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FONG CHARACTERS AND CORRESPONDENCES IN π-SEPARABLE GROUPS

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1. **Introduction.** Let G and S be finite groups. Suppose that S acts on G with $(|G|,|S|) = 1$. If S is solvable, Glauberman showed the existence of a natural bijection from $\mathrm{Irr}_S(G) = \{\chi \in \mathrm{Irr}(G) \mid \chi^s = \chi \text{ for all } s \in S\}$ onto $\mathrm{Irr}(C)$, where $C = C_G(S)$. If S is not solvable, and consequently $|G|$ is odd, Isaacs also proved the existence of a natural bijection between the above set of characters. Finally, Wolf proved that both maps agreed when both were defined ([1], [3], [10]). As in [7], let us denote by $^*\colon \mathrm{Irr}_S(G) \longrightarrow \mathrm{Irr}(C)$ the Glauberman-Isaacs Correspondence.

Recently, there has been some interest in showing what kind of properties and which kind of characters the Glauberman-Isaacs Corresponence preserves.

For instance, we have proved in [7] that the correspondence "commutes" with character induction and restriction. Or in [12], Wolf has described how the Gajendragadkar's π-special characters or Isaacs' B_π characters are mapped under the correspondence.

A celebrated theorem of Fong asserts that in p-solvable groups the projective indecomposable characters are induced from characters of the p-complements. These characters are called Fong characters of G.

If we assume that S acts coprimely on a finite p-solvable group G and we choose H an S-invariant p-complement in G and $\alpha \in \mathrm{Irr}_S(H)$ a Fong character of G, then $H \cap C$ is a p-complement in C and it is natural to ask if α^* is a Fong character of C. Our aim in this paper is to give answer to this question.

We will show that if α induces indecomposably in G, then α^* induces indecomposably in C; i.e., S-invariant Fong characters of G are mapped by the Glauberman-Isaacs Correspondence into the Fong characters of C.

As was pointed out by the referee, it is interesting to speculate whether or not this map has to be onto. However, we have not been able to decide on this question.

In our context, we will also show that it is always possible to find S-invariant Fong characters: each S-invariant element in $\mathrm{IBr}(G)$ has and S-invariant character Fong associated to it.

By means of Isaacs' π-theory, this problem can be settled in the context of π-separable groups, and this is what we are going to do.

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Now assume that S acts coprimely on a π-separable group G, and let $I_\pi(G)$ be the set of Isaacs' π-Brauer characters. Wolf in [12], and Uno in [8] in the classical case $\pi = p'$, showed that the Glauberman-Isaacs Correspondence could be extended to π-Brauer characters for π-separable groups. This bijection from the set of S-invariant elements of $I_\pi(G)$ onto $I_\pi(C)$, which possesses the same properties as the ordinary one, will also be denoted by $*$ with the understanding that the image of a character φ is taken under the Glauberman-Isaacs or under Uno-Wolf Correspondence according if φ is an ordinary or a π-character of G.

If $\varphi \in I_\pi(G)$, let us write (as in [6]) Φ_φ for the *projective indecomposable character* afforded by φ.

Our main result can be stated as follows.

THEOREM A. *Let G be a finite π-separable group and let S act on G coprimely. Let H be an S-invariant Hall π-subgroup of G and $\alpha \in \mathrm{Irr}_S(H)$ such that $\alpha^G = \Phi_\varphi$, for some S-invariant $\varphi \in I_\pi(G)$. Then $(\alpha^*)^C = \Phi_{\varphi^*}$.*

When $\pi = p', I_\pi(G) = \mathrm{IBr}(G)$ and we obtain the result to which we have referred above.

As a corollary, we can prove an extension of our induction and restriction theorem in [7].

COROLLARY B. *Let G be a finite π-separable group and let S act on G coprimely. Let K be an S-invariant subgroup of G and let θ and φ be S-invariant characters of $I_\pi(K)$ and $I_\pi(G)$, respectively.*

(a) If $(\Phi_\theta)^G = \Phi_\varphi$, then $(\Phi_{\theta^*})^C = \Phi_{\varphi^*}$.

(b) If $(\Phi_\varphi)_K = \Phi_\theta$, then $(\Phi_{\varphi^*})_{K\cap C} = \Phi_{\theta^*}$.

2. **Some of π-character theory.** We take notation from [4], [5] and [6].

