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Low-lying zeros of quadratic Dirichlet L-functions: lower order terms for extended support

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Abstract

We study the 1-level density of low-lying zeros of Dirichlet L-functions attached to real primitive characters of conductor at most X. Under the generalized Riemann hypothesis, we give an asymptotic expansion of this quantity in descending powers of $\log X$, which is valid when the support of the Fourier transform of the corresponding even test function ϕ is contained in $(-2,2)$. We uncover a phase transition when the supremum σ of the support of $\widehat{\phi}$ reaches 1, both in the main term and in the lower order terms. A new lower order term appearing at $\sigma=1$ involves the quantity $\widehat{\phi}(1)$, and is analogous to a lower order term which was isolated by Rudnick in the function field case.

1. Introduction

The study of statistics of zeros of L-functions was initiated in Montgomery's seminal paper [Mon73] on the pair correlation of zeros of $\zeta(s)$. This work inspired Özlük and Snyder [ÖS93, ÖS99] to prove related results on the 1-level density of low-lying zeros of Dirichlet L-functions attached to real characters

$$\chi_d(n) := \left(\frac{d}{n}\right),$$

with $d \neq 0$.[1] These low-lying zeros of Dirichlet L-functions are of particular interest since they have strong connections with important problems such as the size of class numbers of imaginary quadratic number fields and Chebyshev's bias for primes in arithmetic progressions. The aforementioned results were extended to the n-level density for general n by Rubinstein [Rub01], and for extended support under the generalized Riemann hypothesis (GRH) by Gao [Gao05, Gao14]. Note that Gao considered the family

$$\mathcal{F}^*(X) := \{L(s,\chi_{8d}) : 1 \leqslant |d| \leqslant X; d \text{ is odd and squarefree}\},$$

which is known to have significant technical advantages over that of all real characters (see also [Sou00]). For several years it was not known how to match Gao's asymptotic with the random matrix theory predictions. However, this was recently established for $n \leqslant 7$ by Levinson and Miller [LM13], and for all n by Entin *et al.* [ERR13]. In addition, we mention that the Ratios conjecture of Conrey *et al.* [CFZ08] has been shown by Conrey and Snaith [CS07] to

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[1] Similar results were obtained independently in an unpublished preprint of Katz and Sarnak [KS97].

predict a precise expression for the 1-level density; this prediction was confirmed up to a power saving error term by Miller [Mil08] for a restricted class of test functions. We further remark that the 1-level density of low-lying zeros has been studied extensively for many other families of L-functions; cf., e.g., [ILS00, You06, SST16] and the references therein.

In this paper we study the low-lying zeros of real Dirichlet L-functions in the family $\mathcal{F}^*(X)$. Our focus will be on lower order terms in the 1-level density, a statistic for low-lying zeros that we now introduce in detail. Throughout, ϕ will denote a real and even Schwartz test function. Given a (large) positive number X, the 1-level density for the single L-function $L(s,\chi_d)$ is the sum

$$D_X(\chi_d;\phi) := \sum_{\gamma_d} \phi\bigg(\gamma_d \frac{L}{2\pi}\bigg),$$

with $\gamma_d := -i(\rho_d - \frac{1}{2})$, where ρ_d runs over the nontrivial zeros of $L(s,\chi_d)$ (i.e. zeros with $0 < \operatorname{Re}(\rho_d) < 1$). Moreover, we set

$$L := \log\bigg(\frac{X}{2\pi e}\bigg). \tag{1.1}$$

We consider a cutoff function $w(t)$, which is an even, nonzero and nonnegative Schwartz function. The corresponding total weight is given by

$$W^*(X) := \sideset{}{^*}\sum_{d \text{ odd}} w\bigg(\frac{d}{X}\bigg).$$

Here and throughout, a star on a sum will denote a restriction to squarefree integers. We then define the 1-level density of the family $\mathcal{F}^*(X)$ as the sum

$$\mathcal{D}^*(\phi;X) := \frac{1}{W^*(X)} \sideset{}{^*}\sum_{d \text{ odd}} w\bigg(\frac{d}{X}\bigg) D_X(\chi_{8d};\phi). \tag{1.2}$$

Our main theorem is an asymptotic expansion of this quantity in descending powers of $\log X$, which is valid when $\operatorname{supp}\widehat{\phi} \subset (-2,2)$. This is a refinement of the results of Özlük and Snyder [ÖS99] and Katz and Sarnak [KS97].

Theorem 1.1. *Fix $K \in \mathbb{N}$, assume the GRH and suppose that $\operatorname{supp}\widehat{\phi} \subset (-2,2)$. Then the 1-level density of low-lying zeros in the family $\mathcal{F}^*$ of quadratic Dirichlet L-functions whose conductor is an odd squarefree multiple of 8 is given by*

$$\mathcal{D}^*(\phi;X) = \widehat{\phi}(0) - \frac{1}{2}\int_{-1}^{1} \widehat{\phi}(u)\,du + \sum_{k=1}^{K} \frac{R_{w,k}(\phi)}{(\log X)^k} + O_{w,\phi,K}\bigg(\frac{1}{(\log X)^{K+1}}\bigg),$$

where the coefficients $R_{w,k}(\phi)$ are linear functionals in ϕ that can be given explicitly in terms of w and the derivatives of $\widehat{\phi}$ at the points 0 and 1. The first coefficient is given by

$$\begin{aligned} R_{w,1}(\phi) = \widehat{\phi}(0)\bigg(\log\bigg(\frac{16}{e^{\gamma+1}}\bigg) - 2\sum_{p \geqslant 3} \frac{\log p}{p(p^2-1)} - 2\int_2^\infty \frac{\theta(t)-t}{t^2}\,dt + \frac{2}{\widehat{w}(0)}\int_0^\infty w(x)(\log x)\,dx\bigg) \\ + 2\widehat{\phi}(1)\int_1^\infty \bigg(H_u h_1(u) + \frac{[u]}{u} h_2(u)\bigg)\,du, \end{aligned} \tag{1.3}$$

where $\theta(t) := \sum_{p \leqslant t} \log p$ is the Chebyshev function, $H_u := \sum_{n \leqslant u} n^{-1}$ is the uth harmonic number and h_1, h_2 are explicit transforms of the weight function w that are defined in § 3.2.

Theorem 1.1 will follow from the more precise Theorem 3.5, which gives an expression for $\mathcal{D}^*(\phi;X)$ with a power saving error term.

We remark that Theorem 1.1 agrees with the Katz–Sarnak prediction [KS99a, KS99b], which states that

$$\lim_{X\to\infty} \mathcal{D}^*(\phi;X) = \widehat{\phi}(0) - \frac{1}{2}\int_{-1}^{1} \widehat{\phi}(u)\,du \tag{1.4}$$

independently of the support of $\widehat{\phi}$. Note that the Katz–Sarnak prediction implies much more, in particular that the family $\mathcal{F}^*$ has symplectic symmetry type (by the work of [SST16], this can be predicted from the fact that the family is homogeneous orthogonal). The asymptotic (1.4), which was already obtained by Özlük–Snyder when $\operatorname{supp}\widehat{\phi} \subset (-2,2)$ (under the GRH), shows that there is a phase transition when the supremum of $\operatorname{supp}\widehat{\phi}$ approaches 1. Such a transition is also present in the lower order terms in Theorem 1.1, because of the terms involving $\widehat{\phi}^{(m)}(1)$.

The Katz–Sarnak prediction originates from the following function field analogue of the family $\mathcal{F}^*(X)$. Consider the family $\mathcal{H}_{n,q}$ of zeta functions of hyperelliptic curves $y^2 = Q(x)$ defined over $\mathbb{F}_q$, where $Q(x)$ is a monic squarefree polynomial of degree n. Note the relations $n = 2g+2$ if n is even and $n = 2g+1$ if n is odd, where g is the genus of the hyperelliptic curve. Using the fact that the monodromy corresponding to the family $\mathcal{H}_{n,q}$ equals the symplectic group $\mathrm{Sp}(2g)$ and an equidistribution theorem of Deligne, Katz and Sarnak proved precise results for the low-lying zeros of the zeta functions in $\mathcal{H}_{n,q}$ in the limit as both q and n tend to infinity (see [KS99a, KS99b]).

The family $\mathcal{H}_{n,q}$ with q fixed and $n = 2g+1$ was also studied by Rudnick [Rud10]. He considered the associated 1-level density in the limit as $n \to \infty$. Note that this limit is expected to be a more direct analogue to number fields than the $q \to \infty$ limit. Restricting to the case when $\operatorname{supp}\widehat{\phi} \subset (-2,2)$, Rudnick gave the following estimate for the 1-level density of low-lying zeros of the zeta functions in $\mathcal{H}_{n,q}$:

$$\widehat{\phi}(0) - \frac{1}{2}\int_{-1}^{1}\widehat{\phi}(u)\,du + \frac{1}{g}\bigg(\widehat{\phi}(0)\bigg(\sum_{P \text{ monic irr.}} \frac{\deg P}{q^{2\deg P}-1} + \frac{1}{2}\bigg) - \widehat{\phi}(1)\frac{q+1}{2(q-1)}\bigg) + o\bigg(\frac{1}{g}\bigg) \tag{1.5}$$

(cf. [Rud10, Corollary 3 and the subsequent paragraph]; see also [BCDGL16, Chi16]). Recall that when translating between function fields and number fields it is customary to set $g = \log X$. Taking this into account, note the striking similarity between the expression in (1.5) and our Theorem 1.1; in particular, they both contain a lower order term involving $\widehat{\phi}(0)$ and $\widehat{\phi}(1)$. Here it is interesting that the prediction from the function field situation indicates not only the main term in the number field case (as in the Katz–Sarnak philosophy), but also lower order terms.

In this connection, we note that a lower order term involving $\widehat{\phi}(1)$ is also present in the 1-level density of the family of Dirichlet L-functions attached to all characters modulo q (see [FM15, Theorem 1.2]). However, in this family this term is of order $X^{-1/2}/\log X$ and is thus much smaller than in the family $\mathcal{F}^*$ of real characters.

