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On Valuations, Places and Graded Rings Associated to $*$-Orderings

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Abstract. We study natural $*$-valuations, $*$-places and graded $*$-rings associated with $*$-ordered rings. We prove that the natural $*$-valuation is always quasi-Ore and is even quasi-commutative (*i.e.,* the corresponding graded $*$-ring is commutative), provided the ring contains an imaginary unit. Furthermore, it is proved that the graded $*$-ring is isomorphic to a twisted semigroup algebra. Our results are applied to answer a question of Cimprič regarding $*$-orderability of quantum groups.

1 Introduction

The notion of a $*$-ordering on a division $*$-ring was introduced by Holland [Ho2] as an analogue to the notion of a total ordering. This theory was developed further by several authors, *e.g.,* by Craven, Chacron [Ch] and Marshall. Marshall [Ma1, Ma2] and Craven–Smith [CS] also extended the theory to $*$-rings and in particular to $*$-domains. Major tools in this theory are valuations and graded rings. To each $*$-ordering of a domain we can associate a natural $*$-valuation and a graded $*$-ring [Ma1, Ma2]. In order to study these objects, we introduce $*$-places motivated by the notion of real places associated with total orderings as introduced and studied by Marshall–Zhang [MZ1]. We show that the natural $*$-valuation v associated with a $*$-ordered domain A is quasi-Ore (for the definition see §2). Furthermore, if A contains a central skew element i satisfying $i^2 = -1$ (we call such an element an *imaginary unit*), then v is quasi-commutative, *i.e.,* the corresponding graded $*$-ring $\operatorname{gr}(A, v)$ is commutative. If A is a $\mathbb{C}$-algebra, this result can be further improved. In this case it is shown that $\operatorname{gr}(A, v)$ is isomorphic to a twisted semigroup ring $\mathbb{C}[\Gamma, c]$ for an ordered cancellative abelian semigroup Γ and a positive symmetric factor set $c \colon \Gamma \times \Gamma \to \mathbb{R}$.

These results are used in the last section to answer a question posed by Cimprič [Ci, §6]. We show that noncommutative quantum affine spaces and quantum Weyl fields do not admit $*$-orderings (independent of the involution chosen).

2 Basic Definitions and Preliminary Results

Throughout this paper A will denote a domain with involution and $\operatorname{Sym} A$ will be the set of its symmetric elements. A subset $P \subseteq A$ is called a $*$-*ordering* provided the

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following hold:

(O_1) $1 \in P$, $P + P \subseteq P$,
(O_2) $rPr^* \subseteq P$ for all $r \in A$,
(O_3) $P \cup -P = \operatorname{Sym} A$,
(O_4) $P \cap -P = \{0\}$,
(O_5) $a, b \in P \ \Rightarrow\ \{a, b\} := ab + ba \in P$.

If Γ is an ordered cancellative abelian semigroup, then an onto mapping $v\colon A \to \Gamma \cup \{\infty\}$ is a $*$-*valuation* if:

(V_1) $v(x) = \infty$ iff $x = 0$,
(V_2) $v(xy) = v(x) + v(y)$ for all $x, y \in A^\times$,
(V_3) $v(x + y) \geqslant \min\{v(x), v(y)\}$ for all $x, y \in A$,
(V_4) $v(x^*) = v(x)$ for all $x \in A$.

Here $A^\times := A \setminus \{0\}$. The *corresponding graded* $*$-*ring* $\operatorname{gr}(A, v)$ is constructed as follows. We form $A_\alpha := \{x \in A \mid v(x) \geqslant \alpha\}$, $A_\alpha^+ := \{x \in A \mid v(x) > \alpha\}$ and $\overline{A}_\alpha := A_\alpha/A_\alpha^+$. Then $\operatorname{gr}(A, v) := \bigoplus_{\alpha\in\Gamma} \overline{A}_\alpha$ is given the componentwise addition, the multiplication induced by $(\overline{a}, \overline{b}) \mapsto \overline{ab}$ for $a \in A_\alpha$ and $b \in A_\beta$ and the involution defined by $\overline{a}^* := \overline{a^*}$. Then v induces a $*$-valuation $\operatorname{gr}(v)\colon \operatorname{gr}(A, v) \to \Gamma \cup \{\infty\}$ given by $\operatorname{gr}(v)(\sum_{\alpha\in\Gamma} \overline{a}_\alpha) = \gamma$, where γ is the least $\gamma \in \Gamma$ such that $\overline{a}_\gamma \neq 0$ if $\sum_{\alpha\in\Gamma} \overline{a}_\alpha \neq 0$.