First of all, we review some of the definitions and results on Isaacs' π-theory that we are going to use.

Let π be a set of prime numbers and let G be a finite π-separable group. Let G° denote the set of π-elements of G. If χ is a class function on G, i.e., $\chi \in \mathrm{cf}(G)$, we write χ° for the restriction of χ to G°.

In [5], if $I_\pi(G) = \{\chi^\circ \mid \chi \in \mathrm{Irr}(G)$ and χ° is not of the form $\chi^\circ = \alpha^\circ + \beta^\circ$, for $\alpha, \beta \in \mathrm{Char}(G)\}$, Isaacs proved that $I_\pi(G)$ is a basis for $\mathrm{cf}(G^\circ)$, the space of class functions on G°. Moreover, if $\chi \in \mathrm{Irr}(G)$, then $\chi^\circ = \sum_\varphi d_{\chi\varphi}\varphi$ where φ runs over $I_\pi(G)$ and the $d_{\chi\varphi}$'s are some uniquely determined nonnegative integers *decomposition numbers*.

The way of proving that $I_\pi(G)$ is a basis for $\mathrm{cf}(G^\circ)$ was to construct for any π-separable group G a canonical set of irreducible complex characters $B_\pi(G)$ with the property that the map $\chi \longrightarrow \chi^\circ$ from $B_\pi(G)$ onto $I_\pi(G)$ is a bijection (10.2, [5]).

If $\varphi \in I_\pi(G)$, we write $\Phi_\varphi = \sum_{\chi\in\mathrm{Irr}(G)} d_{\chi\varphi}\chi$. Of course, when $\pi = p'$, then $I_\pi(G)$ is the set of irreducible Brauer characters of G and Φ_φ is the projective indecomposable character afforded by $\varphi \in \mathrm{IBr}(G)$.

If θ is a class function on G°, then θ is a linear combination of $I_\pi(G)$ and $\varphi \in I_\pi(G)$ is said to be an irreducible constituent of θ if φ occur with some nonzero coefficient.

Let us write $\mathrm{Char}_\pi(G)$ for the set of nonnegative linear combinations of elements of $I_\pi(G)$. If $\psi \in \mathrm{Char}_\pi(G)$ and $\varphi \in I_\pi(G)$, we will write $m(\varphi, \psi)$ for the multiplicity of φ in ψ. Also, observe that if H is a π-subgroup of G, then $\psi_H \in \mathrm{Char}(H)$.

The behaviour of I_π-characters with respect to normal subgroups is as good as possible. Recall that if K is a subgroup of G and $\mu \in I_\pi(K)$, $I_\pi(G|\mu)$ is the set of $\varphi \in I_\pi(G)$ such that μ is a constituent of φ_K. The definition of μ^G is done by using the usual formula for induced characters.

THEOREM 2.1. *Suppose G is π-separable with $N \lhd G$.*

(a) If $\varphi \in I_\pi(G)$, then the irreducible constituents of φ_N form an orbit under G and all have equal multiplicity dividing $|G : N|$.

(b) Let $\theta \in I_\pi(N)$ and write $T = I_G(\theta)$, the stabilizer. Then induction defines a bijection $I_\pi(T|\theta) \longrightarrow I_\pi(G|\theta)$. If $\mu^G = \varphi$ with $\mu \in I_\pi(T|\theta)$, then μ is a constituent of φ_T and the multiplicities of θ in μ_N and φ_N are equal.

PROOF. See Lemma 3.1 and Proposition 3.2 of [6].

The concept of Fong character plays an important role in π-character theory.

Let $H \subseteq G$ be a Hall π-subgroup of G and let $\varphi \in I_\pi(G)$. An irreducible constituent $\alpha \in \mathrm{Irr}(H)$ of φ_H is a Fong character associated with φ provided $\alpha(1) = \varphi(1)_\pi$.

THEOREM 2.2. *Suppose that G is π-separable and let $H \subseteq G$ be a Hall π-subgroup.*

(a) If $\varphi \in I_\pi(G)$, then the irreducible constituents of smallest degree of φ_H are precisely the Fong characters associated with φ.

(b) If $\alpha \in \mathrm{Irr}(H)$ is a Fong character associated with φ, then α is not a constituent of μ_H when $\varphi \neq \mu \in I_\pi(G)$.