Next we study the family of all real characters χ_d ordered by the modulus $|d|$, that is, we consider

$$\mathcal{F}(X) := \{L(s,\chi_d) : 1 \leqslant |d| \leqslant X\}.$$

Note that $\zeta(s) \in \mathcal{F}(X)$ and that, for any $a \in \mathbb{N}$, the functions $L(s,\chi_d)$ and $L(s,\chi_{a^2 d})$ have the same nontrivial zeros. The reason why we allow such repetitions is that it simplifies the analysis and allows one to obtain significantly sharper error terms[2] (compare the error terms in Theorems 3.5 and 1.2).

Similarly as above, we define the 1-level density of the family $\mathcal{F}(X)$ to be the sum

$$\mathcal{D}(\phi; X) := \frac{1}{W(X)} \sum_{d \neq 0} w\left(\frac{d}{X}\right) D_X(\chi_d; \phi), \tag{1.6}$$

where

$$W(X) := \sum_{d \neq 0} w\left(\frac{d}{X}\right).$$

Our second main theorem is an asymptotic formula for $\mathcal{D}(\phi; X)$ valid when $\operatorname{supp} \widehat{\phi} \subset (-2, 2)$. For convenience, we introduce the notation $\sigma := \sup(\operatorname{supp} \widehat{\phi})$.

THEOREM 1.2. *Fix $\epsilon > 0$. Assume the GRH and suppose that $\operatorname{supp} \widehat{\phi} \subset (-2, 2)$. Let*

$$U_1(X) := \frac{1}{2\sqrt{2\pi e} \mathcal{M}w(1)} \left(\mathcal{M}w\left(\frac{1}{2}\right) - \frac{w(0)}{\sqrt{X}}\right) \int_{-1}^{1} \left(\frac{X}{2\pi e}\right)^{(u-1)/2} \widehat{\phi}(u)\, du$$

and

$$U_2(X) := \frac{\frac{1}{2}\mathcal{M}w(\frac{1}{2})}{\sqrt{2\pi e}\mathcal{M}w(1)} \int_0^1 \left(\frac{X}{2\pi e}\right)^{(u-1)/2} \widehat{\phi}(u)\, du \\ - \frac{2}{L\widehat{w}(0)} \int_0^\infty \left(\widehat{\phi}\left(1 + \frac{\tau}{L}\right) e^{\tau/2} \sum_{n \geqslant 1} \widehat{h}(ne^{\tau/2}) + \widehat{\phi}\left(1 - \frac{\tau}{L}\right) \sum_{n \geqslant 1} h(ne^{\tau/2})\right) d\tau,$$

where $\mathcal{M}w$ denotes the Mellin transform of w and $h(t) := \widehat{w}(2\pi e t^2)$. Let further

$$U(X) := \begin{cases} U_1(X) & \text{if } \sigma < 1, \\ U_2(X) & \text{if } 1 \leqslant \sigma < 2. \end{cases}$$

Then the 1-level density of low-lying zeros in the family $\mathcal{F}$ of all quadratic Dirichlet L-functions is given by

$$\begin{aligned} \mathcal{D}(\phi; X) = \widehat{\phi}(0) &+ \int_1^\infty \widehat{\phi}(u)\, du \\ &+ \frac{\widehat{\phi}(0)}{L}\left(\log\left(\frac{e^{1-\gamma}}{2^{2/3}}\right) + \frac{2}{\widehat{w}(0)} \int_0^\infty w(x)(\log x)\, dx + 2\frac{\zeta'(2)}{\zeta(2)} - \frac{\mathcal{M}w(\frac{1}{2})}{\mathcal{M}w(1)} \frac{\zeta(\frac{1}{2})}{X^{1/2}}\right) \\ &- \frac{2}{L} \sum_{j=1}^\infty \sum_p \frac{\log p}{p^j} \left(1 + \frac{1}{p}\right)^{-1} \widehat{\phi}\left(\frac{2j \log p}{L}\right) \\ &+ \frac{1}{L} \int_0^\infty \frac{e^{-x/2} + e^{-3x/2}}{1 - e^{-2x}} \left(\widehat{\phi}(0) - \widehat{\phi}\left(\frac{x}{L}\right)\right) dx \\ &+ U(X) + O_{w,\phi,\varepsilon}(X^{\eta(\sigma)+\varepsilon}), \end{aligned} \tag{1.7}$$

where

$$\eta(\sigma) := \begin{cases} -\dfrac{3}{5} & \text{if } \sigma < 1, \\ \dfrac{\sigma}{2} - 1 & \text{if } 1 \leqslant \sigma < 2. \end{cases}$$

[2] There are several known examples in the literature of families with repetitions having such advantages (cf. [You06, FM15, FPS16, SST15]).

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The term $U(X)$ in Theorem 1.2 is $O(X^{(\sigma-1)/2})$ when $\operatorname{supp}\widehat{\phi} \subset (-1,1)$, but is of order $(\log X)^{-1}$ when $\widehat{\phi}$ has mass in a neighborhood of 1 (see Lemma 4.7). Therefore, this term is responsible for a phase transition at 1. Moreover, Lemma 3.7 shows that

$$-\frac{2}{L}\sum_{j=1}^{\infty}\sum_{p}\frac{\log p}{p^j}\bigg(1+\frac{1}{p}\bigg)^{-1}\widehat{\phi}\bigg(\frac{2j\log p}{L}\bigg) = -\frac{\phi(0)}{2} + O_{\phi}\bigg(\frac{1}{\log X}\bigg)$$

and, combining this with Lemma 4.7 and taking $X \to \infty$ in (1.7), we recover the Katz–Sarnak prediction.

We now briefly describe the tools used in the proofs of Theorems 1.1 and 1.2. The fundamental tool is an application of Poisson summation to the prime sum in the explicit formula, following the work of Katz and Sarnak [KS97]. In contrast with our previous work [FPS16], terms with square index in the resulting sum are now of considerable size and contribute to both the main term and new lower order terms in the 1-level density. The novelty in the present work is to transform the terms of square index with an additional application of Poisson summation which isolates the Katz–Sarnak main term and other tractable terms that are estimated later in the analysis (see Lemma 3.2).

Finally, we observe that for small support we have an even more precise result for the family $\mathcal{F}$ (which is in fact unconditional). It is interesting to note that the error term we obtain in this case is significantly sharper than the error term predicted by the corresponding Ratios conjecture (cf. [CS07, Theorem 3.1]).

Theorem 1.3. *Fix $\epsilon > 0$ and suppose that $\operatorname{supp}\widehat{\phi} \subset (-1,1)$. Then the 1-level density of low-lying zeros in the family $\mathcal{F}$ of all quadratic Dirichlet L-functions is given by*

$$\begin{aligned}\mathcal{D}(\phi;X) = {}& \frac{\widehat{\phi}(0)}{LW(X)}\sideset{}{^*}\sum_{d\neq 0}\widetilde{w}\bigg(\frac{d}{X}\bigg)\log|d| - \frac{\widehat{\phi}(0)}{L}\log(\pi e^{\gamma}2^{5/3}) \\ & -\frac{2}{L}\sum_{j=1}^{\infty}\sum_{p}\frac{\log p}{p^j}\bigg(1+\frac{1}{p}\bigg)^{-1}\widehat{\phi}\bigg(\frac{2j\log p}{L}\bigg) \\ & +\frac{1}{L}\int_0^{\infty}\frac{e^{-x/2}+e^{-3x/2}}{1-e^{-2x}}\bigg(\widehat{\phi}(0)-\widehat{\phi}\bigg(\frac{x}{L}\bigg)\bigg)\,dx \\ & +\frac{1}{2\sqrt{2\pi e}\mathcal{M}w(1)}\bigg(\mathcal{M}w\bigg(\frac{1}{2}\bigg)-\frac{w(0)}{\sqrt{X}}\bigg)\int_{-1}^{1}\bigg(\frac{X}{2\pi e}\bigg)^{(u-1)/2}\widehat{\phi}(u)\,du \\ & + O_{w,\phi,\varepsilon}(X^{\xi(\sigma)+\varepsilon}),\end{aligned}\tag{1.8}$$

where $\widetilde{w}(y) := \sum_{n\geqslant 1} w(n^2y)$ and

$$\xi(\sigma) := \begin{cases} -1+\sigma & \text{if } \dfrac{1}{2m+1} \leqslant \sigma < \dfrac{1}{2m+\frac{1}{2}}, \\ -\dfrac{4m-1}{4m+1} & \text{if } \dfrac{1}{2m+\frac{1}{2}} \leqslant \sigma < \dfrac{1}{2m-1}, \end{cases}$$

for each $m \geqslant 1$.

Remark 1.4. We can also obtain an unconditional result similar to Theorem 1.3 for $\mathcal{D}^*(\phi;X)$. However, in this case the error term will be the weaker $O_{\varepsilon}(X^{(\sigma-1)/2+\varepsilon})$ and hence we choose not

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to provide the details in the present paper. (Note that under the GRH, Proposition 3.1 gives the sharper error term $O_\varepsilon(X^{\max\{\sigma/4-1/2,\,3\sigma/4-3/4\}+\varepsilon})$.) This setting was previously studied by Miller [Mil08, Theorem 1.2], who claimed an error term of size

$$O_\varepsilon(X^{-1/2}+X^{-(1-3\sigma/2)+\varepsilon}+X^{-3(1-\sigma)/4+\varepsilon}).$$

However, going through the proof of [Mil08, Lemma 3.5], we find[3] that the actual error term resulting from [Mil08] is $O_\varepsilon(X^{(\sigma-1)/2+\varepsilon})$.

2. Preliminary results for the family $\mathcal{F}^*(X)$

2.1 Explicit formula and character sums

We will study the 1-level density via the explicit formula for primitive Dirichlet L-functions.

