THE SIMILARITY PROBLEM FOR TENSOR PRODUCTS OF CERTAIN C*-ALGEBRAS

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We prove that every bounded representation of the tensor product of two C*-algebras, one of which is nuclear and contains matrices of any order, is similar to a *-representation.

1. INTRODUCTION

A C*-algebra $\mathcal{A}$ has the similarity property if every bounded representation $\pi : \mathcal{A} \rightarrow B(\mathcal{H})$ is similar to a *-representation, that is, if there exists an invertible operator $S \in B(\mathcal{H})$ such that $S^{-1}\pi S$ is a *-representation. This property was introduced by Kadison in [4], where he conjectured that all C*-algebras have the similarity property. Haagerup [3] proved that a bounded representation is similar to a *-representation if and only if it is completely bounded, and also, that representations with a cyclic vector (or a finite cyclic set) are similar to *-representations. In addition, if $\pi$ is completely bounded, then

$$||\pi||_{cb} = \inf \{ ||S|| ||S^{-1}||; S\pi S^{-1} \text{ is a } *\text{-representation} \}$$

and this infimum is attained ([5]).

Recently ([6, 7, 8]), Pisier introduced the notions of similarity degree and length, which have played a significant role in the study of the similarity problem.

The similarity degree $d(\mathcal{A})$ of a C*-algebra $\mathcal{A}$ is the smallest $\alpha \geq 0$ for which there is a constant $C_\mathcal{A}$ such that every bounded representation $\pi : \mathcal{A} \rightarrow B(\mathcal{H})$ satisfies $||\pi||_{cb} \leq C_\mathcal{A}||\pi||^\alpha$. The length $\ell(\mathcal{A})$ is the smallest integer $d$ for which there is a constant $K$ such that, for any $n$ and any $X \in M_n(\mathcal{A})$, there is an integer $N = N(n, X)$, scalar matrices $\alpha_0 \in M_{n,N}(\mathbb{C})$, $\alpha_1 \in M_N(\mathbb{C}), \ldots, \alpha_{d-1} \in M_N(\mathbb{C})$, $\alpha_d \in M_{N,n}(\mathbb{C})$, and diagonal matrices $D_1, \ldots, D_d \in M_N(\mathcal{A})$ satisfying

$$X = \alpha_0 D_1 \alpha_1 D_2 \cdots D_d \alpha_d$$

$$\prod_{0}^{d} ||\alpha_i|| \prod_{i=1}^{d} ||D_i|| \leq K||X||.$$
A $C^*$-algebra $\mathcal{A}$ has the similarity property if and only if $d(\mathcal{A}) < \infty$ and Pisier [6] proved the striking fact that $d(\mathcal{A}) = \ell(\mathcal{A})$.

Despite all the progress made so far, there are few concrete examples of $C^*$-algebras known to have the similarity property. We list them below, together with their respective lengths:

(i) If $\mathcal{A}$ is nuclear, then $\ell(\mathcal{A}) = 2$ ([1]).
(ii) If $\mathcal{A} = B(H)$, then $\ell(\mathcal{A}) = 3$ ([7]).
(iii) $\ell(\mathcal{A} \otimes \mathcal{K}(H)) \leq 3$, $\mathcal{A}$ is arbitrary ([3, 8]).
(iv) If $\mathcal{M}$ is a type II$_1$ factor with property $\Gamma$, then $\ell(\mathcal{M}) = 3$ ([2]).

In this paper we add to the above list the following result: If $\mathcal{A}$ and $\mathcal{B}$ are unital $C^*$-algebras such that $\mathcal{B}$ is nuclear and contains unital matrix algebras of any order, then $\ell_{\min}(A \otimes B) \leq 5$.

Throughout this paper we shall assume that all $C^*$-algebras and their Hilbert space representations are unital. We denote by $\mathcal{A} \otimes \mathcal{B}$, $\mathcal{A} \otimes_{\min} \mathcal{B}$ and $\mathcal{A} \otimes_{\max} \mathcal{B}$ the algebraic, the spatial (minimal), and the maximal tensor products of two $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively.

