruence distributive variety, as do skew Boolean algebras with intersections.*

Proof. Given a skew lattice with a natural meet, $\langle S; \vee, \wedge, \cap\rangle$, set

$$m(x, y, z) = (x \cap y) \vee (y \cap z) \vee (x \cap z)$$

and notice that $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$. This establishes that $\mathrm{Con}\,\langle S; \vee, \wedge, \cap\rangle$ is distributive. (See [10, Theorem II, Section 12.3].) □

The Green's equivalences, $\mathcal{D}$, $\mathcal{L}$, and $\mathcal{R}$, defined originally for semigroups, are extended to skew lattices as follows. The equivalence $\mathcal{D}$ is defined via the natural preorder $\succeq$ by $x \,\mathcal{D}\, y$ if $x \succeq y \succeq x$. Thus, $x \,\mathcal{D}\, y$ if and only if both $x \wedge y \wedge x = x$ and $y \wedge x \wedge y = y$ and, dually, both $x \vee y \vee x = x$ and $y \vee x \vee y = y$. Applying the Clifford–McLean theorem for bands (semigroups of idempotents), we know (1) that $\mathcal{D}$ is a skew lattice congruence; (2) that $\mathbf{S}/\mathcal{D}$ is the maximal lattice image of $\mathbf{S}$; and (3) that each $\mathcal{D}$-class is a maximal rectangular subalgebra of $\mathbf{S}$. Each $\mathcal{D}$-class is anticommutative in that $x \wedge y = y \wedge x$ (or $x \vee y = y \vee x$) if and only if $x = y$. If D denotes a $\mathcal{D}$-class, then both $\langle D; \wedge\rangle$ and $\langle D; \vee\rangle$ are rectangular bands that jointly satisfy $x \wedge y = y \vee x$. A zero element 0, if it exists, consists of a sole $\mathcal{D}$-class. If $x \wedge y = 0$, then also $y \wedge x = 0$ and, if $\mathbf{S}$ is symmetric, $x \vee y = y \vee x$. In general, if $x \wedge y = 0$ and $u \,\mathcal{D}\, x$ and $v \,\mathcal{D}\, y$, then $u \wedge v = 0$ also.

The equivalence $\mathcal{D}$ is refined by a pair of congruences, $\mathcal{L}$ and $\mathcal{R}$. We have $x \,\mathcal{L}\, y$ if $x \wedge y = x$ and $y \wedge x = y$ or, equivalently, $x \vee y = y$ and $y \vee x = x$. Likewise, $x \,\mathcal{R}\, y$ if $x \wedge y = y$ and $y \wedge x = x$ or, equivalently, $x \vee y = x$ and $y \vee x = y$. The congruence $\mathcal{L} \cap \mathcal{R} = \Delta$, the identity equivalence, while $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \vee \mathcal{R} = \mathcal{D}$. A skew lattice $\mathbf{S}$ is right-handed [respectively left-handed] if and only if $\mathcal{D} = \mathcal{R}$ [respectively $\mathcal{D} = \mathcal{L}$]. In general, $\mathbf{S}/\mathcal{R}$ and $\mathbf{S}/\mathcal{L}$ are the maximal left-handed and right-handed images of $\mathbf{S}$, respectively, and $\mathbf{S}$ is isomorphic to the fibered product $\mathbf{S}/\mathcal{R} \times_{\mathbf{S}/\mathcal{D}} \mathbf{S}/\mathcal{L}$ over their common maximal lattice image $\mathbf{S}/\mathcal{D}$.

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Given a skew lattice $\mathbf{S}$, an *ideal* of $\mathbf{S}$ is a subset I of S such that if $x, y \in I$ and $z \in S$, then $x \vee y$, $z \wedge x$, and $x \wedge z$ are in I. Given any element a in S, the *principal ideal* of a is the set $\langle a \rangle = \{x \in S \mid x \preceq a\}$. Clearly, $x \in \langle a \rangle$ if and only if $x \preceq b$ for all $b \in \mathcal{D}_a$, where $\mathcal{D}_a$ is the $\mathcal{D}$-class of a. If $\mathbf{S}$ has a zero element 0, the *annihilator* of a is the set $\mathrm{ann}(a) = \{x \in S \mid x \wedge a = 0\}$. Alternatively, it is both $\{x \in S \mid a \wedge x = 0\}$ and $\{x \in S \mid a \wedge x \wedge a = 0\}$. By our discussion above, $x \in \mathrm{ann}(a)$ if and only if $x \wedge b = 0 = b \wedge x$ for any (or all) $b \in \mathcal{D}_a$. If $\mathbf{S}$ is distributive, then $\mathrm{ann}(a)$ is easily seen to be an ideal. The ideal $\langle a \rangle$ and annihilator $\mathrm{ann}(a)$ can also be parameterized by the $\mathcal{D}$-class $A = \mathcal{D}_a$ as $\langle A \rangle$ and $\mathrm{ann}(A)$, respectively, since any b in $\mathcal{D}_a$ induces the same pair of sets.

Remarks. (1) Four distributive identities occur in this section: two characterize strong distributivity and two characterize distributivity. All four are equivalent for lattices. No two, however, are equivalent for skew lattices. In particular, the identities characterizing distributivity are not equivalent in general, but are so for symmetric skew lattices. (See [28] and [13].) A duality theory for strongly distributive skew lattices with zero has been developed that extends Priestly duality for distributive lattices with zero, itself an extension of classical Stone duality. (See [2].) (2) Given an SBA, any congruence on its skew lattice reduct is easily seen to be an SBA congruence [29, Proposition 1.4.27]. In particular, the congruences $\mathcal{D}$, $\mathcal{R}$, and $\mathcal{L}$ are SBA congruences; moreover, the canonical factorization $\mathbf{S} \cong \mathbf{S}/\mathcal{R} \times_{\mathbf{S}/\mathcal{D}} \mathbf{S}/\mathcal{L}$ of skew lattices extends to SBAs.

3. Algebras with intersections amongst skew Boolean algebras

If an SBA has intersections, it is because $\cap$ occurs naturally in that $\langle S; \geq \rangle$ has meets. Many SBAs thus have intersections. This includes finite SBAs or, more generally, SBAs with finite maximal lattice images. Even more generally, complete SBAs (where point-wise commuting subsets have suprema *and hence also infima*) have arbitrary intersections. The latter includes partial function algebras $\mathcal{P}(A, B)$. Free skew Boolean algebras have intersections. (See Kudryavtseva and Leech [21].) On the other hand, we have the following example.