LEMMA 2.1 (Explicit formula). *Assume that ϕ is an even Schwartz test function whose Fourier transform has compact support. Then the 1-level density defined in (1.2) is given by the formula*

$$\begin{aligned}\mathcal{D}^*(\phi;X)&=\frac{\widehat{\phi}(0)}{LW^*(X)}\sideset{}{^*}\sum_{d\text{ odd}}w\Big(\frac{d}{X}\Big)\log|d|-\frac{\widehat{\phi}(0)}{L}(\gamma+\log\pi)\\&\quad-\frac{2}{LW^*(X)}\sum_{p,m}\frac{\log p}{p^{m/2}}\widehat{\phi}\Big(\frac{m\log p}{L}\Big)\sideset{}{^*}\sum_{d\text{ odd}}w\Big(\frac{d}{X}\Big)\Big(\frac{8d}{p^m}\Big)\\&\quad+\frac{1}{L}\int_0^\infty\frac{e^{-x/2}+e^{-3x/2}}{1-e^{-2x}}\Big(\widehat{\phi}(0)-\widehat{\phi}\Big(\frac{x}{L}\Big)\Big)\,dx.\end{aligned}\tag{2.1}$$

Proof. Let d be an odd squarefree integer. Then χ_{8d} is a primitive character of conductor $8|d|$. Taking $\widehat{F}(t)=\Phi(\frac{1}{2}+it):=\phi(tL/(2\pi))$ in [MV07, Theorem 12.13] (whose conditions are satisfied by our restrictions on ϕ), we obtain the formula

$$\begin{aligned}D_X(\chi_{8d};\phi)&=\frac{\widehat{\phi}(0)}{L}\Big(\log\Big(\frac{8|d|}{\pi}\Big)+\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{\mathfrak{a}}{2}\Big)\Big)-\frac{2}{L}\sum_{p,m}\frac{\chi_{8d}(p^m)\log p}{p^{m/2}}\widehat{\phi}\Big(\frac{m\log p}{L}\Big)\\&\quad+\frac{2}{L}\int_0^\infty\frac{e^{-(1/2+\mathfrak{a})x}}{1-e^{-2x}}\Big(\widehat{\phi}(0)-\widehat{\phi}\Big(\frac{x}{L}\Big)\Big)\,dx,\end{aligned}$$

where

$$\mathfrak{a}:=\begin{cases}0&\text{if } d\geqslant 0,\\1&\text{if } d<0.\end{cases}$$

Formula (2.1) then follows by summing over d against the weight function w. □

We will need the following estimate on a weighted quadratic character sum.

LEMMA 2.2. *Fix $n\in\mathbb{N}$ and $\varepsilon>0$. Under the Riemann hypothesis (RH), we have the estimate*

$$\sideset{}{^*}\sum_{d\text{ odd}}w\Big(\frac{d}{X}\Big)\Big(\frac{8d}{n}\Big)=\kappa(n)\frac{2X}{3\zeta(2)}\widehat{w}(0)\prod_{p\mid n}\Big(1+\frac{1}{p}\Big)^{-1}+O_{\varepsilon,w}(|n|^{3(1-\kappa(n))/8+\varepsilon}X^{1/4+\varepsilon}),$$

where

$$\kappa(n):=\begin{cases}1&\textit{if } n \textit{ is an odd square},\\0&\textit{otherwise.}\end{cases}$$

[3] The second, third and fourth terms on the right-hand side of [Mil08, (3.39)] should be multiplied by X (see [Mil08, (3.36)]).

Proof. The result follows similarly as in [FPS16, Lemma 2.10]. □

Remark 2.3. Taking $n = 1$ in Lemma 2.2 gives the following conditional estimate for the total weight:

$$W^*(X) = \frac{2X}{3\zeta(2)}\widehat{w}(0) + O_{\varepsilon,w}(X^{1/4+\varepsilon}). \tag{2.2}$$

Let us now evaluate the first sum on the right-hand side of (2.1).

Lemma 2.4. *Fix $\varepsilon > 0$ and assume the RH. We have the estimate*

$$\frac{1}{W^*(X)}\sum_{d \text{ odd}}^{*} w\left(\frac{d}{X}\right)\log|d| = \log X + \frac{2}{\widehat{w}(0)}\int_0^\infty w(x)(\log x)\,dx + O_{\varepsilon,w}(X^{-3/4+\varepsilon}).$$

Proof. The proof is similar to that of [FPS16, Lemma 2.8]. □

The following consequence of the GRH will be central in our analysis.

Lemma 2.5. *Assume the GRH. For $m \in \mathbb{Z}_{\neq 0}$ and $y \geqslant 1$, we have the estimate*

$$S_m(y) := \sum_{p \leqslant y}\left(\frac{m}{p}\right)\log p = \delta_{m=\square}y + O(y^{1/2}\log(2y)\log(2|m|y)).$$

Proof. Write $m = a^2 b$, with $\mu^2(b) = 1$. Then we clearly have that $S_m(y) = S_b(y) + O(\log|a|)$. Applying [IK04, Theorem 5.15], we have that

$$\sum_{\substack{p^e \leqslant y\\ e \geqslant 1}} \log p\left(\frac{b}{p^e}\right) = \delta_{b=1}y + O(y^{1/2}\log(2y)\log(2|b|y)).$$

The result follows by trivially bounding the contribution of prime powers. □

2.2 Poisson summation

In this section we will provide an approximate expression for the prime sum appearing in (2.1). We first separate the odd and the even prime powers, by writing

$$S^*_{\text{odd}} := -\frac{2}{LW^*(X)}\sum_{\substack{p\\ m \text{ odd}}}\frac{\log p}{p^{m/2}}\widehat{\phi}\left(\frac{m\log p}{L}\right)\sum_{d \text{ odd}}^{*} w\left(\frac{d}{X}\right)\left(\frac{8d}{p^m}\right), \tag{2.3}$$

and similarly for S^*_{even}. We will transform (2.3) using Poisson summation.

From now on, we will not necessarily indicate the dependence of the error terms on ϕ and w.

Lemma 2.6. *Fix $\varepsilon > 0$. Assume the GRH and suppose that $\sigma = \sup(\operatorname{supp}\widehat{\phi}) < \infty$. Then, for any $S \geqslant 1$, we have the estimate*

$$\begin{aligned} S^*_{\text{odd}} = &-\frac{2X}{LW^*(X)}\sum_{\substack{s \leqslant S\\ s \text{ odd}}}\frac{\mu(s)}{s^2}\sum_{p \nmid 2s}\frac{\overline{\epsilon_p}\log p}{p}\widehat{\phi}\left(\frac{\log p}{L}\right)\sum_{t \in \mathbb{Z}}\left(\left(\frac{-2t}{p}\right)\widehat{w}\left(\frac{Xt}{s^2 p}\right) - \frac{1}{2}\left(\frac{-t}{p}\right)\widehat{w}\left(\frac{Xt}{2s^2 p}\right)\right)\\ &+ O_\varepsilon(X^{-3/4+\varepsilon} + X^\varepsilon(\log S)^3 S^{-1}), \end{aligned} \tag{2.4}$$

where

$$\epsilon_p := \begin{cases} 1 & \text{if } p \equiv 1 \bmod 4,\\ i & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

Proof. By Lemma 2.2, the contribution of the terms with $m \geqslant 3$ in (2.3) is $O_\varepsilon(X^{-3/4+\varepsilon})$. We transform the sum over d into a sum over all odd integers using the usual convolution identity for the indicator function of squarefree integers. This yields the estimate

$$S^*_{\text{odd}} = -\frac{2}{LW^*(X)} \sum_{\substack{s\in\mathbb{N}\\ s\text{ odd}}} \mu(s) \sum_{\substack{u\in\mathbb{Z}\\ u\text{ odd}}} \sum_{p\nmid 2s} \frac{\log p}{p^{1/2}} \widehat{\phi}\left(\frac{\log p}{L}\right) w\left(\frac{us^2}{X}\right)\left(\frac{8u}{p}\right) + O_\varepsilon(X^{-3/4+\varepsilon}).$$

We now apply Lemma 2.5 and summation by parts. Note that if u is odd, then $8u$ is never a square. It follows that the terms with $s > S$ are

$$\ll \frac{1}{LX} \sum_{\substack{s>S\\ s\text{ odd}}} \sum_{\substack{u\in\mathbb{Z}\\ u\text{ odd}}} w\left(\frac{us^2}{X}\right)(\log(2|u|sX))^3 \ll_\varepsilon X^\varepsilon (\log S)^3 S^{-1}.$$

As for the terms with $s \leqslant S$, we introduce additive characters using Gauss sums, resulting in the estimate

$$\begin{aligned} S^*_{\text{odd}} = &-\frac{2}{LW^*(X)} \sum_{\substack{s\leqslant S\\ s\text{ odd}}} \mu(s) \sum_{p\nmid 2s} \frac{\overline{\epsilon_p}\log p}{p} \widehat{\phi}\left(\frac{\log p}{L}\right) \sum_{b \bmod p} \left(\frac{b}{p}\right) \\ &\times \sum_{u\in\mathbb{Z}} \left(w\left(\frac{us^2}{X}\right) e\left(\frac{8ub}{p}\right) - w\left(\frac{2us^2}{X}\right) e\left(\frac{16ub}{p}\right)\right) \\ &+ O_\varepsilon(X^{-3/4+\varepsilon} + X^\varepsilon(\log S)^3 S^{-1}). \end{aligned}$$

Applying Poisson summation in the inner sum yields the expression

$$\begin{aligned} S^*_{\text{odd}} = &-\frac{2X}{LW^*(X)} \sum_{\substack{s\leqslant S\\ s\text{ odd}}} \frac{\mu(s)}{s^2} \sum_{p\nmid 2s} \frac{\overline{\epsilon_p}\log p}{p} \widehat{\phi}\left(\frac{\log p}{L}\right) \sum_{b \bmod p} \left(\frac{b}{p}\right) \\ &\times \left(\sum_{v_1\in\mathbb{Z}} \widehat{w}\left(\frac{X}{s^2}\left(v_1 - \frac{8b}{p}\right)\right) - \frac{1}{2} \sum_{v_2\in\mathbb{Z}} \widehat{w}\left(\frac{X}{2s^2}\left(v_2 - \frac{16b}{p}\right)\right)\right) \\ &+ O_\varepsilon(X^{-3/4+\varepsilon} + X^\varepsilon(\log S)^3 S^{-1}). \end{aligned}$$