We define a relation $\sim_v$ on $A^\times$ by $x \sim_v y \Leftrightarrow v(x) < v(x - y)$. This is a semigroup congruence and is $*$-invariant (*i.e.*, $x \sim_v y$ implies $x^* \sim_v y^*$). For details we refer the reader to [Ho1].

Definition Let $v\colon A \to \Gamma \cup \{\infty\}$ be a $*$-valuation.

(i) v is *compatible* with a $*$-ordering P of A iff $x \sim_v y \in P$ implies $x \in P$ for all $x, y \in \operatorname{Sym} A^\times$.
(ii) v is called *quasi-commutative* iff for all $a, b \in A^\times$ we have $ab \sim_v ba$. Obviously, v is quasi-commutative iff $v(ab - ba) > v(ab)$ for all $a, b \in A^\times$ and this is the case iff $\operatorname{gr}(A, v)$ is commutative.
(iii) v is *quasi-Ore* iff for all $a, b \in A^\times$ there exist $r, s \in A^\times$ such that $ra \sim_v sb$. Note that this condition is left-right symmetric by the properties of $\sim_v$.
(iv) If Γ is a subsemigroup of $\mathbb{Z}$, then v is called *discrete*.

Remark Clearly, if $\operatorname{gr}(A, v)$ is an Ore domain, then v is quasi-Ore. The converse is false in general, but it holds in special cases, *e.g.*, if v is discrete [Co, Theorem 4.2].

If $v\colon A \to \Gamma \cup \{\infty\}$ is a $*$-valuation, we write $\mathcal{O}_v := A_0$ and $\mathfrak{m}_v := A_0^+$. If A is a division $*$-ring, then $\mathcal{O}_v$ is an *invariant valuation* $*$-*ring* and $\mathfrak{m}_v$ is its maximal $*$-ideal. In general, $\mathfrak{m}_v$ is only a completely prime $*$-ideal of $\mathcal{O}_v$. Hence the *residue* $*$-*ring* $k_v := \mathcal{O}_v/\mathfrak{m}_v$ is only a domain and not necessarily a division ring.

To each $*$-ordering $P \subseteq A$ a *natural* (*order-compatible*) $*$-*valuation* v_P can be associated as follows. The $*$-ordering P gives an order relation $\leqslant$ on $\operatorname{Sym} A$, which induces the archimedean equivalence $\approx$ on $\operatorname{Sym} A$. We extend the latter to the whole A by declaring, for all $a, b \in A^\times$, that $a \prec b$ if $aa^* \leqslant nbb^*$ for some positive integer n, and $a \approx b$ if $a \prec b$ and $b \prec a$. Denote by $v_P(a)$ the equivalence class of $a \in A^\times$ and $v_P(0) := \infty$. Then the relation $\prec$ induces a total ordering of the set $\Gamma_P = v_P(A^\times)$.

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By [Ma1, Theorem 3.3], the binary operation $v_P(a) + v_P(b) := v_P(ab)$ is well defined on Γ_P, so Γ_P becomes an ordered cancellative abelian semigroup. Marshall [Ma1] observed that v_P is a $*$-valuation and $s_1 s_2 \sim_v s_2 s_1$ for all $s_1, s_2 \in \operatorname{Sym} A^\times$. We say that v is *quasi-commutative for symmetric elements*.

3 $*$-Places and Graded $*$-Rings

Proposition 1 *Assume $v\colon A \to \Gamma \cup \{\infty\}$ is a $*$-valuation quasi-commutative for symmetric elements with $v(2) = 0$. Then v is quasi-Ore and symmetric elements of $\operatorname{gr}(A, v)$ commute. If, furthermore, v is discrete, then $\operatorname{gr}(A, v)$ is an Ore domain.*