Example 3.1. Let $S \subseteq \mathcal{P}(\mathbb{N}, \{0, 1\})$ be characterized by $f \in S$ if and only if $\mathrm{dom}(f)$ is either finite or else cofinite in $\mathbb{N}$ (in that $\mathbb{N} \setminus \mathrm{dom}(f)$ is finite). Let φ and ψ in S be defined by letting $\varphi(n) = 0$ for all $n \in \mathbb{N}$ and setting

$$\psi(n) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Then $\varphi \cap \psi$ in $\mathcal{P}(\mathbb{N}, \{0, 1\})$ is the partial function $\xi : \{0, 2, 4, \dots\} \to 0$. The set of all subfunctions of both φ and ψ in S consists of all restrictions of ξ to finite domains. Clearly, $\varphi \cap \psi \notin S$.

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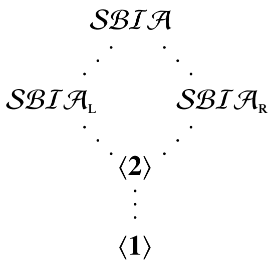
Consider the class of *implicit* SBIAs where $\cap$ is not built into the algebra's signature, that is, the class of their SBA reducts which is a proper subclass of SBAs. One can ask how implicit SBIAs fit into the core theory of SBAs. In particular, what is the relevance of the canonical SBA congruences $\mathcal{D}$, $\mathcal{R}$, and $\mathcal{L}$? And what is the role of left-handed and right-handed implicit SBIAs in their general theory? Here we consider aspects of varieties in the partial lattice of varieties below that directly mirrors the lattice of SBA varieties in the introduction.



Notation	Variety of
$\mathcal{SBIA}$	SBIAs
$\mathcal{SBIA}_{\text{L}}$	Left-handed SBIAs
$\mathcal{SBIA}_{\text{R}}$	Right-handed SBIAs
$\langle\mathbf{2}\rangle$	Generalized Boolean algebras or GBAs ($\wedge$ and $\cap$ merge)
$\langle\mathbf{1}\rangle$	Trivial algebras

We begin by proving two results that may seem to be at odds. The first is little more than an observation.

Proposition 3.2. *Given an SBIA* $\mathbf{S}$*, let* $\mathcal{D}$*,* $\mathcal{R}$*, and* $\mathcal{L}$ *be the Green's relations on* $\mathbf{S}$*. Then:*

(1) *$\mathcal{D}$ is an $\cap$-congruence on $\mathbf{S}$ if and only if $\mathcal{D} = \Delta$ so that $\mathbf{S}$ is a generalized Boolean algebra;*
(2) *$\mathcal{R}$ is an $\cap$-congruence on $\mathbf{S}$ if and only if $\mathcal{R} = \Delta$ so that $\mathbf{S}$ is left-handed;*
(3) *$\mathcal{L}$ is an $\cap$-congruence on $\mathbf{S}$ if and only if $\mathcal{L} = \Delta$ so that $\mathbf{S}$ is right-handed.*

Proof. 'If' is trivial. Conversely, if we say that $\mathcal{D}$ is a $\cap$-congruence but $\mathcal{D} \neq \Delta$, then $a \neq b$ in S exist such that $a \,\mathcal{D}\, b \,\mathcal{D}\, a \cap b$. But then $a \geq a \cap b \leq b$ in this $\mathcal{D}$-class, which is possible only if $a = b$. Thus, if $\mathcal{D}$ is an $\cap$-congruence, then $\mathcal{D} = \Delta$, making $\mathbf{S}$ a generalized Boolean algebra. The $\mathcal{R}$ and $\mathcal{L}$ cases are similar. □

On the other hand, we have the following result.

Theorem 3.3. *A skew Boolean algebra* $\mathbf{S}$ *has intersections if and only if both* $\mathbf{S}/\mathcal{L}$ *and* $\mathbf{S}/\mathcal{R}$ *have them. More generally, a normal skew lattice* $\mathbf{S}$ *has intersections if and only if both* $\mathbf{S}/\mathcal{L}$ *and* $\mathbf{S}/\mathcal{R}$ *have them. (In all cases,* $\mathbf{S}/\mathcal{D}$ *possesses intersections trivially since* $\cap$ *merges with* $\wedge$*.)*

To prove this, we will need the following result.

Proposition 3.4. *If a skew lattice* $\mathbf{S}$ *with intersections has a lattice section (a sublattice meeting each* $\mathcal{D}$*-class in a unique point), then both* $\mathbf{S}/\mathcal{L}$ *and* $\mathbf{S}/\mathcal{R}$ *also have intersections.*

Proof. If $\mathbf{T} \leq \mathbf{S}$ is a lattice section in $\mathbf{S}$, then a copy of $\mathbf{S}/\mathcal{R}$ in $\mathbf{S}$ is given by $\mathbf{T}[\mathcal{L}] = \bigcup_{t\in T} \mathcal{L}_t$. Since $\mathcal{L}$ is a congruence, $\mathbf{T}[\mathcal{L}]$ is a (necessarily maximal) left-handed subalgebra of $\mathbf{S}$; the natural epimorphism from $\mathbf{S}$ to $\mathbf{S}/\mathcal{R}$ moreover restricts to an isomorphism of $\mathbf{T}[\mathcal{L}]$ with $\mathbf{S}/\mathcal{R}$. Given $x \in S$, let t_x be the unique element in $\mathcal{D}_x \cap T$ and set $x_{\mathcal{L}} = x \wedge t_x$ in $T[\mathcal{L}]$. Then $T[\mathcal{L}] = \{x_{\mathcal{L}} \mid x \in S\}$, $x \,\mathcal{R}\, x_{\mathcal{L}} \,\mathcal{L}\, t_x$, and the map $x \to x_{\mathcal{L}}$ is a retraction of $\mathbf{S}$ upon $\mathbf{T}[\mathcal{L}]$ with $x_{\mathcal{L}}$ corresponding to the natural image $\mathcal{R}_x$ of x in $\mathbf{S}/\mathcal{R}$. (See Cvetko-Vah [14].) We prove the proposition by showing that $x, y \in T[\mathcal{L}]$ implies that $x \cap y \in T[\mathcal{L}]$. So, consider $(x \cap y)_{\mathcal{L}} = (x \cap y) \wedge t_{x\cap y}$. We have $x \wedge (x \cap y)_{\mathcal{L}} = x \wedge (x \cap y) \wedge t_{x\cap y} = (x \cap y) \wedge t_{x\cap y} = (x \cap y)_{\mathcal{L}}$ and likewise $y \wedge (x \cap y)_{\mathcal{L}} = (x \cap y)_L$. On the other hand, $(x \cap y)_{\mathcal{L}} \wedge x = (x \cap y) \wedge t_{x\cap y} \wedge x \wedge t_x$ since $x \in T[\mathcal{L}]$. But, since $x \geq t_x, t_{x\cap y}$, regularity gives $t_{x\cap y} \wedge x \wedge t_x = t_{x\cap y} \wedge t_x = t_{x\cap y}$, so that $(x \cap y)_{\mathcal{L}} \wedge x = (x \cap y)_{\mathcal{L}}$ and likewise $(x \cap y)_{\mathcal{L}} \wedge y = (x \cap y)_{\mathcal{L}}$. Thus, $x, y \geq (x \cap y)_{\mathcal{L}}$ and we must have $x \cap y \geq (x \cap y)_{\mathcal{L}}$. But, since $x \cap y \,\mathcal{D}\, (x \cap y)_{\mathcal{L}}$, $x \cap y = (x \cap y)_{\mathcal{L}}$ follows and $\mathbf{T}[\mathcal{L}]$ is closed under intersections. Likewise, the dual subalgebra $\mathbf{T}[\mathcal{R}] = \bigcup_{t\in T} \mathcal{R}_t$ is also closed under intersections, and the proposition follows. □