The sums over b and v_j can be replaced by a single sum over $t_j := v_j p - 2^{j+2} b$ ($j = 1, 2$). Using the fact that for $p > 2$, we have

$$\left(\frac{b}{p}\right) = \left(\frac{2}{p}\right)\left(\frac{8b}{p}\right) = \left(\frac{-2(t_1 - v_1 p)}{p}\right) = \left(\frac{-2t_1}{p}\right)$$

and

$$\left(\frac{b}{p}\right) = \left(\frac{16b}{p}\right) = \left(\frac{v_2 p - t_2}{p}\right) = \left(\frac{-t_2}{p}\right),$$

we end up with the estimate (2.4). □

LEMMA 2.7. *Assume the GRH, fix $\varepsilon > 0$ and suppose that $\sigma = \sup(\operatorname{supp}\widehat{\phi}) < \infty$. Then, for any $1 \leqslant S \leqslant X^2$, we have that*[4]

$$\begin{aligned} S^*_{\text{odd}} = &\frac{2X}{W^*(X)} \sum_{\substack{s\leqslant S\\ s\text{ odd}}} \frac{\mu(s)}{s^2} \int_0^\infty \widehat{\phi}(u) \sum_{m\geqslant 1} \left(\frac{1}{2}\widehat{w}\left(\frac{m^2 X^{1-u}}{2s^2(2\pi e)^{-u}}\right) - \widehat{w}\left(\frac{2m^2 X^{1-u}}{s^2(2\pi e)^{-u}}\right)\right) du \\ &+ O_\varepsilon(X^{-3/4+\varepsilon} + X^\varepsilon S^{-1} + S X^{\sigma/2-1+\varepsilon}). \end{aligned}$$

[4] This range can be replaced by $1 \leqslant S \leqslant X^M$, for any fixed $M \in \mathbb{N}$. However, the important range for our analysis is $1 \leqslant S \leqslant X^2$.

Proof. By the definition of ϵ_p, the second part of the main term in (2.4) equals

$$\frac{X}{LW^*(X)}\sum_{\substack{s\leqslant S\\ s\text{ odd}}}\frac{\mu(s)}{s^2}\sum_{p\nmid 2s}\frac{\log p}{p}\widehat{\phi}\bigg(\frac{\log p}{L}\bigg)\sum_{t\in\mathbb{Z}}\bigg(\frac{1+i}{2}\bigg(\frac{t}{p}\bigg)+\frac{1-i}{2}\bigg(\frac{-t}{p}\bigg)\bigg)\widehat{w}\bigg(\frac{Xt}{2s^2p}\bigg)$$
$$=\frac{X}{LW^*(X)}\sum_{\substack{s\leqslant S\\ s\text{ odd}}}\frac{\mu(s)}{s^2}\sum_{p\nmid 2s}\frac{\log p}{p}\widehat{\phi}\bigg(\frac{\log p}{L}\bigg)\sum_{t>0}\bigg(\bigg(\frac{t}{p}\bigg)+\bigg(\frac{-t}{p}\bigg)\bigg)\widehat{w}\bigg(\frac{Xt}{2s^2p}\bigg).$$

Note that we can add back the primes dividing $2s$ at the cost of an admissible error term.

By Lemma 2.5, we have for $t>0$ and $y\geqslant 1$ that

$$T_t(y):=\sum_{p\leqslant y}\log p\bigg(\bigg(\frac{t}{p}\bigg)+\bigg(\frac{-t}{p}\bigg)\bigg)=\delta_{t=\square}(y-1)+O(y^{1/2}\log(2y)\log(2|t|y)).$$

It then follows that

$$\sum_{t>0}\sum_{p}\frac{\log p}{p}\widehat{\phi}\bigg(\frac{\log p}{L}\bigg)\bigg(\bigg(\frac{t}{p}\bigg)+\bigg(\frac{-t}{p}\bigg)\bigg)\widehat{w}\bigg(\frac{Xt}{2s^2p}\bigg)$$
$$=\sum_{t>0}\int_1^\infty\widehat{\phi}\bigg(\frac{\log y}{L}\bigg)\widehat{w}\bigg(\frac{Xt}{2s^2y}\bigg)\frac{dT_t(y)}{y}$$
$$=-\sum_{t>0}\int_1^\infty\bigg[y^{-1}\widehat{\phi}\bigg(\frac{\log y}{L}\bigg)\widehat{w}\bigg(\frac{Xt}{2s^2y}\bigg)\bigg]'(\delta_{t=\square}(y-1)+O(y^{1/2}\log(2y)\log(2|t|y)))\,dy$$
$$=\sum_{t=\square}\int_1^\infty\widehat{\phi}\bigg(\frac{\log y}{L}\bigg)\widehat{w}\bigg(\frac{Xt}{2s^2y}\bigg)\frac{dy}{y}+O_\varepsilon(s^2(\log(2s))^2X^{\sigma/2-1+\varepsilon}),$$

by an argument similar to that in the proof of [FPS16, Lemma 4.3].[5] As for the first part of the main term of (2.4), it can be analyzed along the same lines; the quantity analogous to $T_t(y)$ is

$$\sum_{p\leqslant y}\log p\bigg(\bigg(\frac{2t}{p}\bigg)+\bigg(\frac{-2t}{p}\bigg)\bigg)=\delta_{t=2\square}(y-1)+O(y^{1/2}\log(2y)\log(2|t|y)).$$

The lemma follows from taking the change of variables $u=\log y/L$ and summing over s. □

3. New lower order terms

In Lemma 2.7, we saw that to understand S^*_{odd} it is important to give a precise estimate of the term

$$I_s(X):=\int_0^\infty\widehat{\phi}(u)\sum_{m\geqslant 1}\widehat{w}\bigg(\frac{2m^2X^{1-u}}{s^2(2\pi e)^{-u}}\bigg)\,du. \tag{3.1}$$

Indeed, the lemma implies that for $S\leqslant X^2$ and under the GRH,

$$S^*_{\text{odd}}=\frac{2X}{W^*(X)}\sum_{\substack{s\leqslant S\\ s\text{ odd}}}\frac{\mu(s)}{s^2}\bigg(\frac{1}{2}I_{2s}(X)-I_s(X)\bigg)+O_\varepsilon(X^{-3/4+\varepsilon}+X^\varepsilon S^{-1}+SX^{\sigma/2-1+\varepsilon}). \tag{3.2}$$

Here and throughout this section we assume that $\sigma=\sup(\operatorname{supp}\widehat{\phi})<\infty$. Our strategy will be to treat the integrals over the intervals $[0,1]$ and $[1,\sigma]$ differently; the former will be computed directly and the latter via an application of Poisson summation.

[5] Note that the integrand in the current paper is zero for $y\geqslant(X/2\pi e)^\sigma$.

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3.1 Small support

In this section we assume that $\sigma < 1$. In this range we will not find new lower order terms; these only appear when σ is at least 1 (see § 3.2).

Proposition 3.1. *Fix $\epsilon > 0$. Assume the GRH and suppose that $\sigma < 1$. Then we have the bound*

$$S^*_{\text{odd}} \ll_\varepsilon X^{\sigma/4-1/2+\varepsilon} + X^{3\sigma/4-3/4+\varepsilon}.$$

Proof. Let

$$T(t) := \sum_{\substack{s\leqslant t\\ s\,\text{odd}}} \frac{\mu(s)}{s^2} = \frac{4}{3\zeta(2)} + O_\varepsilon(t^{-3/2+\varepsilon}).$$

We then have, for $0 \leqslant u \leqslant 1$, that

$$\begin{aligned}\sum_{\substack{s\leqslant S\\ s\,\text{odd}}} \frac{\mu(s)}{s^2}\widehat{w}\bigg(\frac{2m^2X^{1-u}}{s^2(2\pi e)^{-u}}\bigg) &= \int_{0+}^{S} \widehat{w}\bigg(\frac{2m^2X^{1-u}}{t^2(2\pi e)^{-u}}\bigg)\,dT(t)\\ &= \widehat{w}\bigg(\frac{2m^2X^{1-u}}{S^2(2\pi e)^{-u}}\bigg)T(S) + \frac{4m^2X^{1-u}}{(2\pi e)^{-u}}\int_{0+}^{S}\widehat{w}'\bigg(\frac{2m^2X^{1-u}}{t^2(2\pi e)^{-u}}\bigg)\bigg(\frac{4}{3\zeta(2)} + O_\varepsilon\big(t^{-3/2+\varepsilon}\big)\bigg)\frac{dt}{t^3}\\ &\ll_\varepsilon S^{-3/2+\varepsilon}\bigg|\widehat{w}\bigg(\frac{2m^2X^{1-u}}{S^2(2\pi e)^{-u}}\bigg)\bigg| + m^2X^{1-u}\int_{0+}^{S}\bigg|\widehat{w}'\bigg(\frac{2m^2X^{1-u}}{t^2(2\pi e)^{-u}}\bigg)\bigg|\frac{dt}{t^{9/2-\varepsilon}}.\end{aligned}$$

Note that the part of the last integral for $t \in (0, X^{(1-u)/2-\varepsilon}]$ is $O_{N,\varepsilon}((mX)^{-N})$ for any $N \geqslant 1$, by the rapid decay of $\widehat{w}'$. Summing over m and integrating over u, we obtain that

$$\begin{aligned}\sum_{\substack{s\leqslant S\\ s\,\text{odd}}} \frac{\mu(s)}{s^2} I_s(X) &\ll_\varepsilon \int_0^\infty |\widehat{\phi}(u)| \sum_{m\geqslant 1}\bigg(S^{-3/2+\varepsilon}\bigg|\widehat{w}\bigg(\frac{2m^2X^{1-u}}{S^2(2\pi e)^{-u}}\bigg)\bigg|\\ &\quad + m^2X^{1-u}\int_{X^{(1-u)/2-\varepsilon}}^{S}\bigg|\widehat{w}'\bigg(\frac{2m^2X^{1-u}}{t^2(2\pi e)^{-u}}\bigg)\bigg|\frac{dt}{t^{9/2-\varepsilon}}\bigg)\,du + X^{-1}\\ &\ll \int_0^\infty |\widehat{\phi}(u)|\bigg(S^{-3/2+\varepsilon}\frac{S}{X^{(1-u)/2}} + \frac{1}{X^{(1-u)/2}}\int_{X^{(1-u)/2-\varepsilon}}^{S}\frac{dt}{t^{3/2-\varepsilon}}\bigg)\,du + X^{-1}\\ &\ll \frac{X^{(\sigma-1)/2}}{S^{1/2-\varepsilon}} + X^{3\sigma/4-3/4+\varepsilon}.\end{aligned}$$

