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Infinite limits in the iteration of entire functions

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Abstract. If f is a transcendental entire function and D is a non-wandering component of the set of normality of the iterates of f such that $f^n \to \infty$ in D then $\log |f^n(z)| = O(n)$ as $n \to \infty$ for z in D. For a wandering component the convergence of f^n to ∞ in D may be arbitrarily fast.

1. *Introduction*

Denote by f a non-linear entire function and set $N(f) = \{z; (f^n)$ is normal in some neighbourhood of $z\}$, where f^n, $n \in \mathbb{N}$, is the nth iterate of f. The complement of $N(f)$ in the plane is the Julia set $J(f)$. We recall that $J(f)$ is a non-empty perfect set and that $N(f)$ is completely invariant in the sense that z belongs to $N(f)$ if and only if $f(z)$ belongs to $N(f)$ [**6**]. Thus each component D of $N(f)$ is mapped by f into some (possibly different) component of $N(f)$: if each $f^n(D)$ belongs to a different component of $N(f)$ the component D is called 'wandering'; if this is not the case there are integers $m \geq 0$ and $p \geq 1$ such that $f^m(D) \subset D_1$ where D_1 is a component of $N(f)$ such that $f^p(D_1) \subset D_1$.

We wish to study the ways in which $f^n(z)$ may tend to ∞ for $z \in N(f)$ as $n \to \infty$. There are special classes of functions, for example those transcendental f such that all the singularities of f^{-1} lie over a finite number of points, for which this is impossible [**5**]. In general, however, $f^n(z)$ may approach ∞ either for z in a wandering component or for z such that, for some $m \geq 0$, $f^m(z)$ belongs to an unbounded periodic component D_1 of $N(f)$ in which the iterates have limit ∞.

In situations where f^n approaches a constant finite limit in a non-wandering component D the classical theory (see e.g. [**6**]) gives simple asymptotic formulae for $f^n(z)$ as $n \to \infty$. It is perhaps surprising that something can be said even for infinite limits.

THEOREM 1. *If f is a transcendental entire function, if z belongs to a non-wandering component of $N(f)$ and if $f^n(z) \to \infty$, then*

$$\log |f^n(z)| = O(n), \qquad (n \to \infty).$$

THEOREM 2. *If f is transcendental entire and D is a component of $N(f)$ such that $f(D) \subset D$ and $f^n \to \infty$ in D, then there exist a curve Γ, which tends to ∞ in D, and* *positive constants K, L such that* $f(\Gamma) \subset \Gamma$ *and*

$$K|z| < |f(z)| < L|z|, \qquad z \text{ in } \Gamma.$$

THEOREM 3. *If z belongs to a wandering component of* $N(f)$ *then* $|f^n(z)|$ *may approach* ∞ *arbitrarily fast.*

Theorem 1 does not hold for polynomials, where $O(n)$ becomes $O(K^n)$, for some $K>0$. The correctness of the orders of magnitude in Theorems 1 and 2 will be shown by a discussion of the example $f(z) = e^{-z} + kz$, $k > 1$.

If it is known that f has no wandering components, then Theorem 1 may be used to test whether certain points are in $J(f)$. It is also interesting that, according to a recent letter from A. Eremenko to the author, for any non-constant entire function f there exist points z (not necessarily belonging to $N(f)$) such that $f^n(z) \to \infty$.

Theorems 1 and 2 improve recent results in [**3**] and [**9**].

The paper is a more formal version of a lecture given at the Mathematical Institute of Academia Sinica, Beijing on 18th May 1987.

2. *Proof of Theorems* 1 *and* 2

It is sufficient to prove Theorem 1 for a component D of $N(f)$ which is invariant in the sense that $f(D) \subset D$. Since $f^n(z) \to \infty$ for some z in D it then follows that D is unbounded and that $f^n \to \infty$ locally uniformly in D. It was shown in [**2**] that any unbounded component of $N(f)$ is simply-connected. The desired result is thus contained in the following lemma.