Proof Since v is quasi-commutative for symmetric elements, we have $v(s_1 s_2 - s_2 s_1) > v(s_1 s_2)$ for all $s_1, s_2 \in \operatorname{Sym} A^\times$. Now let $a, b \in A^\times$ be arbitrary. Define $r_1 := a^* b b^*$ and $r_2 := b^* a a^*$. Then

$$\begin{aligned} v(ar_1 - br_2) &= v(aa^*bb^* - bb^*aa^*) = v\big((aa^*)(bb^*) - (bb^*)(aa^*)\big) \\ &> v(aa^*bb^*) = v(ar_1) = v(br_2). \end{aligned}$$

Hence v is quasi-Ore. Note that an element $a = \sum \overline{a}_\alpha \in \operatorname{gr}(A, v)$ is symmetric iff $\overline{a}_\alpha \in \overline{A}_\alpha$ is symmetric for every α. By a simple induction argument, to prove that symmetric elements of $\operatorname{gr}(A, v)$ commute, it suffices to show that two symmetric elements of the form $\overline{a}_\alpha$, $\overline{b}_\beta$ commute. Moreover, as $2\overline{a} = 0$ implies $\overline{a} = 0$ for all $\overline{a} \in \operatorname{gr}(A, v)$ by the assumption $v(2) = 0$, it is enough to prove that $2\overline{a}_\alpha$, $2\overline{b}_\beta$ commute. Since $\overline{a}_\alpha$ is symmetric, $a_\alpha \sim_v a_\alpha^*$ and thus $v(a_\alpha + a_\alpha^*) = v(a_\alpha)$. Hence $2\overline{a}_\alpha = \overline{a_\alpha + a_\alpha^*}$ and $a_\alpha + a_\alpha^*$ is symmetric. Similarly, $2\overline{b}_\beta = \overline{b_\beta + b_\beta^*}$. Since v is quasi-commutative for symmetric elements, $(a_\alpha + a_\alpha^*)(b_\beta + b_\beta^*) \sim_v (b_\beta + b_\beta^*)(a_\alpha + a_\alpha^*)$ and so $(2\overline{a}_\alpha)(2\overline{b}_\beta) = (2\overline{b}_\beta)(2\overline{a}_\alpha)$, as desired. Finally, the last statement of the proposition follows from [Co, Theorem 4.2]. ■

Theorem 2 *If A is a $*$-ordered domain and v the natural $*$-valuation, then v is quasi-Ore. If also, A contains an imaginary unit, then v is quasi-commutative.*

Proof By [Ma1, 3.3 Theorem] and Proposition 1, v is quasi-Ore. So let us assume that $i \in A$ is an imaginary unit. Observe that every $*$-ordering of A extends uniquely to a $*$-ordering of the central localization $A_{\mathbb{N}} (\cong A \otimes_{\mathbb{Z}} \mathbb{Q})$. Hence we may assume that $\mathbb{Q} \subseteq A$. For every $x \in A$ we have $x = \frac{x+x^*}{2} + i\,\frac{x-x^*}{2i}$. In other words, $x = x_1 + i\,x_2$ for (uniquely determined) $x_1, x_2 \in \operatorname{Sym} A$. Let $a, b \in A^\times$ be arbitrary. Write $a = a_1 + ia_2$ and $b = b_1 + ib_2$ for symmetric a_1, a_2, b_1, b_2. Then

$$\begin{aligned} ab - ba &= (a_1 + ia_2)(b_1 + ib_2) - (b_1 + ib_2)(a_1 + ia_2) \\ &= (a_1 b_1 - b_1 a_1) + (b_2 a_2 - a_2 b_2) + i(a_1 b_2 - b_2 a_1) + i(a_2 b_1 - b_1 a_2). \end{aligned}$$

Hence by the triangle inequality,

$$v(ab - ba) \geqslant \min\left\{ v(a_1 b_1 - b_1 a_1), v(b_2 a_2 - a_2 b_2), v(a_1 b_2 - b_2 a_1), v(a_2 b_1 - b_1 a_2) \right\}.$$

Now use [Ma1, 3.3(5) Theorem] to get

$$\begin{aligned} v(ab-ba) &> \min\big\{\, v(a_1b_1), v(a_2b_2), v(a_1b_2), v(a_2b_1)\big\} \\ &= \min\{v(a_1), v(a_2)\} + \min\{v(b_1), v(b_2)\}. \end{aligned}$$

By [Ma2, 2.4 Proposition], the right-hand side of the last equation equals $v(a) + v(b) = v(ab)$. Hence $v(ab-ba) > v(ab)$, as required. ■

An application of this result will be given in the next section, where we answer a question posed by Cimprič [Ci]. For another application we refer the reader to [KM].