Hence, from (3.2), it follows that for $S \leqslant X^2$,

$$S^*_{\text{odd}} \ll_\varepsilon \frac{X^{(\sigma-1)/2}}{S^{1/2-\varepsilon}} + X^{3\sigma/4-3/4+\varepsilon} + X^\varepsilon S^{-1} + SX^{\sigma/2-1+\varepsilon}.$$

The result follows by taking $S = X^{1/2-\sigma/4}$. □

3.2 Extended support

In this section we will see that when $\sigma > 1$, the prime sum S^*_{odd} contains terms of considerable size, and we will give an asymptotic expansion of these terms in descending powers of $\log X$. For convenience, we introduce the function

$$g(y) := \widehat{w}(4\pi e y^2). \tag{3.3}$$

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Lemma 3.2. *Suppose that $\sigma = \sup(\operatorname{supp}\widehat{\phi}) < \infty$. Then, for $s \geqslant 1$, the quantity defined in (3.1) satisfies the estimate*

$$I_s(X) = \frac{1}{L}\int_0^\infty \left(\widehat{\phi}\left(1+\frac{\tau}{L}\right) s e^{\tau/2} \sum_{n\geqslant 1} \widehat{g}(sne^{\tau/2}) + \widehat{\phi}\left(1-\frac{\tau}{L}\right)\sum_{m\geqslant 1} g\left(\frac{me^{\tau/2}}{s}\right)\right) d\tau$$
$$+ \frac{s\widehat{g}(0)}{2}\int_1^\infty \left(\frac{X}{2\pi e}\right)^{(u-1)/2} \widehat{\phi}(u)\,du - \frac{\widehat{w}(0)}{2}\int_1^\infty \widehat{\phi}(u)\,du + O(sX^{-1/2}). \tag{3.4}$$

Proof. Extending the integral in (3.1) to $\mathbb{R}$ and making the substitution $\tau = L(u-1)$, we obtain

$$I_s(X) = \frac{1}{L}\int_{-\infty}^\infty \widehat{\phi}\left(1+\frac{\tau}{L}\right)\sum_{m\geqslant 1}\widehat{w}\left(\frac{4\pi m^2 e^{1-\tau}}{s^2}\right) d\tau + O(sX^{-1/2}).$$

We denote the integrals over $(-\infty, 0]$ and $[0,\infty)$ by $I_s^-(X)$ and $I_s^+(X)$, respectively. For the second of these integrals, we apply Poisson summation. We obtain

$$\begin{aligned} I_s^+(X) &= \frac{1}{L}\int_0^\infty \widehat{\phi}\left(1+\frac{\tau}{L}\right)\left(-\frac{\widehat{w}(0)}{2} + \frac{1}{2}\sum_{m\in\mathbb{Z}} g\left(\frac{me^{-\tau/2}}{s}\right)\right) d\tau \\ &= \frac{1}{L}\int_0^\infty \widehat{\phi}\left(1+\frac{\tau}{L}\right)\left(-\frac{\widehat{w}(0)}{2} + \frac{se^{\tau/2}}{2}\sum_{n\in\mathbb{Z}} \widehat{g}(sne^{\tau/2})\right) d\tau \\ &= \frac{1}{L}\int_0^\infty \widehat{\phi}\left(1+\frac{\tau}{L}\right)\left(\frac{se^{\tau/2}\widehat{g}(0)}{2} - \frac{\widehat{w}(0)}{2} + se^{\tau/2}\sum_{n\geqslant 1} \widehat{g}(sne^{\tau/2})\right) d\tau. \end{aligned}$$

For $I_s^-(X)$, we substitute τ with $-\tau$, which gives

$$I_s^-(X) = \frac{1}{L}\int_0^\infty \widehat{\phi}\left(1-\frac{\tau}{L}\right)\sum_{m\geqslant 1} g\left(\frac{me^{\tau/2}}{s}\right) d\tau.$$

The lemma follows by combining the above formulas for $I_s^-(X)$ and $I_s^+(X)$. □

We define the functions

$$h_1(x) := \frac{3\zeta(2)}{\widehat{w}(0)}\sum_{\substack{s\geqslant 1\\ s \text{ odd}}}\frac{\mu(s)}{s}(\widehat{g}(2sx) - \widehat{g}(sx)); \quad h_2(x) := \frac{3\zeta(2)}{\widehat{w}(0)}\sum_{\substack{s\geqslant 1\\ s \text{ odd}}}\frac{\mu(s)}{s^2}\left(\frac{1}{2}g\left(\frac{x}{2s}\right) - g\left(\frac{x}{s}\right)\right).$$

It is a routine exercise to check that $h_1(x)$ and $h_2(x)$ are smooth for $x \in \mathbb{R}_{\neq 0}$ and $x \in \mathbb{R}$, respectively. One can also check that for any fixed $N \geqslant 1$ and $\varepsilon > 0$, we have the bounds $h_1(x) \ll_N x^{-N}$ and (under the RH) $h_2(x) \ll_\varepsilon x^{-3/2+\varepsilon}$.

Remark 3.3. One can show that h_1 is continuous at 0. Indeed, let $f(u) = \widehat{g}(2u) - \widehat{g}(u)$ and write, for $x \neq 0$,

$$\sum_{\substack{s\geqslant 1\\ s \text{ odd}}}\frac{\mu(s)}{s} f(sx) = \int_{1^-}^\infty f(tx)\,dS(t) = -\int_{1^-}^\infty xf'(tx)S(t)\,dt, \tag{3.5}$$

where

$$S(t) = \sum_{\substack{s\leqslant t\\ s \text{ odd}}}\frac{\mu(s)}{s} \ll t^{-1/2+\epsilon}$$

(under the RH). We then apply the (rough) bound $f'(u) \ll |u|^{-3/4}$ ($u \neq 0$) and conclude that the right-hand side of (3.5) is $\ll |x|^{1/4}$, proving continuity.

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COROLLARY 3.4. *Fix $\epsilon > 0$ and assume the GRH. Then we have the estimate*

$$S^*_{\text{odd}} = \int_1^\infty \widehat{\phi}(u)\,du + \frac{1}{L}\int_0^\infty \bigg(\widehat{\phi}\Big(1+\frac{\tau}{L}\Big)e^{\tau/2}\sum_{n\geqslant 1} h_1(ne^{\tau/2}) + \widehat{\phi}\Big(1-\frac{\tau}{L}\Big)\sum_{n\geqslant 1} h_2(ne^{\tau/2})\bigg)\,d\tau$$
$$+ O_\varepsilon(X^{\sigma/6-1/3+\varepsilon}).$$

Proof. We sum the right-hand side of (3.4) over s. By (3.2) and Remark 2.3, in the range $S \leqslant X^2$ this gives the estimate

$$\begin{aligned} S^*_{\text{odd}} = \frac{2X}{LW^*(X)} \sum_{\substack{s\leqslant S\\ s\,\text{odd}}} \frac{\mu(s)}{s^2}\bigg(\frac{1}{2}\int_0^\infty \bigg[\widehat{\phi}\Big(1+\frac{\tau}{L}\Big)2se^{\tau/2}\sum_{n\geqslant 1}\widehat{g}(2sne^{\tau/2}) \\ + \widehat{\phi}\Big(1-\frac{\tau}{L}\Big)\sum_{m\geqslant 1} g\Big(\frac{me^{\tau/2}}{2s}\Big)\bigg]\,d\tau \\ - \int_0^\infty \bigg[\widehat{\phi}\Big(1+\frac{\tau}{L}\Big)se^{\tau/2}\sum_{n\geqslant 1}\widehat{g}(sne^{\tau/2}) + \widehat{\phi}\Big(1-\frac{\tau}{L}\Big)\sum_{m\geqslant 1} g\Big(\frac{me^{\tau/2}}{s}\Big)\bigg]\,d\tau\bigg) \\ + \int_1^\infty \widehat{\phi}(u)\,du + O_\varepsilon(X^{-1/2+\varepsilon} + X^\varepsilon S^{-1} + SX^{\sigma/2-1+\varepsilon}). \end{aligned} \tag{3.6}$$

We can extend the sum over s to all positive odd integers at the cost of the error term $O(X^\varepsilon S^{-1/2})$. Changing the order of summation, taking $S = X^{2/3-\sigma/3}$ and applying Remark 2.3 gives the result. □

We summarize the findings of this section in the following theorem.

