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NORMAL AUTOMORPHISMS OF A FREE METABELIAN NILPOTENT GROUP

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Abstract. An automorphism φ of a group G is said to be normal if $\varphi(H) = H$ for each normal subgroup H of G. These automorphisms form a group containing the group of inner automorphisms. When G is a non-abelian free (or free soluble) group, it is known that these groups of automorphisms coincide, but this is not always true when G is a free metabelian nilpotent group. The aim of this paper is to determine the group of normal automorphisms in this last case.

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1. Preliminary results. In a group G, consider a map $\varphi : G \to G$ of the form

$$\varphi : x \mapsto x[x, u_1]^{\lambda(1)} \ldots [x, u_m]^{\lambda(m)},$$

where $u_1, \ldots, u_m$ are elements of G, the exponents $\lambda(1), \ldots, \lambda(m)$ being integers (as usual, the commutator $[a, b]$ is defined by $[a, b] = a^{-1}b^{-1}ab$). When G is metabelian, using the relation $[xy, u] = y^{-1}[x, u]y[y, u]$, it is easy to see that φ is an endomorphism. These endomorphisms appear in [**4**] (also see [**1**]). Such endomorphisms are not necessarily automorphisms. But in a nilpotent group, each *map* of the form

$$x \mapsto w_0 x^{\lambda(1)} w_1 x^{\lambda(2)} \ldots x^{\lambda(n)} w_n \quad (\text{with } \lambda(1) + \lambda(2) + \cdots + \lambda(n) = \pm 1)$$

is bijective [**2**, Theorem 1]. Hence we have:

Proposition 1.1. *In a metabelian nilpotent group G, every map $\varphi : G \to G$ of the form $\varphi : x \mapsto x \prod_{i=1}^{m} [x, u_i]^{\lambda(i)}$ $(u_i \in G,\ \lambda(i) \in \mathbb{Z})$ is an automorphism.*

For convenience sake, in a metabelian nilpotent group, an automorphism of the form $x \mapsto x \prod_{i=1}^{m} [x, u_i]^{\lambda(i)}$ will be called a *generalized inner automorphism*.

As usual, in a group, the left-normed commutator $[x_1, \ldots, x_n]$ is defined inductively by

$$\begin{aligned}[x_1, \ldots, x_n] &= [x_1, \ldots, x_{n-1}]^{-1}[x_1, \ldots, x_{n-1}]^{x_n} \\ &= [x_1, \ldots, x_{n-1}]^{-1}x_n^{-1}[x_1, \ldots, x_{n-1}]x_n.\end{aligned}$$

The next technical result will be useful in the following.

Proposition 1.2. *In a group G, consider a map $\varphi : G \to G$ of the form*

$$\varphi(x) = x \prod_{i=1}^{n} [x, v_{i,1}, \ldots, v_{i,\sigma(i)}]^{\eta(i)} \quad (\eta(i) \in \mathbb{Z}),$$

for some function $\sigma : \{1, \ldots, n\} \to \mathbb{N} \setminus \{0\}$ and elements $v_{i,j} \in G$ $(1 \le i \le n,\ 1 \le j \le \sigma(i))$. Then $\varphi(x)$ can be written in the form

$$\varphi(x) = x[x, u_1]^{\lambda(1)} \ldots [x, u_m]^{\lambda(m)} \quad (\lambda(i) \in \mathbb{Z},\ u_i \in G).$$

Proof. By induction, using the relation $[x, y, z] = [x, y]^{-1}[x, z]^{-1}[x, yz]$ □

Frequently in this paper we shall make use of well-known commutator identities (see for example [**7**, 5.1.5]). In particular, we have the following relations, valid in a metabelian group G, for any $x, y, z \in G$, $t \in G'$ and $\lambda \in \mathbb{Z}$:

$$[xt, y] = [x, y][t, y], \qquad [t^{\lambda}, y] = [t, y]^{\lambda},$$
$$[x, y, z][y, z, x][z, x, y] = 1, \qquad [t, x, y] = [t, y, x].$$

Proposition 1.3. *The set of generalized inner automorphisms of a metabelian nilpotent group G forms a (normal) subgroup of the group of automorphisms of G.*

