Some long telescoping series

In [1], Greenwell showed that

\[ \sum_{k=b+1}^{\infty} \frac{1}{k^2 - b^2} = \frac{1}{2b} \sum_{k=1}^{2b} \frac{1}{k}. \]  

Notice that the series on the left is an infinite sum, but the series on the right is finite and therefore is the sum viewed as a ‘closed form expression’. This result is an unusual application of the technique from elementary calculus known as ‘telescoping series’. In this short note we will show that (1) is a special case of a more general series.

**Theorem:** Let \( b \) be a positive integer and let the function \( f(x) \) be such that \( f(n) \neq 0 \) for \( n = 1, 2, 3, \ldots \), and \( \lim_{n \to \infty} f(n) = \infty \). Then

\[ \sum_{k=b+1}^{\infty} \frac{f(k + b) - f(k - b)}{f(k + b)f(k - b)} = \frac{1}{2b} \sum_{k=1}^{2b} \frac{1}{f(k)}. \]  

It is easy to see that (1) is the special case of (2) in which \( f(x) = x \).

To prove the theorem we first note that

\[ \frac{f(k + b) - f(k - b)}{f(k + b)f(k - b)} = \frac{1}{f(k - b)} - \frac{1}{f(k + b)}. \]

Consider now the partial sum \( S_N \) of \( N \) terms for the series on the left side of (2), i.e.

\[ S_N = \sum_{k=b+1}^{b+N} \frac{f(k + b) - f(k - b)}{f(k + b)f(k - b)} = \sum_{k=b+1}^{b+N} \left\{ \frac{1}{f(k - b)} - \frac{1}{f(k + b)} \right\}. \]

Writing out the terms we have

\[ S_N = \frac{1}{f(1)} - \frac{1}{f(2b + 1)} + \frac{1}{f(2)} - \frac{1}{f(2b + 2)} + \frac{1}{f(3)} - \frac{1}{f(2b + 3)} + \cdots + \frac{1}{f(2b)} - \frac{1}{f(2b + 2b)} + \frac{1}{f(2b + 1)} - \frac{1}{f(2b + 2b + 1)} + \frac{1}{f(2b + 2)} - \frac{1}{f(2b + 2b + 2)} + \cdots + \frac{1}{f(N)} - \frac{1}{f(2b + N)}. \]
We see that all the terms cancel except the first $2b$ terms of the first column and the last $2b$ terms of the second column. Thus we have

$$S_N = \sum_{k=1}^{2b} \frac{1}{f(k)} - \sum_{k=1}^{2b} \frac{1}{f(k + N)}.$$ 

Since $\lim_{n \to \infty} f(n) = \infty$, it is clear that the second sum above vanishes as $N$ approaches infinity. Thus the theorem is proved.

We end by looking at a few special cases of (2). When $f(x) = x^p$, with $p > 0$ we get

$$\sum_{k=b+1}^{\infty} \frac{(k + b)^p - (k - b)^p}{(k^2 - b^2)^{p/2}} = \sum_{k=1}^{2b} \frac{1}{k^p}.$$

If we take $f(x) = \sinh ax$, with $a > 0$, we get, after a little simplification,

$$\sum_{k=b+1}^{\infty} \frac{\cosh ak}{\sinh a(k + b) \sinh a(k - b)} = \frac{1}{2 \sinh ab} \sum_{k=1}^{2b} \frac{1}{\sinh ak}.$$

With $f(x) = e^{ax}$ and $a > 0$, we get

$$\sum_{k=b+1}^{\infty} e^{-ak} = \frac{1}{2 \sinh ab} \sum_{k=1}^{2b} e^{-ak}.$$

If $f(x) = \log ax$ with $a > 1$, we have

$$\sum_{k=b+1}^{\infty} \frac{\log(k + b)}{\log(k - b)} = \sum_{k=1}^{2b} \frac{1}{\log ak}.$$

Reference


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91.13 A new proof of a curious identity

Recently, Simons [1] proved the identity

$$\sum_{r=0}^{n} \frac{(-1)^r (n + r)! (1 + x)^r}{(n - r)! (r!)^2} = \sum_{r=0}^{n} \frac{(n + r)! x^r}{(n - r)! (r!)^2}.$$ 

Chapman [2] gave a nice, short proof of it; later Prodinger [3] presented another attractive proof by Cauchy’s integral formula. In this note, by an operator method, we establish an equivalent version:

$$\sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} \binom{n + r}{r} (1 + x)^r = \sum_{r=0}^{n} \binom{n}{r} \binom{n + r}{r} x^r.$$  (1)