THEOREM 3.5. *Fix $\epsilon > 0$. Assume the GRH and suppose that $\sigma = \sup(\operatorname{supp}\widehat{\phi}) < 2$. Then the 1-level density of low-lying zeros in the family $\mathcal{F}^*$ of quadratic Dirichlet L-functions whose conductor is an odd squarefree multiple of 8 is given by*

$$\begin{aligned} \mathcal{D}^*(\phi; X) = \widehat{\phi}(0) + \int_1^\infty \widehat{\phi}(u)\,du + \frac{\widehat{\phi}(0)}{L}\bigg(\log(2e^{1-\gamma}) + \frac{2}{\widehat{w}(0)}\int_0^\infty w(x)(\log x)\,dx\bigg) \\ + \frac{1}{L}\int_0^\infty \frac{e^{-x/2}+e^{-3x/2}}{1-e^{-2x}}\bigg(\widehat{\phi}(0) - \widehat{\phi}\Big(\frac{x}{L}\Big)\bigg)\,dx \\ - \frac{2}{L}\sum_{\substack{p>2\\ j\geqslant 1}} \frac{\log p}{p^j}\Big(1+\frac{1}{p}\Big)^{-1}\widehat{\phi}\Big(\frac{2j\log p}{L}\Big) + J(X) + O_\varepsilon(X^{\sigma/6-1/3+\varepsilon}), \end{aligned} \tag{3.7}$$

where

$$J(X) := \frac{1}{L}\int_0^\infty \bigg(\widehat{\phi}\Big(1+\frac{\tau}{L}\Big)e^{\tau/2}\sum_{n\geqslant 1} h_1(ne^{\tau/2}) + \widehat{\phi}\Big(1-\frac{\tau}{L}\Big)\sum_{n\geqslant 1} h_2(ne^{\tau/2})\bigg)\,d\tau.$$

Proof. Combining Lemma 2.1 with Lemma 2.4 and Corollary 3.4, and noting that Lemma 2.2 implies the estimate

$$S^*_{\text{even}} = -\frac{2}{L}\sum_{\substack{p>2\\ j\geqslant 1}} \frac{\log p}{p^j}\Big(1+\frac{1}{p}\Big)^{-1}\widehat{\phi}\Big(\frac{2j\log p}{L}\Big) + O_\varepsilon(X^{-3/4+\varepsilon}), \tag{3.8}$$

we obtain the desired result. □

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Next we show how to deduce Theorem 1.1 from this result. The key is to expand the various terms in the right-hand side of (3.7) in descending powers of $\log X$. Note that the term $J(X)$ is of order $(\log X)^{-1}$ and constitutes a genuine lower order term in the 1-level density $\mathcal{D}^*(\phi; X)$.

LEMMA 3.6. *Assume the RH. Then, for any $K \geqslant 1$, we have the expansion*

$$J(X) = \sum_{k=1}^{K} \frac{c_{w,k}\widehat{\phi}^{(k-1)}(1)}{L^k} + O_K\bigg(\frac{1}{L^{K+1}}\bigg),$$

where the constants $c_{w,k}$ can be given explicitly. The first of these constants is given by

$$c_{w,1} = 2\int_1^{\infty}\bigg(H_u h_1(u) + \frac{[u]}{u}h_2(u)\bigg)\,du,$$

where $H_u := \sum_{n\leqslant u} n^{-1}$ is the uth harmonic number.

Proof. By the decay properties of h_1 and h_2, we have that

$$\begin{aligned} J(X) = \frac{1}{L}\int_0^{L^{1/2}} &\bigg(\widehat{\phi}\bigg(1+\frac{\tau}{L}\bigg)e^{\tau/2}\sum_{n\geqslant 1} h_1(ne^{\tau/2}) + \widehat{\phi}\bigg(1-\frac{\tau}{L}\bigg)\sum_{n\geqslant 1} h_2(ne^{\tau/2})\bigg)\,d\tau \\ &+ O_\varepsilon\bigg(\exp\bigg(-\bigg(\frac{3}{4}-\varepsilon\bigg)\sqrt{L}\bigg)\bigg). \end{aligned}$$

We can now expand $\widehat{\phi}$ in Taylor series, resulting in the expression

$$\begin{aligned} J(X) &= \sum_{k=1}^{K}\frac{\widehat{\phi}^{(k-1)}(1)}{(k-1)!L^k}\int_0^{L^{1/2}}\bigg(\tau^{k-1}e^{\tau/2}\sum_{n\geqslant 1}h_1(ne^{\tau/2}) + (-\tau)^{k-1}\sum_{n\geqslant 1}h_2(ne^{\tau/2})\bigg)\,d\tau \\ &\quad + O_K(L^{-K-1}) \\ &= \sum_{k=1}^{K}\frac{\widehat{\phi}^{(k-1)}(1)}{(k-1)!L^k}\sum_{n\geqslant 1}\int_0^{\infty}(\tau^{k-1}e^{\tau/2}h_1(ne^{\tau/2}) + (-\tau)^{k-1}h_2(ne^{\tau/2}))\,d\tau + O_K(L^{-K-1}). \end{aligned}$$

Finally, the first summand equals

$$\frac{\widehat{\phi}(1)}{L}\sum_{n\geqslant 1}\int_0^{\infty}(e^{\tau/2}h_1(ne^{\tau/2}) + h_2(ne^{\tau/2}))\,d\tau = \frac{2\widehat{\phi}(1)}{L}\sum_{n\geqslant 1}\int_n^{\infty}\bigg(\frac{h_1(u)}{n} + \frac{h_2(u)}{u}\bigg)\,du.$$

The result follows from interchanging the order of summation and integration. □

The final ingredient needed in the proof of Theorem 1.1 is an expansion for S^*_{even} of the same form as that of $J(X)$ in Lemma 3.6.

LEMMA 3.7. *Suppose that $\sigma = \sup(\operatorname{supp}\widehat{\phi}) < \infty$. Then we have the formula*

$$-\frac{2}{L}\sum_{\substack{p>2\\ j\geqslant 1}}\frac{\log p}{p^j}\bigg(1+\frac{1}{p}\bigg)^{-1}\widehat{\phi}\bigg(\frac{2j\log p}{L}\bigg) = -\frac{\phi(0)}{2} + \sum_{k=1}^{K}\frac{d_k\widehat{\phi}^{(k-1)}(0)}{L^k} + O_K\bigg(\frac{1}{L^{K+1}}\bigg),$$

where the coefficients d_k are real numbers that can be given explicitly. In particular, we have

$$d_1 = -2\sum_{p,j\geqslant 3}\frac{\log p}{p^j}\bigg(1+\frac{1}{p}\bigg)^{-1} - 2 + 3\log 2 - 2\int_2^{\infty}\frac{\theta(t)-t}{t^2}\,dt.$$

Proof. Let $\delta = 1/(K+2)$ and set $\xi = (\log X)^{-1+\delta}$. The sum of the terms with $j \geqslant 2$ equals

$$-\frac{2}{L}\sum_{\substack{p^j\leqslant X^\xi\\ p>2\\ j\geqslant 2}} \frac{\log p}{p^j}\bigg(1+\frac{1}{p}\bigg)^{-1}\bigg(\sum_{k=0}^{K}\frac{\widehat{\phi}^{(k)}(0)}{k!}\bigg(\frac{2j\log p}{L}\bigg)^k + O_K\bigg(\bigg(\frac{\log p^{2j}}{L}\bigg)^{K+1}\bigg)\bigg) + O(e^{-(1/2)(\log X)^\delta})$$

$$= -\frac{2}{L}\sum_{k=0}^{K}\frac{\widehat{\phi}^{(k)}(0)}{k!L^k}\sum_{\substack{p^j\leqslant X^\xi\\ p>2\\ j\geqslant 2}}\frac{\log p(2j\log p)^k}{p^j}\bigg(1+\frac{1}{p}\bigg)^{-1} + O_K(L^{-K-2})$$

$$= -\frac{2}{L}\sum_{k=0}^{K}\frac{\widehat{\phi}^{(k)}(0)}{k!L^k}\sum_{\substack{p>2\\ j\geqslant 2}}\frac{\log p(2j\log p)^k}{p^j}\bigg(1+\frac{1}{p}\bigg)^{-1} + O_K(L^{-K-2}),$$

which is of the desired form.

As for the terms with $j=1$, we observe that the sum over $p \leqslant X^\xi$ is given by

$$-\frac{2}{L}\sum_{2<p\leqslant X^\xi}\frac{\log p}{p}\bigg(1+\frac{1}{p}\bigg)^{-1}\bigg(\sum_{k=0}^{K}\frac{\widehat{\phi}^{(k)}(0)}{k!}\bigg(\frac{2\log p}{L}\bigg)^k + O_K\bigg(\bigg(\frac{\log p}{L}\bigg)^{K+1}\bigg)\bigg)$$

$$= -\sum_{k=0}^{K}\frac{\widehat{\phi}^{(k)}(0)}{k!L^{k+1}}\sum_{2<p\leqslant X^\xi}\frac{(2\log p)^{k+1}}{p}\bigg(1+\frac{1}{p}\bigg)^{-1} + O_K(L^{(-1+\delta)(K+2)})$$

$$= -\sum_{k=0}^{K}\frac{\widehat{\phi}^{(k)}(0)}{k!L^{k+1}}\bigg(\frac{(2\xi\log X)^{k+1}}{k+1} + \ell_k + \sum_{n\geqslant 1}(-1)^n\sum_{p>2}\frac{(2\log p)^{k+1}}{p^{n+1}}\bigg) + O_K(L^{-K-1}),$$

where

$$\ell_k := (2\log 2)^k\bigg(2-\frac{k+3}{k+1}\log 2\bigg) - 2^{k+1}\int_2^\infty \frac{(\theta(t)-t)(\log t)^{k-1}(k-\log t)}{t^2}\,dt.$$

The terms with $p > X^\xi$ are handled by writing

$$-\frac{2}{L}\sum_{p>X^\xi}\frac{\log p}{p}\bigg(1+\frac{1}{p}\bigg)^{-1}\widehat{\phi}\bigg(\frac{2\log p}{L}\bigg) = -\frac{2}{L}\int_{X^\xi}^\infty \frac{1}{t}\widehat{\phi}\bigg(\frac{2\log t}{L}\bigg)\,d\theta(t) + O(X^{-\xi})$$

$$= \frac{2}{L}\widehat{\phi}\bigg(\frac{2\xi\log X}{L}\bigg) + \frac{2}{L}\int_{X^\xi}^\infty \bigg[\frac{1}{t}\widehat{\phi}\bigg(\frac{2\log t}{L}\bigg)\bigg]'(t+O(te^{-c\sqrt{\log t}}))\,dt + O(e^{-c(\log X)^{\delta/2}})$$

$$= \frac{2}{L}\widehat{\phi}\bigg(\frac{2\xi\log X}{L}\bigg) + \frac{2}{L}\int_{X^\xi}^\infty \bigg[\frac{1}{t}\widehat{\phi}\bigg(\frac{2\log t}{L}\bigg)\bigg]' t\,dt + O(e^{-(c/2)(\log X)^{\delta/2}})$$

$$= -\frac{2}{L}\int_{X^\xi}^\infty \frac{1}{t}\widehat{\phi}\bigg(\frac{2\log t}{L}\bigg)\,dt + O(e^{-(c/2)(\log X)^{\delta/2}})$$

$$= -\int_{2\xi\log X/L}^\infty \widehat{\phi}(u)\,du + O(e^{-(c/2)(\log X)^{\delta/2}})$$

$$= -\frac{1}{2}\phi(0) + \sum_{k=0}^{K}\frac{\widehat{\phi}^{(k)}(0)}{(k+1)!}\bigg(\frac{2\xi\log X}{L}\bigg)^{k+1} + O_K(L^{(-1+\delta)(K+2)}).$$

The result follows from combining the above estimates. Note in particular that the terms involving ξ cancel. $\square$

We are now ready to prove our main theorem.