Proof of Theorem 3.3. A skew lattice $\mathbf{S}$ has intersections if and only if each principal ideal $S \wedge x \wedge S$ has intersections. Given a normal skew lattice $\mathbf{S}$, then each ideal $S \wedge x \wedge S$ has a lattice section, namely $x \wedge S \wedge x$, with $x \wedge S = (x \wedge S \wedge x)[\mathcal{R}] \cong (S \wedge x \wedge S)/\mathcal{L}$ and $S \wedge x = (x \wedge S \wedge x)[\mathcal{L}] \cong (S \wedge x \wedge S)/\mathcal{R}$. But $(S \wedge x \wedge S)/\mathcal{L}$ and $(S \wedge x \wedge S)/\mathcal{R}$ in turn form the principal ideals of $\mathbf{S}/\mathcal{L}$ and $\mathbf{S}/\mathcal{R}$, respectively. Thus, if $\mathbf{S}$ has intersections, so do all $(S \wedge x \wedge S)/\mathcal{L}$ and $(S \wedge x \wedge S)/\mathcal{R}$ and hence $\mathbf{S}/\mathcal{L}$ and $\mathbf{S}/\mathcal{R}$.

Conversely, let $\mathbf{S}/\mathcal{L}$ and $\mathbf{S}/\mathcal{R}$ have finite intersections, where $\mathbf{S}$ is normal. We represent $\mathbf{S}$ as the fibred product $\mathbf{S}/\mathcal{L} \times_{\mathbf{S}/\mathcal{D}} \mathbf{S}/\mathcal{R}$. So, let both (x', x'') and (y', y'') in S be given, where $x', y' \in S/\mathcal{L}$ and $x'', y'' \in S/\mathcal{R}$. Since $\mathbf{S}/\mathcal{L}$ and $\mathbf{S}/\mathcal{R}$ have finite intersections, $x' \cap y'$ and $x'' \cap y''$ exist in $S/\mathcal{L}$ and $S/\mathcal{R}$, respectively. If $x' \cap y'$ and $x'' \cap y''$ share a common image in $S/\mathcal{D}$, then $(x', x'') \cap (y', y'')$ is just $(x' \cap y', x'' \cap y'')$. In general, let $u_0 \wedge v_0$ be the meet in $\mathbf{S}/\mathcal{D}$ of the respective images u_0 of $x' \cap y'$ and v_0 of $x'' \cap y''$ in $\mathbf{S}/\mathcal{D}$. In the respective $\mathcal{D}$-classes of $\mathbf{S}/\mathcal{L}$ and $\mathbf{S}/\mathcal{R}$ indexed by $u_0 \wedge v_0$, unique elements w' and w'' exist (by normality) such that both $x' \cap y' \geq w'$ in $\mathbf{S}/\mathcal{L}$ and $x'' \cap y'' \geq w''$ in $\mathbf{S}/\mathcal{R}$. The intersection $(x', x'') \cap (y', y'')$ is then precisely (w', w''). □

To complete this picture, here is a second observation (the first being Proposition 3.2).

Proposition 3.5. *Given a skew Boolean algebra* $\mathbf{S}$ *with intersections,* $(x_{\mathrm{L}}, x_{\mathrm{R}}) \cap (y_{\mathrm{L}}, y_{\mathrm{R}}) = (x_{\mathrm{L}} \cap y_{\mathrm{L}}, x_{\mathrm{R}} \cap y_{\mathrm{R}})$ *holds in* $\mathbf{S}/\mathcal{L} \times_{\mathbf{S}/\mathcal{D}} \mathbf{S}/\mathcal{R}$ *if and only if* $\mathbf{S}$ *is a generalized Boolean algebra (making* $\mathcal{D} = \mathcal{R} = \mathcal{L} = \Delta$*).*

Proof. If $\mathbf{S}$ is indeed commutative, then things are trivialized and the identity holds. Otherwise, $x \neq y$ in S exist such that either $x \,\mathcal{L}\, y$ or $x \,\mathcal{R}\, y$, say $x \,\mathcal{L}\, y$. Then $x_{\mathrm{L}} \neq y_{\mathrm{L}}$ in $\mathbf{S}/\mathcal{R}$, so that $x_{\mathrm{L}} \cap y_{\mathrm{L}}$ lies in a lower $\mathcal{D}$-class in $\mathbf{S}/\mathcal{R}$ than that of x_{L} and y_{L}, while $x_{\mathrm{R}} \cap y_{\mathrm{R}} = x_{\mathrm{R}} = y_{\mathrm{R}}$ in a common $\mathcal{D}$-class in $\mathbf{S}/\mathcal{L}$. It follows that $(x_{\mathrm{L}} \cap y_{\mathrm{L}}, x_{\mathrm{R}} \cap y_{\mathrm{R}})$ is in

$\mathbf{S}/\mathcal{R} \times \mathbf{S}/\mathcal{L}$, but not in $\mathbf{S}/\mathcal{L} \times_{\mathbf{S}/\mathcal{D}} \mathbf{S}/\mathcal{R}$. Thus, the identity in the proposition's statement does not hold in the fibered product. □

Since similar identities for $\vee$, $\wedge$, and $\backslash$ do hold, for example $(x_{\mathrm{L}}, x_{\mathrm{R}}) \vee (y_{\mathrm{L}}, y_{\mathrm{R}}) = (x_{\mathrm{L}} \vee y_{\mathrm{L}}, x_{\mathrm{R}} \vee y_{\mathrm{R}})$, we get the following result.