(c) If α is Fong and associated with φ, then $\alpha^G = \Phi_\varphi$.

PROOF. This is Corollary 2.5 in [6].

Observe that condition (c) actually guarantess α to be a Fong character.

LEMMA 2.3. *Suppose that G is π-separable and let $H \subseteq G$ be a Hall π-subgroup. If $\alpha \in \mathrm{Char}(H)$ is such that $\alpha^G = \Phi_\varphi$, for some $\varphi \in I_\pi(G)$, then $\alpha \in \mathrm{Irr}(H)$ is a Fong character associated to φ.*

PROOF. Take $\psi \in B_\pi(G)$ with $\psi^\circ = \varphi$. Note that $\Phi_\varphi(1) = |G|_{\pi'}\varphi(1)_\pi$ by Theorem 2.2, and thus $\alpha(1) = \varphi(1)_\pi$. So all we need is to show that α is an irreducible constituent of φ_H. Now, $[\varphi_H, \alpha] = [\psi_H, \alpha] = [\psi, \alpha^G] = [\psi, \Phi_\varphi] = d_{\psi\varphi} = 1$. If β is an irreducible constituent of φ_H and α, then $\varphi(1)_\pi \leq \beta(1) \leq \alpha(1) = \varphi(1)_\pi$. This proves the lemma.

In the case $\pi = p'$, the next lemma is due to Willems and it appears as Proposition 2.7(b) of [9].

LEMMA 2.4. *Let G be a π-separable group and $K \subseteq G$. Let $\tau \in I_\pi(K)$. If $\tau^G \in I_\pi(G)$, then $(\Phi_\tau)^G = \Phi_{\tau^G}$.*

PROOF. First we claim that τ is a constituent of $(\tau^G)_K$. Let $\psi \in \mathrm{Irr}(K)$ such that $\psi^\circ = \tau$ and write $\chi = \psi^G$. By Frobenius reciprocity, $\chi_K = \psi + \Delta$, where $\Delta \in \mathrm{Char}(K)$.

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Note that $\chi^\circ = (\psi^G)^\circ = (\psi^\circ)^G = \tau^G$. Thus $(\tau^G)_K = (\chi^\circ)_K = (\chi_K)^\circ = \psi^\circ + \Delta^\circ = \tau + \Delta^\circ$, as claimed.

Now, choose H a Hall π-subgroup of G such that $H \cap K$ is a Hall π-subgroup of K and let $\alpha \in \mathrm{Irr}(H \cap K)$ a Fong character of K associated with τ. Since α is under τ, by the claim, α is under τ^G. Thus, let $\beta \in \mathrm{Irr}(H)$ over α and under τ^G.

Then $\beta(1) \geq \left(\tau^G(1)\right)_\pi = |G : K|_\pi \tau(1)_\pi = |H : H \cap K| \alpha(1)$. Since β is an irreducible constituent of α^H, we have that $\beta = \alpha^H \in \mathrm{Irr}(H)$ is a Fong character associated with τ^G. Consequently, $(\Phi_\tau)^G = \alpha^G = (\alpha^H)^G = \Phi_{\tau^G}$.

3. **Preliminaries.** Let us state our hypotheses.

HYPOTHESIS 3.1. *Let G and S be finite groups such that S acts on G with $(|G|, |S|) = 1$. Let $C = C_G(S)$ and $\Gamma = GS$ the semidirect product.*

A key tool for proving our main result and its corollary is the following due to Isaacs and the author which appears as Theorem A in [7].

THEOREM 3.2. *Assume 3.1. Let H be an S-invariant subgroup of G, $\theta \in \mathrm{Irr}_S(H)$ and $\chi \in \mathrm{Irr}_S(G)$.*

(a) If $\theta^G \in \mathrm{Irr}_S(G)$, then $(\theta^*)^C = (\theta^G)^*$.

(b) If $\chi_H \in \mathrm{Irr}_S(H)$, then $(\chi_H)^* = (\chi^*)_{C \cap H}$.

For our purpose, a slight generalization of Proposition 3.4 of [6] is needed.

PROPOSITION 3.3. *Assume 3.1 with G π-separable. Let $\varphi \in I_\pi(G)$ which has not π-degree. Then there exists $N \lhd \Gamma$ contained in G such that the irreducible constituents of φ_N have π-degree and are not invariant in G.*

PROOF. The same as 3.4 of [6].