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Proof of Theorem 1.1. Combining Theorem 3.5 and Lemmas 3.6 and 3.7, we obtain the formula

$$\mathcal{D}^*(\phi;X) = \widehat{\phi}(0) - \frac{\phi(0)}{2} + \int_1^\infty \widehat{\phi}(u)\,du + \frac{\widehat{\phi}(0)}{L}\bigg(\log(2e^{1-\gamma}) + \frac{2}{\widehat{w}(0)}\int_0^\infty w(x)(\log x)\,dx\bigg)$$
$$+ \frac{1}{L}\int_0^\infty \frac{e^{-x/2}+e^{-3x/2}}{1-e^{-2x}}\bigg(\widehat{\phi}(0) - \widehat{\phi}\bigg(\frac{x}{L}\bigg)\bigg)\,dx$$
$$+ \sum_{k=1}^{K} \frac{d_k\widehat{\phi}^{(k-1)}(0)}{L^k} + \sum_{k=1}^{K}\frac{c_{w,k}\widehat{\phi}^{(k-1)}(1)}{L^k} + O_K\bigg(\frac{1}{L^{K+1}}\bigg).$$

Writing the gamma factor as

$$\frac{1}{L}\int_0^\infty \frac{e^{-x/2}+e^{-3x/2}}{1-e^{-2x}}\bigg(\widehat{\phi}(0) - \widehat{\phi}\bigg(\frac{x}{L}\bigg)\bigg)\,dx$$
$$= -\sum_{j=1}^{K-1}\frac{\widehat{\phi}^{(j)}(0)}{j!L^{j+1}}\int_0^\infty \frac{x^j(e^{-x/2}+e^{-3x/2})}{1-e^{-2x}}\,dx + O_K\bigg(\frac{1}{L^{K+1}}\bigg),$$

the desired result clearly follows. □

4. The family $\mathcal{F}(X)$

4.1 Preliminaries

For convenience, we define the even smooth function $\widetilde{w} : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ by

$$\widetilde{w}(x) := \sum_{n\geqslant 1} w(n^2x). \tag{4.1}$$

It follows that $\widetilde{w}(x)$ decays rapidly as $x\to\infty$, and that its Mellin transform satisfies $\mathcal{M}\widetilde{w}(s) = \zeta(2s)\mathcal{M}w(s)$ (see [FPS16, Lemma 2.3]). Moreover, note that $\widetilde{w}(x)$ blows up near $x=0$. Applying the explicit formula, we now give an expression for $\mathcal{D}(\phi;X)$.

Lemma 4.1 (Explicit formula). *Assume that ϕ is an even Schwartz test function whose Fourier transform has compact support. Then the 1-level density defined in (1.6) is given by the formula*

$$\mathcal{D}(\phi;X) = \frac{\widehat{\phi}(0)}{LW(X)}\sideset{}{^*}\sum_{d\neq 0}\widetilde{w}\bigg(\frac{d}{X}\bigg)\log\bigg(\frac{|d|}{\pi}\bigg) - \frac{\widehat{\phi}(0)}{L}\bigg(\gamma + \log 4\bigg(1 - \frac{1}{W(X)}\sideset{}{^*}\sum_{\substack{d>0\\ d\ \mathrm{odd}}}\widetilde{w}\bigg(\frac{2d}{X}\bigg)\bigg)\bigg)$$
$$- \frac{2}{LW(X)}\sum_{p,m}\frac{\log p}{p^{m/2}}\widehat{\phi}\bigg(\frac{m\log p}{L}\bigg)\sideset{}{^*}\sum_{d\neq 0}\widetilde{w}\bigg(\frac{d}{X}\bigg)\bigg(\frac{d}{p^m}\bigg) + \frac{2}{W(X)}\widetilde{w}\bigg(\frac{1}{X}\bigg)\phi\bigg(\frac{iL}{4\pi}\bigg)$$
$$+ \frac{1}{L}\int_0^\infty \frac{e^{-x/2}+e^{-3x/2}}{1-e^{-2x}}\bigg(\widehat{\phi}(0) - \widehat{\phi}\bigg(\frac{x}{L}\bigg)\bigg)\,dx. \tag{4.2}$$

Proof. We first note that $D_X(\chi_{dm^2};\phi) = D_X(\chi_d;\phi)$ for any $m\geqslant 1$. Hence, by the definition of $\widetilde{w}$,

$$W(X)\mathcal{D}(\phi;X) = \sum_{d\neq 0} w\bigg(\frac{d}{X}\bigg)D_X(\chi_d;\phi) = \sideset{}{^*}\sum_{d\neq 0}\widetilde{w}\bigg(\frac{d}{X}\bigg)D_X(\chi_d;\phi). \tag{4.3}$$

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We also note that the conductor of χ_d for d squarefree is given by $4^{\mathfrak{b}}|d|$, where

$$\mathfrak{b} := \begin{cases} 0 & \text{if } d > 0 \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases}$$

As in the proof of Lemma 2.1, we apply [MV07, Theorem 12.13] and obtain the formula

$$D_X(\chi_d;\phi) = \frac{\widehat{\phi}(0)}{L}\left(\log\left(\frac{4^{\mathfrak{b}}|d|}{\pi}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{\mathfrak{a}}{2}\right)\right) - \frac{2}{L}\sum_{p,m}\frac{\chi_d(p^m)\log p}{p^{m/2}}\widehat{\phi}\left(\frac{m\log p}{L}\right)$$
$$+ 2\delta_{d=1}\phi\left(\frac{iL}{4\pi}\right) + \frac{2}{L}\int_0^\infty \frac{e^{-(1/2+\mathfrak{a})x}}{1-e^{-2x}}\left(\widehat{\phi}(0) - \widehat{\phi}\left(\frac{x}{L}\right)\right)dx.$$

Formula (4.2) then follows by summing over squarefree d against the weight function $\widetilde{w}$. □

We now give estimates on sums of the weight function $\widetilde{w}$. Recall that

$$W(X) = \sideset{}{^*}\sum_{d\neq 0}\widetilde{w}\left(\frac{d}{X}\right) = \sum_{d\neq 0} w\left(\frac{d}{X}\right)$$

is the total weight.

Lemma 4.2. *Fix $n \in \mathbb{N}$ and $\varepsilon > 0$. We have the estimates*

$$W(X) = X\widehat{w}(0) + O_{\varepsilon,w}(X^{\varepsilon});$$
$$\sideset{}{^*}\sum_{d\neq 0}\widetilde{w}\left(\frac{d}{X}\right)\left(\frac{d}{n}\right) = \kappa(n)X\widehat{w}(0)\prod_{p\mid n}\left(1+\frac{1}{p}\right)^{-1} + O_{\varepsilon,w}(|n|^{1/2-\kappa(n)/2+\varepsilon}X^{\varepsilon}),$$

where

$$\kappa(n) := \begin{cases} 1 & \text{if } n = \square, \\ 0 & \text{otherwise.} \end{cases}$$

Under the RH, we also have

$$\frac{1}{W(X)}\sideset{}{^*}\sum_{d\neq 0}\widetilde{w}\left(\frac{d}{X}\right)\log|d| = \log X + \frac{2}{\widehat{w}(0)}\int_0^\infty w(x)(\log x)\,dx + 2\frac{\zeta'(2)}{\zeta(2)} - \frac{\mathcal{M}w(\frac{1}{2})}{\mathcal{M}w(1)}\zeta\left(\frac{1}{2}\right)X^{-1/2}$$
$$+ O_{\varepsilon,w}(X^{-3/4+\varepsilon}).$$

Proof. The result follows exactly as in [FPS16, Lemmas 2.5 and 2.8]. □

4.2 Poisson summation

In this section we analyze

$$S_{\text{odd}} := -\frac{2}{LW(X)}\sum_{\substack{p \\ m \text{ odd}}}\frac{\log p}{p^{m/2}}\widehat{\phi}\left(\frac{m\log p}{L}\right)\sideset{}{^*}\sum_{d\neq 0}\widetilde{w}\left(\frac{d}{X}\right)\left(\frac{d}{p^m}\right) \tag{4.4}$$

using Poisson summation (see also § 2.2).[6]

[6] S_{even} is defined analogously.