Corollary 3.6. *The operation $\cap$ cannot be polynomially defined in terms of $\vee$, $\wedge$, and $\backslash$.* □

This, of course, also follows from Proposition 3.2.

Returning to Theorem 3.3, hopefully one can appreciate its simple significance. Just as all SBAs can be constructed to within isomorphism from pairs of left- and right-handed SBAs, $\mathbf{S}/\mathcal{L}$ and $\mathbf{S}/\mathcal{R}$, sharing a common maximal commutative image $\mathbf{B}$ by taking the fibered product $\mathbf{S}/\mathcal{L} \times_{\mathbf{B}} \mathbf{S}/\mathcal{R}$, so also all SBIAs can be constructed to within isomorphism from pairs of left- and right-handed SBIAs, $\mathbf{S}/\mathcal{L}$ and $\mathbf{S}/\mathcal{R}$, whose SBA reducts share a common maximal commutative image $\mathbf{B}$ by forming the fibered product over $\mathbf{B}$. Thus, like SBAs, many (but not all) aspects of SBIAs can be reduced to studying the right-handed cases or their term-equivalent left-handed duals.

An instance of this occurs in constructing the free skew Boolean algebra $\mathbf{F}_{\mathbf{SBA}}(X)$ on a set X. (See [21].) It is more convenient to first construct the free left-handed SBA, $\mathbf{F}_{\mathbf{LSBA}}(X)$. The free right-handed SBA, $\mathbf{F}_{\mathbf{RSBA}}(X)$, is just its left–right dual (where $x \vee * y = y \vee x$ and $x \wedge * y = y \wedge x$, but $x \backslash y$ and 0 remain the same). Both algebras share a maximal generalized Boolean algebra (GBA) image, $\mathbf{F}_{\mathbf{GBA}}(X)$, the free GBA on X. The image $\mathbf{F}_{\mathbf{SBA}}(X)$ is just the fibered product over $\mathbf{F}_{\mathbf{GBA}}(X)$ of the two one-sided free algebras. Since both factor algebras are seen to have intersections, so must $\mathbf{F}_{\mathbf{SBA}}(X)$.

Before going to the next section, observe that Example 3.1 shows that the class of implicit SBIAs is not closed under the processes of taking SBA subalgebras or taking SBA homomorphic images. As an SBA, this example is a subalgebra of $\mathcal{P}(\mathbb{N}, \{0, 1\})$ and the homomorphic image of some free SBA. Implicit SBIAs, however, *are* closed under products.

4. The subdirectly irreducible algebras

When $\mathbf{S}$ is a skew Boolean algebra, given a $\mathcal{D}$-class A, the relation between $\langle A \rangle$ and $\mathrm{ann}(A)$ can be sharpened to give a decomposition of primary importance.

Theorem 4.1. *Given a $\mathcal{D}$-class A of a skew Boolean algebra $\mathbf{S}$, then:*

(1) *both $\langle A \rangle$ and $\mathrm{ann}(A)$ are ideals of $\mathbf{S}$;*
(2) *all elements of $\langle A \rangle$ commute with all elements of $\mathrm{ann}(A)$;*
(3) *in particular, for all $u \in \langle A \rangle$ and all $v \in \mathrm{ann}(A)$, $u \wedge v = 0 = v \wedge u$;*
(4) *the map $\mu : \langle A \rangle \times \mathrm{ann}(A) \to S$ defined by $\mu(e, f) = e \vee f$ is an isomorphism.*

Proof. See Leech [22, Paragraph 1.10 and Lemma 1.11]. □

Given skew lattices $\mathbf{S}$ and $\mathbf{T}$, their product $\mathbf{S} \times \mathbf{T}$ has intersections if and only if $\mathbf{S}$ and $\mathbf{T}$ each do, with $(x, y) \cap (x', y') = (x \cap x', y \cap y')$. We thus have the following result.

COROLLARY 4.2. *Given a principal ideal* $\langle A\rangle$ *in a skew Boolean algebra* $\mathbf{S}$ *and its associated annihilator ideal* $\mathrm{ann}(A)$*,* $\mathbf{S}$ *has intersections if and only if both* $\langle A\rangle$ *and* $\mathrm{ann}(A)$ *have intersections, in which case the map* $\mu : \langle A\rangle \times \mathrm{ann}(A) \to S$ *defined by* $\mu(x, y) = x \vee y$ *is an isomorphism of skew Boolean* $\cap$*-algebras. In particular, given* x, y *in* $\langle A\rangle$ *and* u, v *in* $\mathrm{ann}(A)$, $(x \vee u) \cap (y \vee v) = (x \cap y) \vee (u \cap v)$. □

This holds not just for finite intersections (the meaning of 'having intersections'), but also for arbitrary intersections, as in the case of $\mathcal{P}(A, B)$.

A skew Boolean algebra is *primitive* if it consists of two $\mathcal{D}$-classes: $D > \{0\}$. All such algebras are created by the following process: take a rectangular band D (satisfying $xyz = xz$) and create a rectangular skew lattice by setting $x \wedge y = xy$ and $x \vee y = yx$. Then adjoin an element $0 \notin D$ along with the extended outcomes, $x \wedge 0 = 0 = 0 \wedge x$ and $x \vee 0 = x = 0 \vee x$. Next set $x \backslash y = 0$ if $y \neq 0$, but $= x$ when $y = 0$. This gives a primitive skew Boolean algebra. Theorem 4.1 implies that *the nontrivial directly irreducible skew Boolean algebras are precisely the primitive algebras. Skew Boolean algebras with finitely many* $\mathcal{D}$*-classes are thus direct products of primitive algebras*. Finally, note that *primitive algebras have intersections*: $x \cap y = x$ if $x = y$ and 0 otherwise. Thus, *all skew Boolean algebras with finitely many* $\mathcal{D}$*-classes have intersections*. We also have the following result.