If D is a $*$-ordered division ring and v is the natural $*$-valuation, then $k_v = \mathcal{O}_v/\mathfrak{m}_v$ is a $*$-ordered division subring of $\mathbb{H}$, *cf.* [Ho2]. Hence we have a $*$-homomorphism $\mathcal{O}_v \to \mathbb{H}$. We extend this to a map $D \to \mathbb{H} \cup \{\infty\}$ by mapping $D \setminus \mathcal{O}_v \to \{\infty\}$. This mapping is called a $*$-place. For more on $*$-places on division rings we refer the reader to [Cr].

Proposition 3 *If A is a $*$-ordered domain and v the natural $*$-valuation, then k_v is a $*$-ordered subring of $\mathbb{H}$. If A also contains an imaginary unit, then k_v is a $*$-ordered subring of $\mathbb{C}$.*

Proof Write P for the $*$-ordering of A. Let $\overline{a}, \overline{b} \in k_v^\times$. As $v(aa^*bb^* - bb^*aa^*) > v(aa^*bb^*) = 0$, we have $\overline{a} \cdot \overline{a^*bb^*} = \overline{b} \cdot \overline{b^*aa^*}$. In other words, k_v is an Ore domain. Moreover, P induces an archimedean $*$-ordering $\overline{P}$ of k_v. By [CS, Corollary 2.5], $\overline{P}$ extends to a $*$-ordering Q of $\operatorname{Quot}(k_v)$. Let w denote the natural $*$-valuation of $\operatorname{Quot}(k_v)$. By a result of Holland [Ho2, 4.1], $w(as-sa) > w(as)$ for all $a, s \in \operatorname{Quot}(k_v)$ with $s = s^*$. Obviously, $w|_{k_v}$ is the natural $*$-valuation associated with the $*$-ordering $\overline{P}$ of k_v. Since $\overline{P}$ is archimedean, $w|_{k_v}$ is trivial. This implies that symmetric elements of k_v are central. Furthermore, $\overline{P}$ induces an archimedean total ordering of $\operatorname{Sym} k_v$, hence $\operatorname{Sym} k_v$ is an ordered subring of $\mathbb{R}$. Form the central localization $B := k_v(\operatorname{Sym} k_v^\times)^{-1}$. Clearly, $B \subseteq \operatorname{Quot}(k_v)$, hence Q induces a $*$-ordering of B. From the definition of B it is easy to see that this $*$-ordering is archimedean. Moreover, by results of Herstein [He], B is finite dimensional over its center. As it is also a domain, B must be a division ring. Hence by a theorem due to Holland [Ho1], B is a $*$-ordered division subring of $\mathbb{H}$. In particular, k_v is a $*$-ordered subring of $\mathbb{H}$.

If A contains an imaginary unit, then k_v is commutative by Theorem 2 and thus a $*$-ordered subring of $\mathbb{C}$. ■

Again, by this proposition we have a mapping $A \to \mathbb{H} \cup \{\infty\}$. We call it the *weak $*$-place* associated with P. A $*$-place associated with P will be a mapping $(\operatorname{Sym} A \times \operatorname{Sym} A) \setminus \{(0,0)\} \to \mathbb{R} \cup \{\infty\}$ with certain properties. In order to define it, we need the following classical result.

Lemma 4 ([Fu, Ch. IV]) *Let $(A, +, \leqslant)$ be a totally ordered abelian group and v the natural order-compatible valuation. For $a, b > 0$ and $v(a) = v(b)$ there exists a unique real number $\mu(a,b) \in (0, \infty)$ such that $\mu(a,b) \in \big[\frac{m}{n}, \frac{m+1}{n}\big]$ for any $m, n \in \mathbb{N}$ satisfying $mb \leqslant na \leqslant (m+1)b$.*