Proof. If φ and ψ are generalized inner automorphisms, the fact that $\psi \circ \varphi$ is a generalized inner automorphism follows from Proposition 1.2. It remains to prove that φ^{-1} is a generalized inner automorphism. For that, it suffices to construct for each integer $k \ge 1$ a generalized inner automorphism ψ_k such that $\psi_k \circ \varphi$ is of the form

$$\psi_k \circ \varphi : x \mapsto x \prod_{i=1}^{m} [x, v_{i,1}, \ldots, v_{i,\sigma(i)}]^{\eta(i)}$$

for some function $\sigma : \{1, \ldots, m\} \to \mathbb{N} \setminus \{0\}$ and elements $v_{i,j} \in G$ $(1 \le i \le m,\ 1 \le j \le \sigma(i))$, and where each commutator is of weight $\ge 1 + 2^{k-1}$ (namely, $\sigma(i) \ge 2^{k-1}$ for $i = 1, \ldots, m$). Indeed, since G is nilpotent, this implies that $\psi_k \circ \varphi(x) = x$ for k large enough, thus $\varphi^{-1} = \psi_k$ is a generalized inner automorphism, as required. We argue by induction on k. The result is clear when $k = 1$ by taking for ψ_1 the identity map. Now suppose that for some integer $k \ge 1$, there exists a generalized inner automorphism ψ_k such that $\psi_k \circ \varphi(x) = x \prod_{i=1}^{m} [x, v_{i,1}, \ldots, v_{i,\sigma(i)}]^{\eta(i)}$, with $\sigma(i) \ge 2^{k-1}$ for $i = 1, \ldots, m$. Put $\psi_{k+1} = \psi' \circ \psi_k$, where ψ' is defined by $\psi'(x) = x \prod_{i=1}^{m} [x, v_{i,1}, \ldots, v_{i,\sigma(i)}]^{-\eta(i)}$. We have

$$\psi_{k+1} \circ \varphi(x) = x \prod_{i=1}^{m} [x, v_{i,1}, \ldots, v_{i,\sigma(i)}]^{\eta(i)}$$
$$\times \prod_{j=1}^{m} \left[x \prod_{i=1}^{m} [x, v_{i,1}, \ldots, v_{i,\sigma(i)}]^{\eta(i)}, v_{j,1}, \ldots, v_{j,\sigma(j)} \right]^{-\eta(j)}.$$

Since

$$\prod_{j=1}^{m}\left[x\prod_{i=1}^{m}[x, v_{i,1}, \dots, v_{i,\sigma(i)}]^{\eta(i)}, v_{j,1}, \dots, v_{j,\sigma(j)}\right]^{-\eta(j)}$$
$$= \prod_{i=1}^{m}[x, v_{j,1}, \dots, v_{j,\sigma(j)}]^{-\eta(j)}\prod_{j=1}^{m}\prod_{i=1}^{m}[x, v_{i,1}, \dots, v_{i,\sigma(i)}, v_{j,1}, \dots, v_{j,\sigma(j)}]^{-\eta(i)\eta(j)},$$

we obtain

$$\psi_{k+1}\circ\varphi(x) = x\prod_{j=1}^{m}\prod_{i=1}^{m}[x, v_{i,1}, \dots, v_{i,\sigma(i)}, v_{j,1}, \dots, v_{j,\sigma(j)}]^{-\eta(i)\eta(j)}$$

and this completes the proof of the proposition. □

2. Main result. We recall that a normal automorphism φ of a group G is an automorphism such that $\varphi(H) = H$ for each normal subgroup H of G. These automorphisms form a subgroup of the group of all automorphisms of G. Obviously, this subgroup contains the subgroup of inner automorphisms of G. It happens these subgroups coincide, for instance, when G is a non-abelian free group [**5**], a non-abelian free soluble group [**8**], or a non-abelian free nilpotent group of class 2 [**3**]. On the other hand, the subgroup of inner automorphisms is of infinite index in the group of normal automorphisms when G is a non-abelian free nilpotent group of class $k \geq 3$ [**3**]. Also note there are exactly two normal automorphisms in a (non-trivial) free abelian group: $x \mapsto x$ and $x \mapsto x^{-1}$.

Certainly, in a metabelian nilpotent group, each generalized inner automorphism is a normal automorphism, but a normal automorphism need not to be a generalized inner automorphism. However, our main result states that the converse holds in a non-abelian free metabelian nilpotent group.

THEOREM 2.1. *In a non-abelian free metabelian nilpotent group, the group of normal automorphisms coincides with the group of generalized inner automorphisms.*

3. Proof of Theorem 2.1. In all this section, we consider a fixed set S of cardinality ≥ 2 and we denote by M_k the free metabelian nilpotent group of class $k > 1$ freely generated by S. In other words, $M_k = F/F''\gamma_{k+1}(F)$, where F is the free group freely generated by S and $\gamma_{k+1}(F)$ the $(k+1)$th term of the lower central series of F. The normal closure in a group G of an element a is written $\langle a^G\rangle$.