LEMMA 3.4. *Assume 3.1, with G π-separable. Then there exists an S-invariant Hall π-subgroup of G and all of them are C-conjugate. Each S-invariant π-subgroup of G is contained in an S-invariant Hall π-subgroup of G. If H is any of them, then $H \cap C$ is a Hall π-subgroup of C.*

PROOF. Is a direct consequence of Glauberman's lemma (13.8, 13.9 of [4]) and Hall-Cunihin theorem on π-separable groups (VI. Haupsatz 1.7 of [2]).

Now, let us prove a more or less well-known lemma. It follows easily from Glauberman's lemma.

LEMMA 3.5. *Assume 3.1. Let H and K two S-invariants subgroups of G with $HK = G$. The $(C \cap H)(C \cap K) = C$.*

PROOF. If $c \in C$, apply Glauberman's lemma to the nonempty set $\Omega = \{(h, k) \in H \times K \mid hk = c\}$, and the groups S and $H \cap K$, with the actions $(h, k)^s = (h^s, k^s)$ for $s \in S$ and $(h, k)^x = (hx, x^{-1}k)$ for $x \in H \cap K$.

Finally, we need the analogue of Lemma 2.15 of [11] for the Uno-Wolf Correspondence; i.e., to show that the modular map preserves the Clifford Correspondence.

Recall that if $\varphi \in I_\pi(G)$ is S-invariant, φ^* is defined as follows: take ψ the unique S-invariant character in $B_\pi(G)$ such that $\psi^\circ = \varphi$ and let $\varphi^* = (\psi^*)^\circ$ (see page 934 in [12]).

One can think that to prove briefly the lemma it suffices to work with the B_π-lifting and apply the known result for ordinary characters. This is not true, however, when the order of the group is even. The reason for this is the anomalous behaviour of the B_π-characters with respect to irreducible induction and restriction (M. Isaacs has proved in an unpublished result that the set B_π is closed under irreducible inductions and restrictions when the group is of odd order; this result fails when $|G|$ is even).

When S is a p-group our proof will run parallel to that of Wolf.

For ordinary characters, next lemma appears in [7].

LEMMA 3.6. *Assume 3.1 with G π-separable and S a p-group. Let $H \subseteq G$ be S-invariant and let $\chi \in I_\pi(G)$ and $\theta \in I_\pi(H)$ be S-invariant. Then p divides $m(\theta, \chi_H)$ if and only if p divides $m(\theta^*, \chi^*_{C\cap H})$.*

PROOF. We can write $\chi_C = m(\chi^*, \chi_C)\chi^* + p\Delta$, where $\Delta \in \mathrm{Char}_\pi(C)$, and $\theta_{C\cap H} = m(\theta^*, \theta_{C\cap H})\theta^* + pE$, where $E \in \mathrm{Char}_\pi(C \cap H)$. Also, it is clear that we can write $\chi_H = m(\theta, \chi_H)\theta + B + X$, where all irreducible constituents of B are S-invariant and all irreducible constituents of X lie in nontrivial S-orbits.

Thus $m(\theta^*, \chi_{H\cap C}) \equiv m(\theta^*, \theta_{C\cap H})m(\theta, \chi_H) \bmod p$.

Also, $m(\theta^*, \chi_{H\cap C}) \equiv m(\chi^*, \chi_C)m(\theta^*, \chi^*_{C\cap H}) \bmod p$.

Since $m(\theta^*, \theta_{C\cap H})$ and $m(\chi^*, \chi_C)$ are not divisible by p, the result follows.

Two consequences of this result will be needed in the sequel. First, observe that always χ_H has an irreducible S-invariant constituent with multiplicity prime to p, say μ (otherwise p would divide $\chi(1)$). If C is contained in H, by the lemma above, $\mu^* = \chi^*$ (and consequently μ is unique).

Also, note that if H is a normal subgroup fo G, since $m(\theta, \chi_H)$ divides the order of G, the lemma actually says that $m(\theta, \chi_H) = 0$ if and only if $m(\theta^*, \chi^*_{C\cap H}) = 0$.