2. PRELIMINARY RESULTS

It is well known that, if $\mathcal{A}$ and $\mathcal{B}$ are $C^*$-algebras and $f : \mathcal{A} \to B(H)$ and $g : \mathcal{B} \to B(H)$ are commuting $^*$-representations, then the map $\psi : \mathcal{A} \otimes \mathcal{B} \to B(H)$ defined on elementary tensors by $\psi(a \otimes b) = f(a)g(b)$ is bounded with respect to the maximal $C^*$-norm on $\mathcal{A} \otimes \mathcal{B}$. If, however, $\psi$ is bounded with respect to the spatial norm on $\mathcal{A} \otimes \mathcal{B}$, then it extends to a $^*$-representation of $\mathcal{A} \otimes_{\min} \mathcal{B}$, so $\psi$ is completely contractive. Therefore, boundedness alone with respect to the spatial norm implies automatic complete contractivity. The technical results in this section belong to this circle of ideas.

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital $C^*$-algebras and $\mathcal{X} \subseteq \mathcal{A}$ is an operator system, that is, a closed, self-adjoint, unital vector subspace. Denote by $\mathcal{X} \otimes_{\min} \mathcal{B}$ the closure of $\mathcal{X} \otimes \mathcal{B}$ (elementary operators) in the spatial norm inherited from $\mathcal{A} \otimes_{\min} \mathcal{B}$. Let $\varphi : \mathcal{X} \to B(H)$ be a unital completely positive map and let $\pi : \mathcal{B} \to B(H)$ be a $^*$-representation such that $\varphi(x)\pi(b) = \pi(b)\varphi(x)$ for every $x \in \mathcal{X}, b \in \mathcal{B}$. Suppose, in addition, that the map defined on $\mathcal{X} \otimes \mathcal{B}$ with values in $B(H)$ taking $\sum_{i=1}^{n} x_i \otimes b_i$ to $\sum_{i=1}^{n} \varphi(x_i)\pi(b_i)$ is bounded with respect to the spatial norm on $\mathcal{X} \otimes \mathcal{B}$, so it extends to a bounded map $\omega$ on $\mathcal{X} \otimes_{\min} \mathcal{B}$.

Under these hypotheses we have

**Lemma 2.1.** If $\mathcal{B}$ is nuclear, then the map $\omega$ is completely positive on $\mathcal{X} \otimes_{\min} \mathcal{B}$.
PROOF: The map taking \( \sum_{i=1}^{n} x_i \otimes b_i \) to \( \sum_{i=1}^{n} \varphi(x_i) \otimes b_i \) is completely positive from \( \mathcal{X} \otimes B \) to \( B(\mathcal{H}) \otimes B \), and the map taking \( \sum_{i=1}^{n} y_i \otimes b_i \) to \( \sum_{i=1}^{n} y_i \otimes \pi(b_i) \) is completely positive from \( B(\mathcal{H}) \otimes B \) to \( B(\mathcal{H}) \otimes \pi(B) \). Then the map taking \( \sum_{i=1}^{n} x_i \otimes b_i \) to \( \sum_{i=1}^{n} \varphi(x_i) \otimes \pi(b_i) \) is completely positive from \( \mathcal{X} \otimes B \) to \( B(\mathcal{H}) \otimes \pi(B) \), as the composition of the two previous maps. Since the ranges of \( \varphi \) and \( \pi \) commute, the latter map's range is included in \( \pi(B)' \otimes \pi(B) \). Note that, since \( B \) is nuclear, so is \( \pi(B) \). The map from \( \pi(B)' \otimes \pi(B) \) into \( B(\mathcal{H}) \) taking \( \sum_{i=1}^{n} y_i \otimes z_i \) to \( \sum_{i=1}^{n} y_i z_i \) extends to a \( \ast \)-representation of \( \pi(B)' \otimes \pi(B) \) max min \( \pi(B) \) \( \pi(B) \). This shows that \( \omega \), as a composition of three completely positive maps, is completely positive on \( \mathcal{X} \otimes B \).

PROPOSITION 2.2 Let \( A \) and \( B \) be unital \( C^\ast \)-algebras, \( B \) nuclear. Let \( \varphi : A \rightarrow B(\mathcal{H}) \) be a complete contraction and let \( \pi : B \rightarrow B(\mathcal{H}) \) be a \( \ast \)-representation such that \( \varphi(a) \pi(b) = \pi(b) \varphi(a) \) for every \( a \in A, b \in B \). If the map \( \Theta : A \otimes B \rightarrow B(\mathcal{H}) \), defined on elementary tensors by \( \Theta(a \otimes b) = \varphi(a) \pi(b) \), is bounded with respect to the spatial norm on \( A \otimes B \), then it extends to a complete contraction on \( A \otimes B \).