LEMMA 3.1. *For any distinct elements $a, b \in S$ and any integer t, the subgroup $\langle (a^t b)^{M_k}\rangle \cap \gamma_k(M_k) \trianglelefteq M_k$ is generated by the set of elements of the form $[a^t b, c_1, \dots, c_{k-1}]$, with $c_1, \dots, c_{k-1} \in S$.*

Proof. First suppose that $t = 0$ and consider an element $w \in \langle b^{M_k}\rangle \cap \gamma_k(M_k)$. Hence w is a product of elements of the form $[c_0, c_1, \dots, c_{k-1}]^{\pm 1}$, with $c_i \in S$. More precisely, we can write $w = w_0 w_1$, where w_0 (resp. w_1) is a product of elements of the form $[c_0, c_1, \dots, c_{k-1}]^{\pm 1}$ with $c_i \in S \setminus \{b\}$, (resp. with $c_i \in S$, the element b occurring once at least in $[c_0, c_1, \dots, c_{k-1}]$). In fact, substituting 1 for the indeterminate b in the relation $w = w_0 w_1$ and using the fact that w lies in $\langle b^{M_k}\rangle$, we obtain $w_0 = 1$.

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Thus w is a product of elements of the form $[c_0, c_1, \dots, c_{k-1}]^{\pm 1}$, with $c_i = b$ for some $i \in \{0, \dots, k-1\}$. If $i = 1$, we can write $[c_0, b, \dots, c_{k-1}] = [b, c_0, \dots, c_{k-1}]^{-1}$. If $i > 1$, we have $[c_0, c_1, \dots, b, \dots, c_{k-1}] = [c_0, c_1, b, \dots, c_{k-1}]$ and it follows:

$$[c_0, c_1, \dots, b, \dots, c_{k-1}] = [b, c_1, c_0, \dots, c_{k-1}][b, c_0, c_1, \dots, c_{k-1}]^{-1}.$$

Thus we have shown that any element of $\langle b^{M_k} \rangle \cap \gamma_k(M_k)$ is a product of elements of the form $[b, c_1, \dots, c_{k-1}]^{\pm 1}$, with $c_i \in S$. Since $[b, c_1, \dots, c_{k-1}] \in \langle b^{M_k} \rangle \cap \gamma_k(M_k)$, the lemma is proved when $t = 0$.

Now consider the general case. Actually, since clearly $S' = \{a^t b\} \cup S \setminus \{b\}$ is a free basis of M_k, we can use the result obtained in the particular case. It follows that $\langle (a^t b)^{M_k} \rangle \cap \gamma_k(M_k)$ is generated by the set of elements of the form $[a^t b, c_1, \dots, c_{k-1}]$, with $c_i \in S'$. But, in fact, we may take $c_i \in S$ and so conclude, since

$$[a^t b, c_1, \dots, a^t b, \dots, c_{k-1}] = [a^t b, c_1, \dots, a, \dots, c_{k-1}]^t [a^t b, c_1, \dots, b, \dots, c_{k-1}].$$

□

As usual, the expression $[x,_n y]$ is defined in a group by $[x,_0 y] = x$ and $[x,_n y] = [[x,_{n-1} y], y]$ for each positive integer n.

For a fixed subset $\{a_0, \dots, a_r\} \subseteq S$ and a function $\Delta : \{0, \dots, r\} \to \mathbb{N}$, we define in M_k the symbol $[x, y, \Delta]$ $(x, y \in M_k)$ by

$$[x, y, \Delta] = [x, y,_{\Delta(0)} a_0,_{\Delta(1)} a_1, \dots,_{\Delta(r)} a_r].$$

Note that for any sequence $b_1, \dots, b_k$ of elements of $\{a_0, \dots, a_r\}$, there is a function $\Delta : \{0, \dots, r\} \to \mathbb{N}$ such that $[x, y, b_1, \dots, b_k] = [x, y, \Delta]$, with $\Delta(0) + \cdots + \Delta(r) = k$. If j, j' are distinct given integers in $\{0, \dots, r\}$ and if $\Delta(j) \neq 0$, we define the function $\Delta_{(j)}^{(j')} : \{0, \dots, r\} \to \mathbb{N}$ by

$$\Delta_{(j)}^{(j')}(j) = \Delta(j) - 1, \quad \Delta_{(j)}^{(j')}(j') = \Delta(j') + 1 \text{ and}$$
$$\Delta_{(j)}^{(j')}(i) = \Delta(i) \text{ for all } i \in \{0, \dots, r\} \setminus \{j, j'\}.$$

When Δ is not the zero-function, we shall denote by $m(\Delta)$ the least integer j such that $\Delta(j) \neq 0$.