LEMMA 1. *Suppose that D is an unbounded simply-connected plane domain and that g is a function which is analytic in D and such that* $g(D) \subset D$, $g^n \to \infty$ *in D. Then for any* $z \in D$ *we have*

$$\log |g^n(z)| = O(n) \qquad (n \to \infty).$$

Further, for any $z' \in D$

$$\log |g^n(z')| - \log |g^n(z)| = O(1) \qquad (n \to \infty).$$

Proof. Take a finite $\alpha \in \partial D$. Denote the hyperbolic distance of points z, z' of D with respect to D by $[z, z', D]$. Choose any point $z = z_0$ in D and put $z_n = g^n(z_0)$, $n \in \mathbb{N}$. Denote $H = [z_0, z_1, D]$ which is certainly positive since $g^n(z_0) \to \infty$. The map does not increase hyperbolic distance, so that $[z_n, z_{n+1}, D] \le H$ for all n.

Note that if ξ, η are in D and $[\xi, \eta, D] = \delta$, then $|\xi - \alpha| \le |\eta - \alpha| \cdot \exp(4\pi\delta)$. For the function $w = \log(z - \alpha)$ maps D onto a simply-connected domain G which contains no vertical segment of length $> 2\pi$. Since any w in G has distance at most π from ∂G, it follows from the Koebe distortion theorem as in e.g. [**8**, p. 6] that the Poincaré metric $\rho_G|dw|$ of G satisfies

$$\rho_G(w) \ge 1/\{4d(w, \partial G)\} \ge 1/(4\pi).$$

Thus if $\xi' = \log(\xi - \alpha)$, $\eta' = \log(\eta - \alpha)$, we have

$$\delta = [\xi, \eta, D] = [\xi', \eta', G] \ge (1/(4\pi)) \operatorname{Re}(\xi' - \eta'),$$

which gives $|\xi - \alpha| \le |\eta - \alpha| \exp(4\pi\delta)$.

In particular we have

$$|z_{n+1}-\alpha|<K|z_n-\alpha|,$$

where $K=\exp(4\pi H)$ and so $|z_n-\alpha|<K^n|z_0-\alpha|$, which implies $\log|z_n|=O(n)$ as $n\to\infty$.

Further if z' is another point of D and $z'_n=g^n(z')$ we put $\varepsilon=[z', z, D]\geq[z'_n, z_n, D]$ and obtain

$$L^{-1}<\left|\frac{z'_n-\alpha}{z_n-\alpha}\right|<L, \qquad L=\exp(4\pi\varepsilon),$$

which implies

$$\log|z'_n|-\log|z_n|=O(1) \qquad \text{as } n\to\infty.$$

The result of Theorem 2 is contained in the slightly more general lemma:

LEMMA 2. *If g and D satisfy the assumptions of Lemma 1, then there exists a curve Γ, which tends to ∞ in D, and positive constants K, L, such that $g(\Gamma)\subset\Gamma$ and*

$$K|z|<|g(z)|<L|z|, \qquad z \text{ in } \Gamma.$$

Proof. In the proof of the previous lemma we join z_0 to $z_1=g(z_0)$ by a path γ in D and change the constant H to $H'=\sup[z, z', D]$ for z, z' in γ. Then $\gamma_n=g^n\gamma$ joins z_n to z_{n+1} and $\Gamma=\bigcup\gamma_n$ is a path. For z in γ_n we have $[z, z_n, D]\leq H'$, $[z, z_{n+1}, D]\leq H'$ and hence, by the argument of the previous lemma

$$K^{-1}|z_n-\alpha|<|z-\alpha|<\min(K|z_n-\alpha|, K|z_{n+1}-\alpha|),$$

where $K=\exp(4\pi H')$. Thus $\Gamma\to\infty$. Moreover, for z in γ_n we have $g(z)$ in γ_{n+1}, so that

$$|g(z)-\alpha|\leq K|z_{n+1}-\alpha|\leq K^2|z-\alpha|$$

and similarly $|g(z)-\alpha|\geq K^{-2}|z-\alpha|$.

The proof is complete. □

COROLLARY. *If in Lemma 2 we replace the assumption that $g(D)\subset D$ by $g^p(D)\subset D$ for some $p\in\mathbb{N}$, then we obtain a path Γ on which $K|z|<|g^p(z)|<L|z|$. Assuming that g is an entire function we can at least say that $g(z)\to\infty$ on Γ.*

3. *The example $f(z)=e^{-z}+kz$, $k>1$*

For fixed $k>1$ there is an x_0 such that $kx-e^{-x}>x$ for $x\geq x_0$. If H denotes the half plane $\{\operatorname{Re} z>x_0\}$ we have $f(H)\subset H$. Moreover it is clear that for real $x>x_0$, $f^n(x)\to\infty$. Thus H is contained in a component D of $N(f)$ such that $f^n\to\infty$ in D, $f(D)\subset D$. Further, D must be simply-connected.