LEMMA 3.7. *Assume 3.1, with G π-separable. Let N be a normal subgroup of Γ contained in G. Let $\theta \in I_\pi(N)$ S-invariant and $T = I_G(\theta)$. Then $(\psi^G)^* = (\psi^*)^C$ for S-invariant $\psi \in I_\pi(T|\theta)$.*

PROOF. First assume that $|S|$ is odd. Then S is solvable and by Theorem 4.5 of [12], we may assume, by induction, that $|S| = p$, a prime number. Suppose $NC = G$. In this case, it is clear that $T \cap C = I_C(\theta^*)$. We may write $\psi_{T\cap C} = m(\psi^*, \psi_{T\cap C})\psi^* + pE$, where $E \in \mathrm{Char}_\pi(T \cap C)$. Note that $\psi^* \in I_\pi\big(I_C(\theta^*)|\theta^*\big)$ and consequently $\psi^{*^C} \in I_\pi(C)$.

Now, $(\psi^G)_C = (\psi^{TC})_C = (\psi_{T\cap C})^C = m(\psi^*, \psi_{T\cap C})\psi^{*^C} + pE^C$. Therefore, $(\psi^G)^* = (\psi^*)^C$.

Consequently, we may assume that $NC < G$. Let $K/N = [G/N, S]$ and choose $N \subseteq L \subseteq K$ where K/L is a chief factor of Γ. Let $H = LC < G$ and let β be the unique irreducible S-invariant constituent of $(\psi^G)_H$ with multiplicity prime to p. We know that $\beta^* = (\psi^G)^*$.

Since $(\psi^G)^*$ is over θ^*, then β is over θ. Thus, by Theorem 2.1 we can find $\gamma \in I_\pi(T \cap H|\theta)$ such that $\gamma^H = \beta$. Observe that γ is S-invariant.

We claim that γ is the unique irreducible S-invariant constituent of $\psi_{T\cap H}$ with multiplicity prime to p. If the claim is true, then $\gamma^* = \psi^*$ and by induction the result will follow. Let $\chi = \psi^G$.

By Theorem 2.1, we may write $\chi_T = \psi + \Pi$ where $\Pi \in \mathrm{Ch}_\pi(T)$ with $m(\psi, \Pi) = 0$ and $m(\theta, \Pi_N) = 0$. Therefore, $m(\gamma, \Pi_{T\cap H}) = 0$ and $m(\gamma, \chi_{T\cap H}) = m(\gamma, \psi_{T\cap H})$.

Write $\chi_H = m(\beta, \chi_H)\beta + A$, where $A \in \mathrm{Char}_\pi(H)$.

Next, we show that if $\kappa \in I_\pi(H|\gamma)$, then $\kappa = \gamma^H = \beta$. This is true because if κ is over γ, then κ is over θ. If δ is its Clifford correspondent, then by Theorem 2.1, the multiplicity of θ in κ_N and δ_N is the same. This forces $\gamma = \delta$ and consequently, $\kappa = \beta$.

Since $m(\beta, A) = 0$, by the preceding comment it follows $m(\gamma, A_{H\cap T}) = 0$.

Write $\beta_{H\cap T} = \gamma + B$, with $m(\gamma, B) = 0$. Then $\chi_{H\cap T} = m(\beta, \chi_H)\gamma + X$ con $m(\gamma, X) = 0$. Thus, $m(\gamma, \psi_{H\cap T}) = m(\gamma, \chi_{H\cap T}) = m(\beta, \chi_H)$ which is not divisible by p.

Suppose now that $|G|$ is odd and let $\alpha \in B_\pi(G)$, $\eta \in B_\pi(T)$ and $\varepsilon \in B_\pi(N)$ such that $\alpha^\circ = \chi$, $\eta^\circ = \psi$, and $\varepsilon^\circ = \theta$. We show that $\alpha = \eta^G$. First observe that $(\eta^G)^\circ \in I_\pi(G)$, since $(\eta^G)^\circ = (\eta^\circ)^G = \psi^G$. By Theorems 12.1 and 12.3 of [5], we have that $\eta^G \in B_\pi(G)$ and lifts ψ^G. Thus, by unicity, $\alpha = \eta^G$.

Since ψ is over θ, by Corollary 7.5 of [5], we have that $\eta \in \mathrm{Irr}(I_G(\varepsilon)|\varepsilon)$.

Now, $\psi^{G*} = (\eta^{G*})^\circ = (\eta^{*C})^\circ = (\eta^{*\circ})^C = (\psi^*)^C$.