If S is ordered, we may define in M_k *basic commutators* (see for example [**6**, Chapter 3]). Recall that a basic commutator of weight k' $(2 \leq k' \leq k)$ is a commutator of the form $[b_1, b_2, \dots, b_{k'}]$ $(b_i \in S)$, with $b_1 > b_2$ and $b_2 \leq b_3 \leq \cdots \leq b_{k'}$. Any set of these commutators freely generates a free abelian subgroup of M'_k.

In the next lemma, we aim to express a product of commutators of the form $[a_s, a_i, \Delta]$ as a product where only basic commutators occur.

LEMMA 3.2. *Let $\{a_0, \dots, a_r\}$ be a finite subset of S $(r > 0)$. Choose an integer $s \in \{0, \dots, r\}$ and consider an element $w \in M_{k+2}$ $(k > 0)$ of the form*

$$w = \prod_{i,\Delta} [a_s, a_i, \Delta]^{\epsilon(i,\Delta)} \quad (\epsilon(i, \Delta) \in \mathbb{Z}),$$

where the product is taken over all integers $i \in \{0, \dots, r\} \setminus \{s\}$ and all functions $\Delta : \{0, \dots, r\} \to \mathbb{N}$ such that $\Delta(0) + \cdots + \Delta(r) = k$. Then:

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(i) *We have*

$$w = \prod_{i<s,\, i\le m(\Delta)} [a_s, a_i, \Delta]^{\epsilon(i,\Delta)} \prod_{s<i,\, s\le m(\Delta)} [a_i, a_s, \Delta]^{-\epsilon(i,\Delta)}$$
$$\times \prod_{m(\Delta)<s,\, m(\Delta)<i} \left[a_i, a_{m(\Delta)}, \Delta^{(s)}_{(m(\Delta))}\right]^{-\epsilon(i,\Delta)} \prod_{m(\Delta)<s,\, m(\Delta)<i} \left[a_s, a_{m(\Delta)}, \Delta^{(i)}_{(m(\Delta))}\right]^{\epsilon(i,\Delta)}$$

(*in all these products, i lies in* $\{0, \ldots, r\} \setminus \{s\}$).
(ii) *We have* $w = 1$ *only if all exponents* $\epsilon(i, \Delta)$ *with* $i \in \{0, \ldots, r\} \setminus \{s\}$ *occurring in the expression of* w *are zero.*

Proof. (i) First we write w as a product of two factors:

$$w = \prod_{i\le m(\Delta)} [a_s, a_i, \Delta]^{\epsilon(i,\Delta)} \prod_{m(\Delta)<i} [a_s, a_i, \Delta]^{\epsilon(i,\Delta)}.$$

The first factor can be expressed in the form

$$\prod_{i\le m(\Delta)} [a_s, a_i, \Delta]^{\epsilon(i,\Delta)} = \prod_{i<s,\, i\le m(\Delta)} [a_s, a_i, \Delta]^{\epsilon(i,\Delta)} \prod_{s<i\le m(\Delta)} [a_s, a_i, \Delta]^{\epsilon(i,\Delta)}$$
$$= \prod_{i<s,\, i\le m(\Delta)} [a_s, a_i, \Delta]^{\epsilon(i,\Delta)} \prod_{s<i\le m(\Delta)} [a_i, a_s, \Delta]^{-\epsilon(i,\Delta)}.$$

In the same way, we have

$$\prod_{m(\Delta)<i} [a_s, a_i, \Delta]^{\epsilon(i,\Delta)} = \prod_{s\le m(\Delta)<i} [a_s, a_i, \Delta]^{\epsilon(i,\Delta)} \prod_{m(\Delta)<s,\, m(\Delta)<i} [a_s, a_i, \Delta]^{\epsilon(i,\Delta)}$$
$$= \prod_{s\le m(\Delta)<i} [a_i, a_s, \Delta]^{-\epsilon(i,\Delta)} \prod_{m(\Delta)<s,\, m(\Delta)<i} [a_s, a_i, \Delta]^{\epsilon(i,\Delta)}.$$