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Assume $P \subseteq A$ is a $*$-ordering. Then $(\operatorname{Sym} A, +)$ is an abelian group and P is a total ordering of $\operatorname{Sym} A$. Hence, using Lemma 4 we can define a map $\wp : (\operatorname{Sym} A \times \operatorname{Sym} A) \setminus \{(0,0)\} \to \mathbb{R} \cup \{\infty\}$ as follows

$$\wp(a,b) = \begin{cases} \infty & \text{if } v(a) < v(b), \\ \mu(|a|,|b|) & \text{if } v(a) = v(b), \{a,b\} \in P, \\ -\mu(|a|,|b|) & \text{if } v(a) = v(b), -\{a,b\} \in P, \\ 0 & \text{if } v(a) > v(b). \end{cases}$$

This mapping is the $*$-*place* associated with P. Let us note some properties of $\wp$. For all $a, b \in \operatorname{Sym} A$, not both zero, we have

(P_1) $\wp(a,b) = \infty$ iff $\wp(b,a) = 0$,
(P_2) if $\wp(a,b), \wp(b,c) \neq \infty$, then $\wp(a,b)\wp(b,c) = \wp(a,c)$,
(P_3) if $\wp(a,c), \wp(b,c) \neq \infty$, then $\wp(a,c) + \wp(b,c) = \wp(a+b,c)$,
(P_4) $\wp(a,b) = \wp\big(\{a,c\},\{b,c\}\big)$ for all $c \in \operatorname{Sym} A^{\times}$,
(P_5) $\wp(a,b) = \wp(r^*ar, r^*br)$ for all $r \in A^{\times}$.

In case $i \in A$ is an imaginary unit, we can extend $\wp$ to a mapping $(A \times A) \setminus \{(0,0)\} \to \mathbb{C}$ as follows. As before, we assume $\mathbb{Q} \subseteq A$. We first extend $\wp$ to $(A \times \operatorname{Sym} A) \setminus \{(0,0)\} \to \mathbb{C}$ by $\wp(a_1 + i\,a_2, b) := \wp(a_1, b) + i\,\wp(a_2, b)$ for $a_1, a_2, b \in \operatorname{Sym} A$. This is well defined since every $a \in A$ can be written uniquely as $a = a_1 + i\,a_2$ for symmetric a_1, a_2. For the second step, we define $\wp(a,b) := \frac{1}{2}\wp(ab^* + b^*a, bb^*)$ for $a, b \in A$, not both zero. We claim that this is well defined. Let $a \in A$ and $s \in \operatorname{Sym} A$. We have to show that $\wp(a,s) = \frac{1}{2}\wp(as+sa, s^2)$. Let $a = a_1 + i\,a_2$ for $a_j \in \operatorname{Sym} A$. Then $\wp(a,s) = \wp(a_1,s) + i\,\wp(a_2,s)$ and $\wp(as+sa,s^2) = \wp(a_1s+sa_1,s^2) + i\,\wp(a_2s+sa_2,s^2)$. Thus we may assume without loss of generality that a is symmetric as well. But then our claim follows from (P_4).

Proposition 5 ([MZ2, §1.3 Notes]) *Assume A is a $*$-ordered domain and a $\mathbb{C}$-algebra and let v denote the natural $*$-valuation. If for $a, b \in A$, $v(a) \geqslant v(b)$, then $\wp(a,b) = \mu$ iff $v(a - \mu b) > v(b)$.*

Proof It is easy to see that $\wp(a,b) = \mu$ iff $\wp(a - \mu b, b) = 0$. So it suffices to prove the statement for $\mu = 0$. By definition, $\wp(a,b) = \frac{1}{2}\wp(ab^* + b^*a, bb^*)$. Write $ab^* + b^*a = c_1 + i\,c_2$ for symmetric c_1, c_2. Then $\wp(a,b) = \frac{1}{2}\wp(c_1, bb^*) + \frac{i}{2}\wp(c_2, bb^*)$. By [Ma2, 2.4 Proposition], $v(ab^* + b^*a) = \min\{v(c_1), v(c_2)\}$. On the other hand, $v(ab^* + b^*a) = v(ab^*)$ by Theorem 2. If $v(a) > v(b)$, then $v(c_j) > v(bb^*)$ for $j = 1, 2$. Thus $\wp(a,b) = 0$ by the definition of $\wp$. Conversely, if $\wp(a,b) = 0$, then $\wp(c_j, bb^*) = 0$ for $j = 1, 2$. Hence $v(c_j) > v(bb^*)$ and so $v(ab^*) = v(ab^* + b^*a) > v(bb^*)$. This implies $v(a) > v(b)$, as desired. ■

In the rest of this section we sharpen Theorem 2 for $*$-ordered $\mathbb{C}$-algebras by showing that the corresponding graded ring is isomorphic to a twisted semigroup ring. As our semigroups are abelian and written additively, we use the exponential notation for twisted semigroup rings.