PROOF: Consider the operator system of \( A \otimes M_2 \)

\[
\mathcal{X} = \left\{ \begin{pmatrix} \lambda I & x \\ y & \mu I \end{pmatrix} ; \ x, y \in A \right\}
\]

and define \( \Phi : \mathcal{X} \rightarrow B(\mathcal{H} \oplus \mathcal{H}) \) by

\[
\Phi \left( \begin{pmatrix} \lambda I & x \\ y & \mu I \end{pmatrix} \right) = \begin{pmatrix} \lambda I & \varphi(x) \\ \varphi(y) & \mu I \end{pmatrix}.
\]

It is well-known that \( \varphi \) is completely contractive if and only if \( \Phi \) is completely positive. Define also \( \tilde{\pi} : B \rightarrow B(\mathcal{H} \oplus \mathcal{H}) \) by

\[
\tilde{\pi}(b) = \begin{pmatrix} \pi(b) & 0 \\ 0 & \pi(b) \end{pmatrix}.
\]

Finally, define \( T : \mathcal{X} \otimes B \rightarrow B(\mathcal{H} \oplus \mathcal{H}) \) by \( T(\mathcal{X} \otimes b) = \Phi(\mathcal{X}) \tilde{\pi}(b) \). Notice that \( \Phi \) and \( \tilde{\pi} \) commute, \( T \) is bounded with respect to the spatial norm inherited by \( \mathcal{X} \otimes B \) from \( (A \otimes M_2) \otimes B \), and its norm satisfies

\[
\|T\| \leq 2\|\Theta\| + 2.
\]
From Lemma 2.1, $T$ is completely positive and, by Arveson’s extension theorem, $T$ has a unital completely positive extension to $(A \otimes M_2) \otimes B$, denoted by $\tilde{T}$. As a unital completely positive map on a $C^*$-algebra, $\tilde{T}$ is completely contractive. The map

$$j : A \otimes B \rightarrow (A \otimes M_2) \otimes B$$

defined on elementary tensors by

$$j(a \otimes b) = \left( \begin{array}{cc} 0 & a \\ 0 & 0 \end{array} \right) \otimes b = \left( \begin{array}{cc} 0 & a \otimes b \\ 0 & 0 \end{array} \right)$$

is completely isometric and

$$\tilde{T} \left( \left( \begin{array}{cc} 0 & a \\ 0 & 0 \end{array} \right) \otimes b \right) = T \left( \left( \begin{array}{cc} 0 & a \\ 0 & 0 \end{array} \right) \otimes b \right) = \left( \begin{array}{cc} 0 & \Theta(a \otimes b) \\ 0 & 0 \end{array} \right).$$

We conclude that $\Theta$ is a complete contraction.

**COROLLARY 2.3.** Let $A$ and $B$ be unital $C^*$-algebras, $B$ nuclear. If $\pi$ is a bounded representation of $A \otimes B$ such that $\pi|_A$ is completely bounded and $\pi|_B$ is self-adjoint, then $\pi$ is completely bounded and $\|\pi\|_{cb} \leq \|\pi|_A\|_{cb}$.

3. THE MAIN RESULT

We are ready to prove the main result of this paper.

**PROPOSITION 3.1.** Let $A$ and $B$ be unital $C^*$-algebras such that $B$ is nuclear and contains unital matrix algebras of any order. If $\pi$ is a bounded representation of $A \otimes B$, then $\pi$ is completely bounded and $\|\pi\|_{cb} \leq \|\pi\|^5$.

**PROOF:** There exists $S \in B(H)$ invertible such that $\|S\| \cdot \|S^{-1}\| \leq \|\pi\|^2$ and $\rho = S\pi S^{-1}$ is self-adjoint on $B$ [3]. Since $\rho$ is unital and $B$ contains matrices of any order, then so does $\rho(B)$. Since $\rho(A)$ and $\rho(B)$ commute, we have $\|\rho|_A \otimes \text{Id}_{M_n}\| \leq \|\rho\|$, which shows that $\rho|_A$ is completely bounded and $\|\rho|_A\|_{cb} \leq \|\rho\| \leq \|\|\|^3$. It follows from Corollary 2.3 that $\rho$ is completely bounded and $\|\rho\|_{cb} \leq \|\|\|^3$. This shows that $\pi = S^{-1}\rho S$ is completely bounded and

$$\|\pi\|_{cb} \leq \|S\| \cdot \|S^{-1}\| \cdot \|\rho\|_{cb} \leq \|\|\|^5.$$

**REFERENCES**


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