Therefore Lemma 3.2(i) is proved if we show the relation

$$\prod_{m(\Delta)<s,\, m(\Delta)<i} [a_s, a_i, \Delta]^{\epsilon(i,\Delta)} = \prod_{m(\Delta)<s,\, m(\Delta)<i} \left[a_i, a_{m(\Delta)}, \Delta^{(s)}_{(m(\Delta))}\right]^{-\epsilon(i,\Delta)}$$
$$\times \prod_{m(\Delta)<s,\, m(\Delta)<i} \left[a_s, a_{m(\Delta)}, \Delta^{(i)}_{(m(\Delta))}\right]^{\epsilon(i,\Delta)}. \tag{1}$$

For that, write more explicitly the commutator $[a_s, a_i, \Delta]$ (in the following equalities, we write m instead of $m(\Delta)$):

$$[a_s, a_i, \Delta] = [a_s, a_i, {}_{\Delta(0)}\, a_0, \ldots, {}_{\Delta(r)}\, a_r]$$
$$= [a_s, a_i, {}_{\Delta(m)}\, a_m, \ldots, {}_{\Delta(r)}\, a_r]$$
$$= [a_s, a_i, a_m, {}_{\Delta(m)-1}\, a_m, \ldots, {}_{\Delta(r)}\, a_r].$$

Since $[a_s, a_i, a_m] = [a_i, a_m, a_s]^{-1} [a_m, a_s, a_i]^{-1} = [a_i, a_m, a_s]^{-1} [a_s, a_m, a_i]$, we obtain

$$[a_s, a_i, \Delta] = \left[a_i, a_m, \Delta^{(s)}_{(m)}\right]^{-1} \left[a_s, a_m, \Delta^{(i)}_{(m)}\right]. \tag{2}$$

Relation (1) is now an immediate consequence of (2).

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(ii) Choose an ordering of S such that a_s is the lowest element. Then, since $[a_s, a_i, \Delta] = [a_i, a_s, \Delta]^{-1}$, all commutators involved in w are inverses of basic commutators, and the basic commutators that occur are distinct. The result follows. □

LEMMA 3.3. *Let φ be a normal automorphism of M_{k+2} $(k > 0)$ acting trivially on $M_{k+2}/\gamma_{k+2}(M_{k+2})$. Then, for all distinct elements $a, b \in S$, there exists a generalized inner automorphism ψ of M_{k+2} such that $\varphi(a) = \psi(a)$ and $\varphi(b) = \psi(b)$.*

Proof. Let a, b be two distinct elements of S. Then $a^{-1}\varphi(a)$ and $b^{-1}\varphi(b)$ belong to $\langle a^{M_{k+2}}\rangle \cap \gamma_{k+2}(M_{k+2})$ and $\langle b^{M_{k+2}}\rangle \cap \gamma_{k+2}(M_{k+2})$ respectively. By Lemma 3.1, there is a finite subset $\{a = a_0, a_1, \dots, a_r = b\} \subseteq S$ such that

$$\varphi(a) = \varphi(a_0) = a_0 \prod_{i,\Delta} [a_0, a_i, \Delta]^{\alpha(i,\Delta)} \quad (\alpha(i, \Delta) \in \mathbb{Z}),$$

$$\varphi(b) = \varphi(a_r) = a_r \prod_{i,\Delta} [a_r, a_i, \Delta]^{\beta(i,\Delta)} \quad (\beta(i, \Delta) \in \mathbb{Z}),$$

where the two products are taken over all integers $i \in \{0, \dots, r\}$ and all functions $\Delta : \{0, \dots, r\} \to \mathbb{N}$ with $\Delta(0) + \cdots + \Delta(r) = k$ (as in Lemma 3.2, $[a_s, a_i, \Delta]$ is defined by $[a_s, a_i, \Delta] = [a_s, a_i, {}_{\Delta(0)}\, a_0, \dots, {}_{\Delta(r)}\, a_r]$). Note that if $|S| = 2$ (and so $r = 1$), Lemma 3.3 is easily verified by taking the generalized inner automorphism ψ defined by

$$\psi(x) = x \prod_{\Delta} [x, a_1, \Delta]^{\alpha(1,\Delta)} \prod_{\Delta} [x, a_0, \Delta]^{\beta(0,\Delta)}.$$