Proposition 6 ([MZ2, 2.3 Example(1)]) *Let Γ be an ordered cancellative abelian semigroup. Consider the twisted semigroup ring $\mathbb{C}[\Gamma, c]$ with the twisting given by $t^\alpha t^\beta = c(\alpha, \beta) t^{\alpha+\beta}$, where $c\colon \Gamma \times \Gamma \to \mathbb{R}$ is a positive symmetric factor set,* i.e.,

(FS_1) $c(\alpha, \beta) > 0$ *for all* $\alpha, \beta \in \Gamma$,
(FS_2) $c(\alpha, 0) = c(0, \alpha) = 1$ *for all* $\alpha \in \Gamma$,
(FS_3) $c(\alpha, \beta)c(\alpha + \beta, \gamma) = c(\alpha, \beta + \gamma)c(\beta, \gamma)$ *for all* $\alpha, \beta, \gamma \in \Gamma$,
(FS_4) $c(\alpha, \beta) = c(\beta, \alpha)$ *for all* $\alpha, \beta \in \Gamma$.

The involution on $\mathbb{C}[\Gamma, c]$ fixes Γ pointwise and $z^* = \bar{z}$ for $z \in \mathbb{C}$, and $\mathbb{C}[\Gamma, c]$ is a graded commutative $*$-domain. A $*$-valuation $v\colon \mathbb{C}[\Gamma, c] \to \Gamma \cup \{\infty\}$ is defined as follows: if $a = \sum a_\alpha t^\alpha \neq 0$, then $v(a)$ is the smallest $\alpha \in \Gamma$ such that $a_\alpha \neq 0$. $*$-orderings of $\mathbb{C}[\Gamma, c]$ compatible with v are in a natural one-to-one correspondence $\sigma \mapsto P_\sigma$ with semigroup homomorphisms $\Gamma \to \{-1, 1\}$. A nonzero $a = \sum a_\alpha t^\alpha$ is symmetric iff $a_\alpha \in \mathbb{R}$ for all $\alpha \in \Gamma$ such that $a_\alpha \neq 0$. Such an element a is positive with respect to P_σ iff $\sigma(\alpha)a_\alpha > 0$, where $\alpha = v(a)$. Also, v is the natural $*$-valuation associated with P_σ.

Proof This is straightforward. For example, to prove that $\sigma \mapsto P_\sigma$ is a bijection, we proceed as follows. $*$-orderings of $\mathbb{C}[\Gamma, c]$ compatible with v are total orderings of $\operatorname{Sym} \mathbb{C}[\Gamma, c] = \mathbb{R}[\Gamma, c]$ compatible with v that are closed under $*$-conjugation. But total orderings of $\mathbb{R}[\Gamma, c]$ compatible with v are all of the form P_σ and these are closed under $*$-conjugation. ■

Theorem 7 ([MZ2, 2.3 Example(2)]) *Suppose P is a $*$-ordering of a $\mathbb{C}$-algebra A and $v\colon A \to \Gamma \cup \{\infty\}$ the natural $*$-valuation.*

(i) *For each $\alpha \in \Gamma$ there exists $s_\alpha \in P$ with $v(s_\alpha) = \alpha$.*
(ii) *The mapping $\overline{A}_\alpha \to \mathbb{C}$ defined by $\bar{a} \mapsto \wp(a, s_\alpha)$, where $\wp : (A \times A) \setminus \{(0, 0)\} \to \mathbb{C}$ is the $*$-place associated with P, is an isomorphism.*
(iii) *The mapping $c\colon \Gamma \times \Gamma \to \mathbb{R}$ defined by $c(\alpha, \beta) := \wp\big(\{s_\alpha, s_\beta\}, s_{\alpha+\beta}\big)$ is a positive symmetric factor set. Furthermore, $\sum \bar{a}_\alpha \mapsto \sum \wp(a_\alpha, s_\alpha)t^\alpha$ defines a $*$-isomorphism between $\operatorname{gr}(A, v)$ and $\mathbb{C}[\Gamma, c]$.*

Proof For $\alpha \in \Gamma$ choose $x \in A$ satisfying $v(x) = \alpha$. If $x = x_1 + ix_2$ for symmetric x_1, x_2, then $v(x) = \min\{v(x_1), v(x_2)\}$ by [Ma2, 2.4 Proposition]. Say $v(x_1) = \alpha$. If $x_1 \in P$, then $s_\alpha := x_1$. Otherwise $-x_1 \in P$, and we take $s_\alpha := -x_1$. This proves (i).