THEOREM 4.3. *In the variety of skew Boolean* $\cap$*-algebras, the following hold:*

(1) *the primitive algebras are the nontrivial simple algebras;*

(2) *the primitive algebras are the nontrivial subdirectly irreducible algebras.*

PROOF. By Theorem 4.1, nontrivial simple (respectively, subdirectly irreducible) skew Boolean algebras must be primitive. Conversely, let $\mathbf{S}$ be a primitive algebra and let θ be a congruence on $\mathbf{S}$. Suppose that $e\,\theta\,f$ with $e \neq f$ in S. Then e and f are also θ-congruent to $e \cap f = 0$. Since either $e \neq 0$ or $f \neq 0$, this forces every element of the primitive algebra to be congruent to 0. Hence, θ is the universal congruence. The only other congruence possible is thus the identity congruence Δ. Thus, all primitive algebras are both simple and hence also subdirectly irreducible. □

COROLLARY 4.4. *Given a primitive SBA* $\mathbf{P}$*, any* $\cap$*-preserving homomorphism* φ *from* $\mathbf{P}$ *to an SBIA* $\mathbf{S}$ *is either the* 0*-homomorphism or an embedding.*

5. The lattice of subvarieties

The following notation is observed. For all $n \leq \aleph_0$, $\mathbf{n_L}$ [or $\mathbf{n_R}$] denotes the left-handed [right-handed] primitive skew Boolean $\cap$-algebra on $\mathbf{n} = \{0, 1, 2, \ldots, n-1\}$ or on $\{0, 1, 2, \ldots\}$ if $n = \aleph_0$, with 0 the zero element. In the left-handed case, $x \wedge y = x$ and $x \vee y = y$ for both $x, y \neq 0$. In the right-handed case, $x \wedge y = y$ and $x \vee y = x$ if both $x, y \neq 0$. Given primitive algebras $\mathbf{A}$ and $\mathbf{B}$, we denote their fibered product $\mathbf{A} \times_2 \mathbf{B}$, also a primitive algebra, by $\mathbf{A} \bullet \mathbf{B}$. If $\mathbf{A}$ and $\mathbf{B}$ have $\mathcal{D}$-class structures $A' > \{0\}$ and $B' > \{0\}$, then $\mathbf{A} \bullet \mathbf{B}$ has the $\mathcal{D}$-class structure $A' \times B' > \{(0, 0)\}$, where $\mathbf{A}' \times \mathbf{B}'$ is the direct product of rectangular skew lattices $\mathbf{A}'$ and $\mathbf{B}'$, and $(0, 0)$ is the zero element, often replaced by 0. Our interest is in algebras $\mathbf{m_L} \bullet \mathbf{m_R}$ for which $\mathbf{m_L} \bullet \mathbf{n_R}/\mathcal{L} \cong \mathbf{n_R}$ while $\mathbf{m_L} \bullet \mathbf{n_R}/\mathcal{R} \cong \mathbf{m_L}$. *Each finite primitive algebra is a copy of some* $\mathbf{m_L} \bullet \mathbf{n_R}$. Finally, given any skew Boolean $\cap$-algebra $\mathbf{A}$, $\langle \mathbf{A} \rangle^{\cap}$ denotes the principal subvariety of all skew Boolean $\cap$-algebras generated by $\mathbf{A}$ in that they satisfy all identities satisfied by $\mathbf{A}$. Consider the following lattice Ω of subalgebras of the primitive algebra $\aleph_{0\mathbf{L}} \bullet \aleph_{0\mathbf{R}}$ with all viewed as skew Boolean $\cap$-algebras.

$$\Omega: \quad \begin{tikzcd}
 & & & \aleph_{0\mathbf{L}} \bullet \aleph_{0\mathbf{R}} & & & \\
 & \aleph_{0\mathbf{L}} \bullet \mathbf{3_R} \arrow[urr, dotted] & & & & \mathbf{3_L} \bullet \aleph_{0\mathbf{R}} \arrow[ull, dotted] & \\
\aleph_{0\mathbf{L}} \arrow[ur] & & & & & & \aleph_{0\mathbf{R}} \arrow[ul] \\
 & \mathbf{4_L} \arrow[ul, dotted] \arrow[ur, dotted] & & \mathbf{3_L} \bullet \mathbf{3_R} \arrow[ul, dotted] \arrow[ur, dotted] & & \mathbf{4_R} \arrow[ul, dotted] \arrow[ur, dotted] & \\
 & & \mathbf{3_L} \arrow[ul] \arrow[ur] & & \mathbf{3_R} \arrow[ul] \arrow[ur] & & \\
 & & & \mathbf{2} \arrow[ul] \arrow[ur] & & & \\
 & & & \mathbf{1} \arrow[u] & & &
\end{tikzcd}$$

In this diagram, each $\mathbf{n_L}$ is identified with the trivial fibered product, $\mathbf{n_L} \bullet \mathbf{2}$, and each $\mathbf{n_R}$ is likewise identified with the trivial fibered product, $\mathbf{2} \bullet \mathbf{n_R}$. The embeddings $\rightarrow$ are induced from the standard chain of inclusions: $\{0\} \subset \{0, 1\} \subset \{0, 1, 2\} \subset \{0, 1, 2, 3\} \subset \cdots$.

Proposition 5.1. *The map* $\mathbf{A} \rightarrow \langle \mathbf{A} \rangle^{\cap}$ *applied to the above diagram of inclusions induces a corresponding diagram of strict inclusions of the respective varieties, with:*

(1) $\langle \aleph_{0\mathbf{L}} \bullet \mathbf{n_R} \rangle^{\cap} = \bigcup \{\langle \mathbf{m_L} \bullet \mathbf{n_R} \rangle^{\cap} \mid m < \aleph_0\}$ *for all* $n < \aleph_0$*;*
(2) $\langle \mathbf{m_L} \bullet \aleph_{0\mathbf{R}} \rangle^{\cap} = \bigcup \{\langle \mathbf{m_L} \bullet \mathbf{n_R} \rangle^{\cap} \mid n < \aleph_0\}$ *for all* $m < \aleph_0$*;*
(3) $\langle \aleph_{0\mathbf{L}} \bullet \aleph_{0\mathbf{R}} \rangle^{\cap} = \bigcup \{\langle \mathbf{m_L} \bullet \mathbf{n_R} \rangle^{\cap} \mid m, n < \aleph_0\}$.