4. **Proofs.** In the proof of our main result, we use repeatedly a fact which appears as Corollary 2.6 of [6]. If H is a Hall π-subgroup of the π-separable group G, then restriction defines an injection between $\{\varphi \in I_\pi(G) | \varphi(1) \text{ is a } \pi\text{-number}\}$ and $\mathrm{Irr}(H)$.

THEOREM A. *Let G be a finite π-separable group and let S act on G coprimely. Let H be an S-invariant Hall π-subgroup of G and $\alpha \in \mathrm{Irr}_S(H)$ such that $\alpha^G = \Phi_\varphi$, for some S-invariant $\varphi \in I_\pi(G)$. Then $(\alpha^*)^C = \Phi_{\varphi^*}$.*

PROOF. Induction on $|G|$.

Suppose first that $\varphi(1)$ is not a π-number. By Proposition 3.3, let N be a normal S-invariant subgroup of G such that the irreducible constituents of φ_N have π-degree and are not G-invariant.

By Glauberman's Lemma and Theorem 2.1, we may choose η be an S-invariant irreducible constituent of $\alpha_{N\cap H}$. Since α lies under φ, η lies under φ. Thus, we may choose $\theta \in I_\pi(N)$ under φ and over η.

Since θ has π-degree, by Corollary 2.6 of [6], we have that $\theta_{N\cap H} = \eta$. Moreover, by the same result, θ is S-invariant and $I_G(\theta) \cap H = I_H(\eta)$. Write $T = I_G(\theta)$.

Let $\beta \in \mathrm{Irr}_S(T \cap H|\eta)$ such that $\beta^H = \alpha$ and let J be an S-invariant Hall π-subgroup of T containing $T \cap H$.

Let $\mu \in I_\pi(T)$ an irreducible constituent of φ_T which lies over β. Then μ lies over η and some irreducible constituent of μ_N lies over η. By Corollary 2.6 of [6], necessarily it has to be θ. Thus, $\mu \in I_\pi(T|\theta)$ and $\mu^G = \varphi$. Note also that μ is S-invariant.

Now, let $\tau \in \mathrm{Irr}(J)$ an irreducible constituent of μ_J which lies over β. We claim that $\tau = \beta^J$.

To see this, observe that $\tau(1) \geq \mu(1)_\pi = \varphi(1)_\pi / |G{:}\,T|_\pi = \alpha(1)/|G{:}\,T|_\pi = (|H{:}\,T \cap H|\beta(1))/|G{:}\,T|_\pi = |J{:}\,T \cap H|\beta(1) = \beta^J(1)$.

Since τ is an irreducible constituent of β^J, we have equality throughout and thus the claim is proved. Also note that τ is an irreducible constituent of μ_J with $\tau(1) = \mu(1)_\pi$ and consequently, τ is an S-invariant Fong character of T associated with μ.

By induction, we have $(\tau^*)^{T\cap C} = \Phi_{\mu^*}$ and by Theorem 3.2.a, $(\beta^*)^{J\cap C} = \tau^*$.

By Lemma 3.7, observe that $(\mu^*)^C = \varphi^*$ and by Lemma 2.4, $(\Phi_{\mu^*})^C = \Phi_{\varphi^*}$. Also, since $\beta^H = \alpha$, we have $(\beta^*)^{H\cap C} = \alpha^*$.

Now, $(\alpha^*)^C = (\beta^*)^C = (\tau^*)^C = (\Phi_{\mu^*})^C = \Phi_{\varphi^*}$.

Thus, we may assume that $\varphi(1)$ is a π-number. Note that $\varphi_H = \alpha$.

Let $\chi \in B_\pi(G)$ such that $\chi^\circ = \varphi$. By Lemma 5.4 of [5], note that χ is a π-special character. Consequently, by Theorem 3.9 of [12], $(\chi_H)^* = (\chi^*)_{H\cap C}$.

Now, $(\varphi^*)_{H\cap C} = \left((\chi^*)^\circ\right)_{H\cap C} = \left((\chi^*)_{H\cap C}\right)^\circ = (\chi^*)_{H\cap C} = (\chi_H)^* = (\varphi_H)^* = \alpha^*$.