To prove (ii), let $a = a_1 + i\, a_2$. If $v(a_1) \neq v(a_2)$, then $\bar{a} = \bar{a}_j$ with j such that $v(a_j) = \min\{v(a_1), v(a_2)\}$. In this case $\wp(a, s_\alpha) = \wp(a_j, s_\alpha)$ is a real number and furthermore, every real number can be obtained in this way since A is a $\mathbb{C}$-algebra. If $v(a_1) = v(a_2)$, then $\wp(a, s_\alpha) = \wp(a_1, s_\alpha) + i\wp(a_2, s_\alpha)$. By the same reasoning as above, $\wp(-, s_\alpha)$ maps A_α onto $\mathbb{C}$. Now if $\wp(a, s_\alpha) = \wp(b, s_\alpha)$ for some $a, b \in A_\alpha$, then $\wp(a - b, s_\alpha) = 0$. Hence $v(a - b) > \alpha$ and thus $\bar{a} = \bar{b}$ in $\overline{A}_\alpha$. This shows that the mapping $\overline{A}_\alpha \to \mathbb{C}$, given by $\bar{a} \mapsto \wp(a, s_\alpha)$, is injective. Since it is obviously a homomorphism, it is an isomorphism, as desired.

(FS_1) and (FS_2) for c follow immediately from the definition of $\wp$ and so does (FS_4) since Γ is abelian. The long and tedious calculation needed to prove (FS_3) is left to the interested reader as an exercise. The rest of (iii) then follows from (ii). ■

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Note that Proposition 6 and Theorem 7 combined with [Ma2, 2.5 Proposition] yield a Krull–Baer type result. Namely, if $P \subseteq A$ is a $*$-ordering, where A is a $\mathbb{C}$-algebra and $v\colon A \to \Gamma \cup \{\infty\}$ is the natural $*$-valuation, then the set of all $*$-orderings of A compatible with v is in a natural one-to-one correspondence with the set of all semigroup homomorphisms $\Gamma \to \{-1, 1\}$.

4 $*$-Orderability of Quantum Groups

In [Ci] Cimprič studied orderability and real spectra of certain classes of quantum groups. At the end of that paper he asked for a similar characterization of $*$-orderings of these quantum groups, see [Ci, §6]. We give an answer to his question by showing that quantum Weyl fields "rarely" admit $*$-orderings.

Proposition 8 *Assume A is a k-algebra containing elements x, y satisfying $yx = \sigma xy$ for $\sigma \in k \setminus \{0, 1\}$. Then A does not admit a quasi-commutative valuation that is trivial on k.*

Proof Assume otherwise and let v be a quasi-commutative valuation of A that is trivial on k. Then $v(xy) < v(xy - yx) = v(xy - \sigma xy) = v(1 - \sigma) + v(xy)$. Hence $v(1 - \sigma) > 0$, a contradiction. ■

Let $\mathbf{q} = (q_{ij})_{i=1,\dots,n}^{j=i+1,\dots,n}$ be a sequence of nonzero complex numbers. The *quantum affine space* $\mathbb{C}_{\mathbf{q}}[x_1, \dots, x_n]$ is the $\mathbb{C}$-algebra on n generators $x_1, \dots, x_n$ subject to relations $x_i x_j = q_{ij} x_j x_i$ for $1 \leqslant i < j \leqslant n$. It is well known that $\mathbb{C}_{\mathbf{q}}[x_1, \dots, x_n]$ is an Ore domain. Its division ring of fractions is denoted by $\mathbb{C}_{\mathbf{q}}(x_1, \dots, x_n)$ and called the *quantum Weyl field*.