Thus we shall assume in the following that $|S| > 2$. By Lemma 3.1, for any positive integer t, $(a_0^t a_r)^{-1}\varphi(a_0^t a_r)$ can be expressed as a product of elements of the form $\left[a_0^t a_r, c_1, \dots, c_{k+1}\right]^{\pm 1}$, with $c_1, \dots, c_{k+1} \in S$. But $(a_0^t a_r)^{-1}\varphi(a_0^t a_r) = (a_0^t a_r)^{-1}\varphi(a_0)^t\varphi(a_r)$ belongs to $\langle a_0, a_1, \dots, a_r\rangle$. Therefore, substituting 1 for all indeterminates in $S \setminus \{a_0, a_1, \dots, a_r\}$ in the expression of $(a_0^t a_r)^{-1}\varphi(a_0^t a_r)$, we can assume that $c_1, \dots, c_{k+1} \in \{a_0, a_1, \dots, a_r\}$. It follows that $\varphi(a_0^t a_r)$ can be expressed in the form

$$\begin{aligned}\varphi(a_0^t a_r) &= a_0^t a_r \prod_{i,\Delta} \left[a_0^t a_r, a_i, \Delta\right]^{\eta_t(i,\Delta)} \quad (\eta_t(i, \Delta) \in \mathbb{Z}) \\ &= a_0^t a_r \prod_{i,\Delta} [a_0, a_i, \Delta]^{t\eta_t(i,\Delta)} \prod_{i,\Delta} [a_r, a_i, \Delta]^{\eta_t(i,\Delta)}.\end{aligned}$$

Thus the relation $\varphi(a_0^t a_r) = \varphi(a_0)^t\varphi(a_r)$ implies

$$\begin{aligned}&\prod_{i,\Delta} [a_0, a_i, \Delta]^{t\eta_t(i,\Delta)} \prod_{i,\Delta} [a_r, a_i, \Delta]^{\eta_t(i,\Delta)} \\ &\qquad = \prod_{i,\Delta} [a_0, a_i, \Delta]^{t\alpha(i,\Delta)} \prod_{i,\Delta} [a_r, a_i, \Delta]^{\beta(i,\Delta)}.\end{aligned} \tag{3}$$

Choose an order in S such that $a_0 < a_1 < \cdots < a_r$ and express each product in (3) as a product of basic commutators (or their inverses). Considering, for instance, the left-hand side of (3) (the treatment of the righthand side is similar), we have

$$\prod_{i,\Delta} [a_0, a_i, \Delta]^{t\eta_t(i,\Delta)} = \prod_{i \neq 0,\, \Delta} [a_i, a_0, \Delta]^{-t\eta_t(i,\Delta)}$$

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and, by using Lemma 3.2(i) with $s = r$,

$$\prod_{i,\Delta}[a_r, a_i, \Delta]^{\eta_t(i,\Delta)} = \prod_{i\neq r,\, i\leq m(\Delta)} [a_r, a_i, \Delta]^{\eta_t(i,\Delta)} \times \prod_{i\neq r,\, m(\Delta)<i} [a_i, a_{m(\Delta)}, \Delta^{(r)}_{(m(\Delta))}]^{-\eta_t(i,\Delta)} \prod_{i\neq r,\, m(\Delta)<i} [a_r, a_{m(\Delta)}, \Delta^{(i)}_{(m(\Delta))}]^{\eta_t(i,\Delta)}.$$

Thus relation (3) can be written in the form

$$\begin{aligned}
&\prod_{i\neq 0,\,\Delta} [a_i, a_0, \Delta]^{-t\eta_t(i,\Delta)} \prod_{i\neq r,\, i\leq m(\Delta)} [a_r, a_i, \Delta]^{\eta_t(i,\Delta)} \\
&\quad\times \prod_{i\neq r,\, m(\Delta)<i} \left[a_i, a_{m(\Delta)}, \Delta^{(r)}_{(m(\Delta))}\right]^{-\eta_t(i,\Delta)} \prod_{i\neq r,\, m(\Delta)<i} \left[a_r, a_{m(\Delta)}, \Delta^{(i)}_{(m(\Delta))}\right]^{\eta_t(i,\Delta)} \\
&= \prod_{i\neq 0,\,\Delta} [a_i, a_0, \Delta]^{-t\alpha(i,\Delta)} \prod_{i\neq r,\, i\leq m(\Delta)} [a_r, a_i, \Delta]^{\beta(i,\Delta)} \\
&\quad\times \prod_{i\neq r,\, m(\Delta)<i} \left[a_i, a_{m(\Delta)}, \Delta^{(r)}_{(m(\Delta))}\right]^{-\beta(i,\Delta)} \prod_{i\neq r,\, m(\Delta)<i} \left[a_r, a_{m(\Delta)}, \Delta^{(i)}_{(m(\Delta))}\right]^{\beta(i,\Delta)}. \qquad (4)
\end{aligned}$$