(The variety $\langle \aleph_{0\mathbf{L}} \bullet \aleph_{0\mathbf{R}} \rangle^{\cap}$ *is, of course, the variety of all SBIAs, also denoted by* $\mathcal{SBIA}$*.)*

Proof. Clearly, we have a diagram of inclusions. Since any equation in the operations of the signature contains only finitely many variables, the final three assertions are clear. Thus, we need only show the inclusions of the induced subvarieties to be proper in the case of the finite primitive algebras. We first show $\langle \mathbf{n_L} \rangle^\cap \subset \langle \mathbf{n+1_L} \rangle^\cap$ to be proper for all finite n. To begin, $x \approx y$ holds on $\langle \mathbf{1} \rangle^\cap$ but not on $\langle \mathbf{2} \rangle^\cap$. Next, for $n \geq 2$, we set

$$\Phi_n(x_1, x_2, \dots, x_n) = (x_1 \backslash (x_1 \cap x_2)) \wedge (x_2 \backslash (x_2 \cap x_3)) \wedge \cdots \wedge (x_n \backslash (x_n \cap x_1))$$

and

$$\begin{aligned}\Psi_n(x_1, x_2, \dots, x_n) = x_1 \backslash [(x_1 \cap x_2) \vee \cdots \vee (x_1 \cap x_n) \vee \\ (x_2 \cap x_3) \vee \dots (x_2 \cap x_n) \vee \cdots \vee (x_{n-1} \cap x_n)].\end{aligned}$$

Since $\Phi_n(a_1, a_2, \dots, a_n) \neq 0$ only if none of $a_1, a_2, \dots, a_n$ is 0,

$$\Phi_n(x_1, x_2, \dots, x_n) \wedge \Psi_n(x_1, x_2, \dots, x_n) = 0$$

holds on $\mathbf{n_L}$ but not on $\mathbf{n+1_L}$ (respectively, on $\mathbf{n_R}$ but not on $\mathbf{n+1_R}$) for all $n \geq 2$. All inclusions at least along the two lower sides of the above diagram are seen to be proper. But this forces all links in the above diagram to be proper. For instance, $\langle \mathbf{m_L} \bullet \mathbf{n_R} \rangle^\cap \subseteq \langle \mathbf{m+1_L} \bullet \mathbf{n_R} \rangle$ for $m < \aleph_0$ and $n \leq \aleph_0$ is proper since

$$\Phi_m(x_1 \wedge y, x_2 \wedge y, \dots, x_m \wedge y) \wedge \Psi_m(x_1 \wedge y, x_2 \wedge y, \dots, x_m \wedge y) \approx 0$$

must hold in $\langle \mathbf{m_L} \bullet \mathbf{n_R} \rangle^\cap$ but not in $\langle \mathbf{m+1_L} \bullet \mathbf{n_R} \rangle$. □

REMARK. The part in the proof regarding the left-handed (or right-handed) case was essentially in Bignall's 1976 dissertation [3].

Thus, Ω and its induced array $\Omega^{\langle\rangle}$ of principal varieties possess lattice isomorphic structures (if partially ordered by inclusion). The latter, however, is not the full lattice of varieties for SBIAs. But, before proceeding on that front, we first have the following result.

THEOREM 5.2. *Skew Boolean $\cap$-algebras are locally finite.*

PROOF. Given an SBIA $\mathbf{S}$ generated from a finite set X of size n, if $\varphi : \mathbf{S} \to \mathbf{P}$ is a nontrivial homomorphism from $\mathbf{S}$ to a primitive algebra $\mathbf{P}$, then $\varphi[\mathbf{S}]$ is a primitive subalgebra $\mathbf{P}'$ of $\mathbf{P}$ that is isomorphic to a subalgebra of $\mathbf{n+1_L} \bullet \mathbf{n+1_R}$. It follows that a homomorphism of $\varphi' : \mathbf{S} \to \mathbf{n+1_L} \bullet \mathbf{n+1_R}$ exists inducing the same congruence on $\mathbf{S}$ that φ has. Moreover, only finitely many distinct homomorphisms from $\mathbf{S}$ to $\mathbf{n+1_L} \bullet \mathbf{n+1_R}$ are possible since $\mathbf{S}$ is generated from X. Thus, $\mathbf{S}$ can be embedded in a finite power of $\mathbf{n+1_L} \bullet \mathbf{n+1_R}$, making $\mathbf{S}$ itself finite. □

Theorem 5.2 also follows directly from Theorem 4.3 and [27, Theorem 1].

THEOREM 5.3. *A (quasi-)identity of signature $\langle \vee, \wedge, \backslash, \cap, 0 \rangle$ in n variables holds for all skew Boolean $\cap$-algebras if and only if it holds in $\mathbf{n+1_L} \bullet \mathbf{n+1_R}$, itself an algebra with n generators. Likewise, the (quasi-)identity holds for all left-handed [respectively right-handed] skew Boolean $\cap$-algebras if and only if it holds in $\mathbf{n+1_L}$ [respectively in $\mathbf{n+1_R}$]. The question of when a given (quasi-)identity holds for all (left-handed or right handed) skew Boolean $\cap$-algebras is thus decidable.*

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Proof. A (quasi-)identity in variables $x_1, \ldots, x_n$ holds for all SBIAs if and only if it holds on all such algebras with $\leq n$ generators. From the previous proof, this happens if and only if the (quasi-)identity holds on $\mathbf{n+1_L} \bullet \mathbf{n+1_R}$. The left-handed and right-handed cases are similar. □

In light of the results of this paper, Theorem 5.3 may also be deduced directly from the quasi-equational version of Harrop's theorem [8, Lemma 3.13].

We denote by Ω_0 and $\Omega_0^{\langle\rangle}$ the isomorphic sublattices of Ω and $\Omega^{\langle\rangle}$ determined by the finite cases $\mathbf{m_L} \bullet \mathbf{n_R}$ and the varieties $\langle \mathbf{m_L} \bullet \mathbf{n_R} \rangle^{\cap}$ they generate. At this stage, a result of Davey is relevant. To begin, given a partially ordered set $\mathbf{P} = \langle P; \geq \rangle$, an *order ideal* of $\mathbf{P}$ is any nonempty subset I of P satisfying the implication: if $x \in I$ and $x \geq y$ in P, then $y \in I$ also. Our interest here is in Ω_0 or its isomorphic copy $\Omega_0^{\langle\rangle}$ when ordered by inclusion: $A \geq B$ if and only if $A \supseteq B$. The theorem is as follows. (See [18, Theorems 3.3 and 3.5].)