Corollary 9 *If $q_{ij} \neq 1$ for some i, j, then the quantum affine space $\mathbb{C}_{\mathbf{q}}[x_1, \dots, x_n]$ is not $*$-orderable (independent of the involution chosen). The same holds true for the quantum Weyl field $\mathbb{C}_{\mathbf{q}}(x_1, \dots, x_n)$.*

Proof This follows easily from Proposition 8 and Theorem 2. ■

Remark

(i) A special case of Corollary 9 was given in [CKM]. The authors proved that the complex quantum plane $\mathbb{C}\langle X, Y\rangle/(XY - qYX)$ for $q \neq 0, 1$ does not admit $*$-orderings for certain kinds of involutions.

(ii) If $q_{ij} = 1$ for all i, j, then $\mathbb{C}_{\mathbf{q}}[x_1, \dots, x_n]$ is the ordinary polynomial algebra over $\mathbb{C}$ in n commuting variables and $\mathbb{C}_{\mathbf{q}}(x_1, \dots, x_n)$ is its quotient field. Existence of $*$-orderings of $\mathbb{C}[x_1, \dots, x_n]$, resp., $\mathbb{C}(x_1, \dots, x_n)$ depends on the involution chosen. In the case $x_i^* = x_i$ and $*$ is conjugation on $\mathbb{C}$, $*$-orderings of $\mathbb{C}[x_1, \dots, x_n]$, resp., $\mathbb{C}(x_1, \dots, x_n)$ are precisely total orderings of $\mathbb{R}[x_1, \dots, x_n]$, resp., $\mathbb{R}(x_1, \dots, x_n)$. These have been fully classified, see [KKMZ].

(iii) The quantum Gelfand–Kirillov conjecture states that the division ring of fractions of a quantum group is always a quantum Weyl field. Even though it is known to be false in general, it does hold in a variety of cases. By Corollary 9, these quantum groups are never $*$-orderable.

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References

[Ch] M. Chacron, *c-orderable division rings with involution.* J. Algebra **75**(1982), no. 2, 495–522.
[Ci] J. Cimprič, *Real spectra of quantum groups.* J. Algebra **277**(2004), no. 1, 282–297.
[CKM] J. Cimprič, M. Kochetov, and M. Marshall, *Orderings and $*$-orderings on cocommutative Hopf algebras.* To appear in Algebr. Represent. Theory, preprint available at http://vega.fmf.uni-lj.si/srag.
[Co] P. M. Cohn, *On the embedding of rings in skew fields.* Proc. London Math. Soc. **11**(1961), 511–530.
[Cr] T. Craven, *Places on $*$-fields and the real holomorphy ring.* Comm. Algebra **18**(1990), no. 9, 2791–2820.
[CS] T. Craven and T. Smith, *Ordered $*$-rings.* J. Algebra **238**(2001), no. 1, 314–327.
[Fu] L. Fuchs, *Partially Ordered Algebraic Systems.* Pergamon Press, Oxford, 1963.
[He] I. N. Herstein, *Special simple rings with involution.* J. Algebra **6**(1967), no. 3, 369–375.
[Ho1] S. S. Holland, *Orderings and square roots in $*$-fields.* J. Algebra **46**(1977), no. 1, 207–219.
[Ho2] ———, *Strong orderings on $*$-fields.* J. Algebra **101**(1986), no. 1, 16–46.
[KM] I. Klep and P. Moravec, *On $*$-orderable groups.* J. Pure Appl. Algebra **200**(2005), no. 1-2, 25–35.
[KKMZ] F.-V. Kuhlmann, S. Kuhlmann, M. Marshall, and M. Zekavat, *Embedding ordered fields in formal power series fields.* J. Pure Appl. Algebra **169**(2002), no. 1, 71–90.
[Ma1] M. Marshall, *$*$-orderings on a ring with involution.* Comm. Algebra **28**(2000), no. 3, 1157–1173.
[Ma2] ———, *$*$-orderings and $*$-valuations on algebras of finite Gelfand-Kirillov dimension.* J. Pure Appl. Algebra **179**(2003), no. 3, 255–271.
[MZ1] M. Marshall, and Y. Zhang, *Orderings, real places and valuations on noncommutative integral domains.* J. Algebra **212**(1999), no. 1, 190–207.
[MZ2] ———, *Orderings and valuations on twisted polynomial rings.* Comm. Algebra **28**(2000), no. 8, 3763–3776.

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