Now consider an integer $i \in \{1, \ldots, r-1\}$ and a function $\Delta : \{0, \ldots, r\} \to \mathbb{N}$, with $\Delta(0) + \cdots + \Delta(r) = k$ (we can always suppose that $r > 1$ since $|S| > 2$). By identifying the exponents of the basic commutator $[a_i, a_0, \Delta]$ of each side of relation (4), it is easy to see that

$$t\eta_t(i, \Delta) + \eta_t\left(i, \Delta^{(0)}_{(r)}\right) = t\alpha(i, \Delta) + \beta\left(i, \Delta^{(0)}_{(r)}\right) \qquad (5)$$

if $\Delta(r) > 0$, and $\eta_t(i, \Delta) = \alpha(i, \Delta)$ if $\Delta(r) = 0$. We prove by induction on $\Delta(r)$ that actually, we have always the equality $\eta_t(i, \Delta) = \alpha(i, \Delta)$. At first observe that if $\Delta(r) > 0$, we have $\Delta^{(0)}_{(r)}(r) = \Delta(r) - 1$ and so $\eta_t(i, \Delta^{(0)}_{(r)}) = \alpha(i, \Delta^{(0)}_{(r)})$ by induction. Hence relation (5) implies

$$\alpha\left(i, \Delta^{(0)}_{(r)}\right) - \beta\left(i, \Delta^{(0)}_{(r)}\right) = t\left\{\alpha(i, \Delta) - \eta_t(i, \Delta)\right\}.$$

Consequently, each positive integer t divides the integer $\alpha(i, \Delta^{(0)}_{(r)}) - \beta(i, \Delta^{(0)}_{(r)})$, which is independent of t. It follows that $\alpha(i, \Delta^{(0)}_{(r)}) = \beta(i, \Delta^{(0)}_{(r)})$ and so $\eta_t(i, \Delta) = \alpha(i, \Delta)$, as required.

Using theses relations and taking $t = 1$, relation (3) implies

$$\prod_{\Delta}[a_0, a_r, \Delta]^{\eta(r,\Delta)} \prod_{i,\Delta}[a_r, a_i, \Delta]^{\eta(i,\Delta)} = \prod_{\Delta}[a_0, a_r, \Delta]^{\alpha(r,\Delta)} \prod_{i,\Delta}[a_r, a_i, \Delta]^{\beta(i,\Delta)}$$

(we write η for η_1) and so

$$\prod_{i,\Delta}[a_r, a_i, \Delta]^{\beta(i,\Delta)} = \prod_{i,\Delta}[a_r, a_i, \Delta]^{\eta(i,\Delta)} \prod_{\Delta}[a_r, a_0, \Delta]^{\alpha(r,\Delta)-\eta(r,\Delta)}.$$

Since $\varphi(a_r) = a_r \prod_{i,\Delta}[a_r, a_i, \Delta]^{\beta(i,\Delta)}$, we obtain

$$\varphi(a_r) = a_r \prod_{i,\Delta}[a_r, a_i, \Delta]^{\eta(i,\Delta)} \prod_{\Delta}[a_r, a_0, \Delta]^{\alpha(r,\Delta)-\eta(r,\Delta)} \qquad (6)$$

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Now consider the generalized inner automorphism ψ defined by

$$\psi(x) = x \prod_{i=1,\dots,r,\,\Delta} [x, a_i, \Delta]^{\alpha(i,\Delta)} \prod_{\Delta} [x, a_0, \Delta]^{\alpha(r,\Delta)+\eta(0,\Delta)-\eta(r,\Delta)}.$$

We have

$$\psi(a_0) = a_0 \prod_{i=1,\dots,r,\,\Delta} [a_0, a_i, \Delta]^{\alpha(i,\Delta)} = \varphi(a_0).$$

In the same way,

$$\begin{aligned}\psi(a_r) &= a_r \prod_{i=1,\dots,r-1,\,\Delta} [a_r, a_i, \Delta]^{\alpha(i,\Delta)} \prod_{\Delta} [a_r, a_0, \Delta]^{\alpha(r,\Delta)+\eta(0,\Delta)-\eta(r,\Delta)} \\ &= a_r \prod_{i=1,\dots,r-1,\,\Delta} [a_r, a_i, \Delta]^{\eta(i,\Delta)} \prod_{\Delta} [a_r, a_0, \Delta]^{\alpha(r,\Delta)+\eta(0,\Delta)-\eta(r,\Delta)} \\ &= a_r \prod_{i,\,\Delta} [a_r, a_i, \Delta]^{\eta(i,\Delta)} \prod_{\Delta} [a_r, a_0, \Delta]^{\alpha(r,\Delta)-\eta(r,\Delta)}\end{aligned}$$

and so $\psi(a_r) = \varphi(a_r)$ by (6). This completes the proof of Lemma 3.3. □