Theorem 5.4 (Davey [18]). *Let $\mathcal{V}$ be a locally finite, congruence distributive variety. Then its lattice of subvarieties is completely distributive and is isomorphic to the lattice of order ideals of its partially ordered set of principal subvarieties generated by finite, subdirectly irreducible algebras and ordered by subvariety inclusion.* □

This plus Proposition 5.1, Theorem 5.2, and our remarks gives us the following result.

Theorem 5.5. *The lattice of varieties of skew Boolean $\cap$-algebras is completely distributive and is isomorphic to the lattice of order ideals of the lattice Ω_0 of finite primitive algebras including $\mathbf{1}$, when partially ordered by inclusion.*

The lattices of varieties of left-handed skew Boolean $\cap$-algebras, of their right-handed duals, and of generalized Boolean algebras are respectively the lattice

$$\langle \mathbf{1} \rangle^{\cap} < \langle \mathbf{2} \rangle^{\cap} < \langle \mathbf{3}_{\mathrm{L}} \rangle^{\cap} < \cdots < \langle \aleph_{0\mathbf{L}} \rangle^{\cap}$$

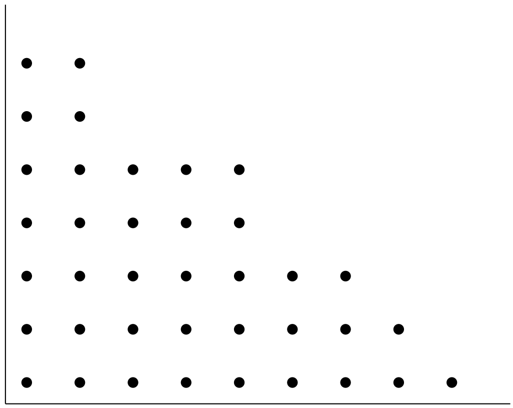
its right-handed counterpart, and their common subchain $\{\langle \mathbf{1} \rangle^{\cap} < \langle \mathbf{2} \rangle^{\cap}\}$. □

Remarks. (1) Again, the left- and right-handed cases go back to Bignall [3]. (2) In general, various proper inclusions occur (for example, $\langle \mathbf{3_L}, \mathbf{3_R} \rangle^{\cap} \subset \langle \mathbf{3_L} \bullet \mathbf{3_R} \rangle^{\cap}$) that take us beyond $\Omega^{\langle\rangle}$. (3) Interestingly, the lattice of varieties of SBAs in Section 1 is also completely distributive and isomorphic to a lattice of order ideals. (4) $\langle \mathbf{3_R} \rangle^{\cap}$ plays a special role in a construction studied by Leech and Spinks [25].

The sublattice of nontrivial varieties containing at least $\langle \mathbf{2} \rangle$ is described in Cartesian fashion using abbreviated notation in the array below. Here m, n represents $(\mathbf{m+1})_{\mathbf{L}} \bullet (\mathbf{n+1})_{\mathbf{R}}$ with m and n counting the sizes of both non-0 $\mathcal{D}$-classes. (m is thus the 'y-variable' and n the 'x variable'.) One has $m, n \geq p, q$ when $m \geq p$ and $n \geq q$, so that p, q lies nonstrictly to the lower left of m, n, which represents the fact that $(\mathbf{p+1})_{\mathbf{L}} \bullet (\mathbf{q+1})_{\mathbf{R}}$ is a subalgebra of $(\mathbf{m+1})_{\mathbf{L}} \bullet (\mathbf{n+1})_{\mathbf{R}}$.

$$
\begin{array}{c|cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
5 & 5,1 & 5,2 & 5,3 & 5,4 & 5,5 & \cdots \\
4 & 4,1 & 4,2 & 4,3 & 4,4 & 4,5 & \cdots \\
3 & 3,1 & 3,2 & 3,3 & 3,4 & 3,5 & \cdots \\
2 & 2,1 & 2,2 & 2,3 & 2,4 & 2,5 & \cdots \\
1 & 1,1 & 1,2 & 1,3 & 1,4 & 1,5 & \cdots \\
\hline
 & 1 & 2 & 3 & 4 & 5 & \cdots
\end{array}
$$

In the quadrant, a nontrivial ideal corresponds to a nonincreasing array of the terraced form below.



As such, it is described by a nonstrictly decreasing function f from the set $\{1, 2, 3, \ldots\}$ to the set $\{0, 1, 2, 3, 4, \ldots, \aleph_0\}$, where $f(n)$ measures the highest dot in the n-column, which must correspond to $(\mathbf{f(n)+1})_{\mathbf{L}} \bullet (\mathbf{n+1})_{\mathbf{R}}$. The function f for the above array is thus

$$
f = \begin{array}{c|ccccccccccc}
n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\
\hline
f(n) & 7 & 7 & 5 & 5 & 5 & 3 & 3 & 2 & 1 & 0 & \cdots
\end{array}
$$

corresponding to the variety

$$
\langle \mathbf{8_L} \bullet \mathbf{3_R} \rangle^{\cap} \cup \langle \mathbf{6_L} \bullet \mathbf{6_R} \rangle^{\cap} \cup \langle \mathbf{4_L} \bullet \mathbf{8_R} \rangle^{\cap} \cup \langle \mathbf{3_L} \bullet \mathbf{9_R} \rangle^{\cap} \cup \langle \mathbf{2_L} \bullet \mathbf{10_R} \rangle^{\cap},
$$

since the algebra $\mathbf{8_L} \bullet \mathbf{3_R}$ has paired non-0 $\mathcal{D}$-classes of sizes 7 and 2, $\mathbf{6_L} \bullet \mathbf{6_R}$ has paired non-0 $\mathcal{D}$-classes of sizes 5 and 5, and so forth. Since $\mathbf{8_L} \bullet \mathbf{3_R}$, $\mathbf{6_L} \bullet \mathbf{6_R}$, etc, belong to the variety, so do all the $(\mathbf{m+1})_{\mathbf{L}} \bullet (\mathbf{n+1})_{\mathbf{R}}$ represented by dots to the (nonstrictly) lower left of each of these dots. Hence, $f(1) = 7 = f(2)$, $f(3) = 5 = f(4) = f(5)$, etc. Since nontrivial varieties are determined by the algebras of type $(\mathbf{m+1})_{\mathbf{L}} \bullet (\mathbf{n+1})_{\mathbf{R}}$ that they contain, the join of two such varieties is determined by the union of their sets of such algebras and their intersection by all common algebras of this form.