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Lemma 4.3. *Assume that $\sigma = \sup(\operatorname{supp} \widehat{\phi}) < 1$, and let $m \in \mathbb{N}$ be such that $1/(2m+1) \leqslant \sigma < 1/(2m-1)$. Then, for any fixed $\varepsilon > 0$, we have the bound*

$$S_{\text{odd}} \ll_\varepsilon X^{-\max\{(4m-1)/(4m+1), 1-\sigma\}+\varepsilon}.$$

Furthermore, if $1 \leqslant \sigma < 2$, then under the GRH we have that

$$S_{\text{odd}} = -\frac{2X}{W(X)} \int_0^\infty \widehat{\phi}(u) \sum_{m \geqslant 1} \widehat{w}\bigg(\frac{m^2 X^{1-u}}{(2\pi e)^{-u}}\bigg)\, du + O_\varepsilon(X^{\sigma/2-1+\varepsilon}). \tag{4.5}$$

Proof. We proceed as in [FPS16, § 3]. Applying the identity

$$\sideset{}{^*}\sum_{d \neq 0} \widetilde{w}\bigg(\frac{d}{X}\bigg)\bigg(\frac{d}{p}\bigg) = \sum_{k \geqslant 0} \sum_{d \neq 0} w\bigg(\frac{d}{X/p^{2k}}\bigg)\bigg(\frac{d}{p}\bigg)$$

and arguing as in [FPS16, Lemmas 3.2 and 3.3], we see that

$$S_{\text{odd}} = -\frac{2X}{LW(X)} \sum_{0 \leqslant k \leqslant 10 \log X} \sum_{p > X^{(1-\varepsilon)/(2k+1)}} \frac{\overline{\epsilon_p} \log p}{p^{1+2k}} \widehat{\phi}\bigg(\frac{\log p}{L}\bigg) \sum_{t \in \mathbb{Z}} \bigg(\frac{-t}{p}\bigg) \widehat{w}\bigg(\frac{Xt}{p^{1+2k}}\bigg) + O_\varepsilon(X^{-1+\varepsilon}),$$

where

$$\epsilon_p = \begin{cases} 1 & \text{if } p \equiv 1 \bmod 4, \\ i & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

If $\sigma < 1$, then the proof is similar to that of [FPS16, Proposition 3.6]. As for the case $\sigma < 2$, we argue as in [FPS16, Lemma 3.9] and see that the terms with $k \geqslant 1$ are $O_\varepsilon(X^{-2/3+\varepsilon})$. Finally, in the terms with $k = 0$ we can add back the primes $p \leqslant X^{1-\varepsilon}$ at the cost of a negligible error term. The resulting sum is handled in a similar way to Lemma 2.7, and the estimate (4.5) follows. □

4.3 The new lower order terms

In this section we treat the new lower order terms that appear in the family $\mathcal{F}$.

Lemma 4.4. *Suppose that $\sigma = \sup(\operatorname{supp} \widehat{\phi}) < \infty$. Then we have the estimate*[7]

$$\begin{aligned}\int_0^\infty \widehat{\phi}(u) \sum_{m \geqslant 1} \widehat{w}\bigg(\frac{m^2 X^{1-u}}{(2\pi e)^{-u}}\bigg)\, du &= \frac{\widehat{h}(0)}{2} \int_1^\infty \bigg(\frac{X}{2\pi e}\bigg)^{(u-1)/2} \widehat{\phi}(u)\, du - \frac{\widehat{w}(0)}{2} \int_1^\infty \widehat{\phi}(u)\, du \\ &\quad + \frac{1}{L} \int_0^\infty \bigg(\widehat{\phi}\bigg(1 + \frac{\tau}{L}\bigg) e^{\tau/2} \sum_{n \geqslant 1} \widehat{h}(ne^{\tau/2}) + \widehat{\phi}\bigg(1 - \frac{\tau}{L}\bigg) \sum_{n \geqslant 1} h(ne^{\tau/2})\bigg)\, d\tau + O(X^{-1/2}),\end{aligned} \tag{4.6}$$

where $h(y) := \widehat{w}(2\pi e y^2)$.

Proof. The proof is similar to that of Lemma 3.2. □

We now give an estimate for the fourth term on the right-hand side of (4.2). For $1 \leqslant \sigma < 2$, we will see that the term that arose from principal characters in (4.2) will essentially cancel the main term in Lemma 4.4.

[7] Note that the first term on the right-hand side equals $X^{\sigma/2-1/2+o(1)}$.

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Lemma 4.5. *Fix $\epsilon > 0$. Then, for $\sigma < 1$, we have that*

$$\frac{2}{W(X)}\widetilde{w}\bigg(\frac{1}{X}\bigg)\phi\bigg(\frac{iL}{4\pi}\bigg)=\frac{1}{2\sqrt{2\pi e}\mathcal{M}w(1)}\bigg(\mathcal{M}w\bigg(\frac{1}{2}\bigg)-\frac{w(0)}{\sqrt{X}}\bigg)\int_{-1}^{1}\bigg(\frac{X}{2\pi e}\bigg)^{(u-1)/2}\widehat{\phi}(u)\,du+O_{\varepsilon}(X^{\varepsilon-1}).$$

As for $1 \leqslant \sigma < 2$, we have

$$\frac{2}{W(X)}\widetilde{w}\bigg(\frac{1}{X}\bigg)\phi\bigg(\frac{iL}{4\pi}\bigg)-\frac{\widehat{h}(0)X}{W(X)}\int_{1}^{\infty}\bigg(\frac{X}{2\pi e}\bigg)^{(u-1)/2}\widehat{\phi}(u)\,du$$
$$=\frac{\frac{1}{2}\mathcal{M}w(\frac{1}{2})}{\sqrt{2\pi e}\mathcal{M}w(1)}\int_{0}^{1}\bigg(\frac{X}{2\pi e}\bigg)^{(u-1)/2}\widehat{\phi}(u)\,du+O(X^{\sigma/2-1}).$$

Proof. First, an application of Poisson summation shows that, for $X \geqslant 1$ and arbitrary $N \geqslant 1$,

$$\widetilde{w}\bigg(\frac{1}{X}\bigg)=\frac{X^{1/2}}{2}\int_{\mathbb{R}}w(t^2)\,dt-\frac{w(0)}{2}+O_N(X^{-N}). \tag{4.7}$$

Moreover, we have that

$$\phi\bigg(\frac{iL}{4\pi}\bigg)=\int_{\mathbb{R}}\bigg(\frac{X}{2\pi e}\bigg)^{u/2}\widehat{\phi}(u)\,du.$$

By trivially bounding the integral on the interval $(-\infty, 0]$, it follows that for $1 \leqslant \sigma < 2$ we have

$$\frac{2}{W(X)}\widetilde{w}\bigg(\frac{1}{X}\bigg)\phi\bigg(\frac{iL}{4\pi}\bigg)=\frac{X\int_{\mathbb{R}}w(t^2)\,dt}{\sqrt{2\pi e}W(X)}\int_{0}^{\infty}\bigg(\frac{X}{2\pi e}\bigg)^{(u-1)/2}\widehat{\phi}(u)\,du+O(X^{\sigma/2-1}).$$

The last step is to apply the Fourier identity

$$\int_{\mathbb{R}}w(t^2)\,dt=\int_{\mathbb{R}}\widehat{w}(t^2)\,dt,$$

which follows from combining Plancherel's identity with the fact that $|x|^{-1/2}$ is its own Fourier transform. Finally, in the case $\sigma < 1$ we use a similar argument but we keep the secondary term in the expansion (4.7). The result follows. □

Proof of Theorem 1.3. The proof is obtained by combining Lemmas 4.1–4.3 and 4.5, with the expression for S_{even} analogous to (3.8). □

The rest of the section is devoted to the proof of Theorem 1.2.

Corollary 4.6. *Fix $\epsilon > 0$. Assume the GRH and suppose that $1 \leqslant \sigma = \sup(\operatorname{supp}\widehat{\phi}) < 2$. Then we have the estimate*

$$S_{\text{odd}}+\frac{2}{W(X)}\widetilde{w}\bigg(\frac{1}{X}\bigg)\phi\bigg(\frac{iL}{4\pi}\bigg)=\int_{1}^{\infty}\widehat{\phi}(u)\,du+U_2(X)+O_{\varepsilon}(X^{\sigma/2-1+\varepsilon}),$$

where

$$U_2(X):=\frac{\frac{1}{2}\mathcal{M}w(\frac{1}{2})}{\sqrt{2\pi e}\mathcal{M}w(1)}\int_{0}^{1}\bigg(\frac{X}{2\pi e}\bigg)^{(u-1)/2}\widehat{\phi}(u)\,du$$
$$-\frac{2}{L\widehat{w}(0)}\int_{0}^{\infty}\bigg(\widehat{\phi}\bigg(1+\frac{\tau}{L}\bigg)e^{\tau/2}\sum_{n\geqslant 1}\widehat{h}(ne^{\tau/2})+\widehat{\phi}\bigg(1-\frac{\tau}{L}\bigg)\sum_{n\geqslant 1}h(ne^{\tau/2})\bigg)\,d\tau.$$

Proof. The proof follows by combining Lemmas 4.3–4.5. □

In the final lemma we give an expansion of $U_2(X)$ in descending powers of $\log X$, which, by Theorem 1.2, shows that such an expansion is also possible for the 1-level density $\mathcal{D}(\phi;X)$.

Lemma 4.7. *For any $K \geqslant 1$, we have the expansion*

$$U_2(X) = \sum_{k=1}^{K} \frac{v_{w,k}\widehat{\phi}^{(k-1)}(1)}{L^k} + O_K(L^{-K-1}),$$

where the constants $v_{w,k}$ can be given explicitly. The first of these constants is given by

$$v_{w,1} = \frac{\mathcal{M}w(\frac{1}{2})}{\sqrt{2\pi e}\mathcal{M}w(1)} - \frac{4}{\widehat{w}(0)}\int_1^\infty \bigg(H_u\widehat{h}(u) + \frac{[u]}{u}h(u)\bigg)\,du,$$

where $H_u = \sum_{n\leqslant u} n^{-1}$ is the uth harmonic number.

Proof. The second term in $U_2(X)$ can be expanded exactly as in the proof of Lemma 3.6. As for the first term, we have

$$\begin{aligned}\int_0^1 \bigg(\frac{X}{2\pi e}\bigg)^{(u-1)/2}\widehat{\phi}(u)\,du &= \int_{1-L^{-1/2}}^1 \bigg(\frac{X}{2\pi e}\bigg)^{(u-1)/2}\bigg(\sum_{k=1}^{K}\widehat{\phi}^{(k-1)}(1)\frac{(u-1)^{k-1}}{(k-1)!} + O((u-1)^K)\bigg)\,du \\ &\quad + O(e^{-(1/2)\sqrt{L}}) \\ &= -\sum_{k=1}^{K}\frac{(-2)^k\widehat{\phi}^{(k-1)}(1)}{L^k} + O_K(L^{-K-1}),\end{aligned}$$

which completes the proof. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. The proof is achieved by combining Lemmas 4.1–4.3 and 4.5, with the expression for S_{even} analogous to (3.8) and Corollary 4.6. □

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