For real $x>x_0$ we have $x_n=f^n(x)>k^nx$ and $x_{n+1}-kx_n=\exp(-x_n)$. Thus, if $t_n=x_n/k^n$, then

$$t_{n+1}-t_n<\{\exp(-x_0k^n)\}/k^{n+1},$$

and so $\sum(t_{n+1}-t_n)<\infty$. Thus

$$\lambda=\lim x_nk^{-n}$$

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is finite and

$$\log(f^n(x)) = n \log k + O(1). \tag{1}$$

By the final statement of Lemma 1 (1) also holds if x is replaced by any z in D.

4. *Proof of Theorem* 3.

In **[1]** some entire f were constructed for which $N(f)$ has wandering components in which $f^n(z) \to \infty$ much faster than in Theorem 1. To obtain the result of Theorem 3 we modify a rather different argument in **[4]** to construct an example with the following properties.

Suppose that $10 < a_1 < a_2 < \cdots$, where a_n is a sequence with no other restriction except that $a_{n+1} - a_n > 4$, so that a_n may increase as fast as we please. Let $A_m = \{z : |z - a_m| \le 1\}$. Then there is an entire function g such that $g(A_m) \subset A_{m+1}$, and each A_m belongs to a different, simply-connected, wandering component of $N(g)$. A_{m+1} lies entirely to the right of $\{\mathrm{Re}\, z = \frac{1}{2}(a_m + a_{m+1})\}$.

First we recollect the following facts, also used in **[4]**. If F denotes a closed subset of $\mathbb{C}$ and $C_a(F)$ the functions which are continuous on F and analytic in $\mathring{F}$, then F is called a Carleman set (for $\mathbb{C}$) if, for any g in $C_a(F)$ and for any positive continuous function ε on F, there is an entire function f such that $|g(z) - f(z)| < \varepsilon(z)$, $z \in F$. By Arakelyan's theorem (e.g. [7, p. 137]) we have (i) $\hat{\mathbb{C}} \backslash F$ must be connected and also locally connected at ∞. If in addition to (i) we have (ii) for each compact K the union $W(K)$ of those components of $\mathring{F}$ which meet K is relatively compact in $\mathbb{C}$, then F is indeed a Carleman set ([7, p. 157]).

To construct the example introduce

$$L_m = \{z : \mathrm{Re}\, z = \tfrac{1}{2}(a_m + a_{m+1})\}, \quad m \in \mathbb{N},$$
$$B = \{z : |z + 6| \le 1\},$$

and let δ, δ_m be positive numbers so small that $|w - \pi i - \log 6| < \delta$ implies $|e^w + 6| < \frac{1}{2}$, and $|w - \log a_{m+1}| < \delta_m$ implies $|e^w - a_{m+1}| < \frac{1}{2}$.

Since the set $F = B \cup \{\bigcup_m (A_m \cup L_m)\}$ is a Carleman set there exists an entire function h such that

$$|h(z) - \pi i - \log 6| < \delta, \qquad z \in L_m,$$
$$|h(z) - \pi i - \log 6| < \delta, \qquad z \in B,$$
$$|h(z) - \log a_{m+1}| < \delta_{m+1}, \qquad z \in A_m.$$

Then $g = e^h$ is entire and satisfies $g(A_m) \subset A_{m+1}$, and so $g^n \to \infty$ in each A_m, uniformly. Thus $A_m \in N(g)$.

Now g maps B into $\{z; |z+6| < \frac{1}{2}\}$, so that B contains an attractive fixed point ξ and $g^n \to \xi$ in B. Further $g(L_m) \subset B$ so that $g^n \to \xi$ in L_m and each L_m belongs to a component of $N(g)$ different from the A_m. Thus the A_m are wandering components of $N(g)$, as is, indeed, apparent from the rate at which $g^n \to \infty$ in A_m.

Finally, since by construction $g^n \ne 0$, we have that the $1/g^n$ are entire functions which converge to the limit 0 in A_m and so in the whole component D_m of $N(g)$ to which A_m belongs. Hence D_m is simply-connected.

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