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In general, each variety is the union of a finite number of 'upper right corner subvarieties' with the latter including the possibilities of $\langle \aleph_{0\mathbf{L}} \bullet \mathbf{n_R} \rangle^{\cap}$ and $\langle \mathbf{m_L} \bullet \aleph_{0\mathbf{R}} \rangle^{\cap}$. The variety of all SBIAs is of course $\langle \aleph_{0\mathbf{L}} \bullet \aleph_{0\mathbf{R}} \rangle^{\cap}$. Thus, since only finitely many strict decreases in the output are possible, the number of such functions (ideals) is countably infinite. Even the trivial ideal $\langle \mathbf{1} \rangle^{\cap}$ can be represented as the zero function: $z(n) = 0$ for all n. The lattice operations are evaluated in point-wise fashion: $(f \vee g)(n) = \max\{f(n), g(n)\}$ and $(f \wedge g)(n) = \min\{f(n), g(n)\}$. We thus have the following result.

Theorem 5.6. *The lattice of varieties of skew Boolean algebras with intersections is isomorphic to the lattice of nonstrictly decreasing functions from the set* $\{1, 2, 3, \ldots\}$ *to the set* $\{0, 1, 2, 3, \ldots, \aleph_0\}$ *with the join and meet operations given point-wise as above. In either the right- or left-handed cases, the lattice of varieties is isomorphic to the usual lattice ordering on* $\{1, 2, 3, \ldots, \aleph_0\}$. □

Another consequence of the above observation of 'only finitely many strict decreases' is the fact that *every variety is principal* in that it is generated from a single algebra. In the case of the example above, the variety is generated by the direct product:

$$\mathbf{8_L} \bullet \mathbf{3_R} \times \mathbf{6_L} \bullet \mathbf{6_R} \times \mathbf{4_L} \bullet \mathbf{8_R} \times \mathbf{3_L} \bullet \mathbf{9_R} \times \mathbf{2_L} \bullet \mathbf{10_R}.$$

When viewed as SBIAs, $\langle (\mathbf{n+1})_\mathbf{L} \bullet (\mathbf{n+1})_\mathbf{R} \rangle^{\cap}$ includes all (necessarily finite) skew Boolean algebras on $\leq n$ generators, of which $(\mathbf{n+1})_\mathbf{L} \bullet (\mathbf{n+1})_\mathbf{R}$ is an instance.

What about the lattice of quasi-varieties? This leads us to recall an important aspect of SBIAs. The (ternary) discriminator on a set A is a function $t : A^3 \to A$ defined by

$$t(a, b, c) = \begin{cases} c & \text{if } a = b, \\ a & \text{otherwise.} \end{cases}$$

A variety $\mathcal{V}$ is a *discriminator variety* if a term $t(x, y, z)$ in the language of $\mathcal{V}$ exists whose canonical interpretation on any subdirectly irreducible member of $\mathcal{V}$ is the ternary discriminator. SBIAs form a discriminator variety since the following term realizes the ternary discriminator on any subdirectly irreducible SBIA (see Bignall and Leech [4, Theorem 4.4]):

$$t(x, y, z) = (x \backslash (x \cap y)) \vee (z \backslash ((x \backslash (x \cap y)) \vee (y \backslash (y \cap x)))).$$

For a variety $\mathcal{V}$, let $\mathbf{\Lambda}^v(\mathcal{V})$ [respectively $\mathbf{\Lambda}^q(\mathcal{V})$] denote the lattice of subvarieties [respectively the lattice of sub-quasi-varieties] of $\mathcal{V}$. In general, $\mathbf{\Lambda}^v(\mathcal{V}) \subseteq \mathbf{\Lambda}^q(\mathcal{V})$. We shall exploit the following result of Blanco, Campercholi, and Vaggione. In its statement, $\mathbf{2}$ denotes the two-element chain (considered as a lattice).

Theorem 5.7 [7, Theorem 1(a)]. *Let* $\mathcal{V}$ *be a discriminator variety having the property that the class*

$$\mathcal{B}[\mathcal{V}] = \{\mathbf{A} \in \mathcal{V} \mid \mathbf{A} \textit{ is simple with no trivial subalgebra}\}$$

is closed under the formation of ultraproducts. Then $\mathbf{\Lambda}^q(\mathcal{V}) = \mathbf{\Lambda}^v(\mathcal{V})$ *if and only if either* $\mathcal{B}[\mathcal{V}] = \emptyset$ *or* $\mathbf{\Lambda}^v(\mathcal{V}) \cong \mathbf{2}$. *(Thus, for a discriminator variety* $\mathcal{V}$, $\mathbf{\Lambda}^q(\mathcal{V}) = \mathbf{\Lambda}^v(\mathcal{V})$ *if* $\mathcal{B}[\mathcal{V}] = \emptyset$.*)* □

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As a consequence, we have the following result.

Theorem 5.8. $\mathbf{\Lambda}^q(\mathcal{SBIA}) = \mathbf{\Lambda}^v(\mathcal{SBIA})$. *In particular, every sub-quasi-variety of $\mathcal{SBIA}$ is actually a subvariety of $\mathcal{SBIA}$.*

Proof. In the discriminator variety $\mathcal{SBIA}$, the simple algebras are precisely the trivial algebras along with the primitive algebras, the latter all having trivial subalgebras. Thus, $\mathcal{B}[\mathcal{SBIA}]$ is empty, making $\mathbf{\Lambda}^q(\mathcal{SBIA}) = \mathbf{\Lambda}^v(\mathcal{SBIA})$. □

Remarks. (1) In our definition of a discriminator variety we used a three-variable discriminator term. At the beginning of [7], the authors used an alternative four-variable term. Both approaches, however, are equivalent as both terms are term-equivalent. (2) For different reasons, the SBA version of Theorem 5.8 also holds: every sub-quasi-variety of an SBA is actually a subvariety of the SBA. For the right-handed case, see [17, Theorem 12]. The term-equivalent left-handed case thus also holds. From both, the general case must follow. (3) Let $\mathcal{V}$ be a variety of SBIAs and let $\mathcal{S}(\mathcal{V}, \mathbf{0})$ be the canonical deductive system inherent in $\mathcal{V}$, namely the $\mathbf{0}$-assertional logic of $\mathcal{V}$. (See Blok and Raftery [9].) Then Theorem 5.8 asserts that $\mathcal{S}(\mathcal{V}, \mathbf{0})$ is *hereditarily structurally complete* in the sense of Olson et al. [26]. For a recent study of structural completeness in the context of discriminator varieties, see Campercholi *et al.* [11]. For studies of the canonical deductive systems inherent in varieties of skew Boolean $\cap$-algebras, see [30] and [31].

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