Then α^* is a Fong character of C associated with φ^* and by Theorem 2.2.c, $(\alpha^*)^C = \Phi_{\varphi^*}$.

Before proving Corollary B, we would like to show that conditions in our Theorem A hold every time we consider S-invariant elements in $I_\pi(G)$.

THEOREM 4.1. *Assume hypothesis 3.1 with G π-separable. Let $\varphi \in I_\pi(G)$ be S-invariant, and let H be an S-invariant Hall π-subgroup of G. Then there exists $\alpha \in$* $\mathrm{Irr}_S(H)$ *such that $\alpha^G = \Phi_\varphi$.*

PROOF. We may assume that $\varphi(1)$ is not a π-number, because otherwise $\varphi_H \in \mathrm{Irr}_S(H)$ and $(\varphi_H)^G = \Phi_\varphi$.

By Proposition 3.3, let N be a normal S-invariant subgroup of G such that the irreducible constituents of φ_N have π-degree and are not G-invariant. Choose θ an S-invariant irreducible constituent of φ_N such that, if $T = I_G(\theta)$, then $T \cap H$ is a Hall π-subgroup of T.

Let $\tau \in I_\pi(T|\theta)$ such that $\tau^G = \varphi$ and note that τ is S-invariant. By induction, $\Phi_\tau = \gamma^T$, where $\gamma \in \mathrm{Irr}_S(T \cap H)$. Since $\gamma^G = \Phi_\varphi$, by Lemma 2.3 we have that $\gamma^H \in \mathrm{Irr}_S(H)$ is a Fong character of G associated with φ and the theorem is proved.

COROLLARY B. *Let G be a finite π-separable group and let S act on G coprimely. Let K be an S-invariant subgroup of G and let θ and φ be S-invariant characters of $I_\pi(K)$ and $I_\pi(G)$, respectively.*

(a) If $(\Phi_\theta)^G = \Phi_\varphi$, then $(\Phi_{\theta^*})^C = \Phi_{\varphi^*}$.

(b) If $(\Phi_\varphi)_K = \Phi_\theta$, then $(\Phi_{\varphi^*})_{K\cap C} = \Phi_{\theta^*}$.

PROOF. Choose H an S-invariant Hall π-subgroup of G such that $H \cap K$ is a Hall π-subgroup of K.

(a) By Theorem 4.1, let $\alpha \in \mathrm{Irr}_S(H \cap K)$ a Fong character associated with θ. Then $\alpha^G = \Phi_\varphi$, and thus $\beta = \alpha^H \in \mathrm{Irr}_S(H)$ is a Fong character associated with φ.

Therefore, by Theorem A, $(\alpha^*)^{K\cap C} = \Phi_{\theta^*}$ and $(\beta^*)^C = \Phi_{\varphi^*}$.

By Theorem 3.2.a, $(\alpha^*)^{H\cap C} = \beta^*$. Then, $(\Phi_{\theta^*})^C = (\alpha^*)^C = (\beta^*)^C = \Phi_{\varphi^*}$.

(b) Since $(\Phi_\varphi)_K = \Phi_\theta$, we have that $|G|_{\pi'} = |K|_{\pi'}$. Therefore, $|G{:}\, K|$ is a π-number and consequently $HK = G$. This implies, by Lemma 3.5, that $(C\cap H)(C\cap K) = C$.

Let $\beta \in \mathrm{Irr}_S(H)$ a Fong character of G associated with φ. Then $\Phi_\theta = (\beta^G)_K = (\beta_{H\cap K})^K$, and $\beta_{H\cap K} \in \mathrm{Irr}_S(H\cap K)$ is a Fong character associated with θ.

By Theorem A, $(\beta^*)^C = \Phi_{\varphi^*}$ and $\left((\beta_{H\cap K})^*\right)^{K\cap C} = \Phi_{\theta^*}$.

By Theorem 3.2.b, $(\beta^*)_{K\cap H\cap C} = (\beta_{H\cap K})^*$. Thus, $(\Phi_{\varphi^*})_{K\cap C} = \left((\beta^*)^C\right)_{K\cap C} = \left((\beta^*)^{(C\cap H)(C\cap K)}\right)_{K\cap C} = \left((\beta^*)_{K\cap H\cap C}\right)^{K\cap C} = \left((\beta_{H\cap K})^*\right)^{K\cap C} = \Phi_{\theta^*}$.

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