LEMMA 3.4. *Let φ be a normal automorphism of M_{k+2} ($k > 0$) acting trivially on $M_{k+2}/\gamma_{k+2}(M_{k+2})$. Then φ is a generalized inner automorphism of M_{k+2}.*

Proof. We can assume that $|S| > 2$ (otherwise Lemma 3.4 is a consequence of Lemma 3.3). Consider two distinct elements $a, b \in S$. According to Lemma 3.3, there exists a generalized inner automorphism ψ such that $\varphi(a) = \psi(a)$ and $\varphi(b) = \psi(b)$. It suffices to prove that for any $c \in S \setminus \{a, b\}$, we have $\varphi(c) = \psi(c)$. For that, apply again Lemma 3.3: there are generalized inner automorphisms ψ', ψ'' such that $\varphi(a) = \psi'(a)$, $\varphi(c) = \psi'(c)$ and $\varphi(b) = \psi''(b)$, $\varphi(c) = \psi''(c)$. There exists a finite subset $\{a_0, \dots, a_r\} \subseteq S$, containing a, b, c, such that ψ, ψ', ψ'' can be defined by the equations

$$\psi(x) = x \prod_{i,\,\Delta} [x, a_i, \Delta]^{\epsilon(i,\Delta)}$$
$$\psi'(x) = x \prod_{i,\,\Delta} [x, a_i, \Delta]^{\epsilon'(i,\Delta)}$$
$$\psi''(x) = x \prod_{i,\,\Delta} [x, a_i, \Delta]^{\epsilon''(i,\Delta)}$$

(the products are taken over all integers $i \in \{0, \dots, r\}$ and all functions $\Delta : \{0, \dots, r\} \to \mathbb{N}$ with $\Delta(0) + \cdots + \Delta(r) = k$). Since $\psi(a) = \psi'(a)$, we have

$$a \prod_{i,\,\Delta} [a, a_i, \Delta]^{\epsilon(i,\Delta)} = a \prod_{i,\,\Delta} [a, a_i, \Delta]^{\epsilon'(i,\Delta)}$$

and so

$$\prod_{i,\,\Delta} [a, a_i, \Delta]^{\epsilon(i,\Delta)-\epsilon'(i,\Delta)} = 1.$$

Applying Lemma 3.2(ii), we obtain $\epsilon(i, \Delta) = \epsilon'(i, \Delta)$ for all functions Δ and all integers $i \in \{0, \dots, r\}$ such that $a_i \neq a$. Similarly, we have $\epsilon(i, \Delta) = \epsilon''(i, \Delta)$ if $a_i \neq b$

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and $\epsilon'(i, \Delta) = \epsilon''(i, \Delta)$ if $a_i \neq c$. It follows that $\epsilon(i, \Delta) = \epsilon'(i, \Delta)$ for all function Δ and all integer $i \in \{0, \ldots, r\}$, hence $\psi = \psi'$. Thus $\varphi(c) = \psi'(c) = \psi(c)$, as required. □

Proof of Theorem 2.1. We argue by induction on the nilpotency class k of M_k. If $k = 2$, the result follows from [**3**, Theorem 2(ii)] (in this case, each normal automorphism is inner). Now consider a normal automorphism φ of M_k, with $k > 2$. Then φ induces a normal automorphism on the quotient group $M_k/\gamma_k(M_k)$. By induction, since this quotient is isomorphic to M_{k-1}, there exists a generalized inner automorphism $\psi : M_k \to M_k$ such that $\varphi(x) = \psi(x)\theta(x)$, where $\theta(x)$ is an element of $\gamma_k(M_k)$. It follows that $\psi^{-1}(\varphi(x)) = x\psi^{-1}(\theta(x))$. Thus $\psi' := \psi^{-1} \circ \varphi$ is a normal automorphism of M_k acting trivially on $M_k/\gamma_k(M_k)$. By Lemma 3.4, ψ' is a generalized inner automorphism, and so is $\varphi = \psi \circ \psi'$. This completes the proof of Theorem 2.1. □

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