MAPS INTO DYNKIN DIAGRAMS ARISING FROM REGULAR MONOIDS

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Abstract

It has been shown by one of the authors that the system of idempotents of monoids on a group $G$ of Lie type with Dynkin diagram $\Gamma$ can be classified by the following data: a partially ordered set $\mathcal{Z}$ with maximum element 1 and a map $\lambda: \mathcal{Z} \to 2^\Gamma$ with $\lambda(1) = \Gamma$ and with the property that for all $J_1, J_2, J_3 \in \mathcal{Z}$ with $J_1 < J_2 < J_3$, any connected component of $\lambda(J_2)$ is contained in either $\lambda(J_1)$ or $\lambda(J_3)$. In this paper we show that $\lambda$ comes from a regular monoid if and only if the following conditions are satisfied:

1. $\mathcal{Z}$ is a $\wedge$-semilattice;
2. If $J_1, J_2 \in \mathcal{Z}$, then $\lambda(J_1) \cap \lambda(J_2) \subseteq \lambda(J_1 \wedge J_2)$;
3. If $\theta \in \Gamma$, $J \in \mathcal{Z}$, then $\max\{J_1 \in \mathcal{Z} : J_1 \leq J, \theta \in \lambda(J_1)\}$ exists;
4. If $J_1, J_2 \in \mathcal{Z}$ with $J_1 < J_2$ and if $X$ is a two element discrete subset of $\lambda(J_1) \cup \lambda(J_2)$, then $X \subseteq \lambda(J)$ for some $J \in \mathcal{Z}$ with $J_1 \leq J \leq J_2$.


By a Coxeter group $W = (W, \Gamma)$ is meant a group $W$ generated by a subset $\Gamma$ of elements of order 2, such that $W$ has a presentation by the relations $(\sigma \theta)^m(\sigma, \theta) = 1$, for $\sigma, \theta \in \Gamma$. We assume that the rank $|\Gamma| < \infty$. If $\sigma, \theta \in \Gamma$, define $\sigma \theta^m$ if $m = m(\sigma, \theta) \geq 3$. In this way $\Gamma$ becomes a graph, called the Coxeter graph of $W$. Note that $\sigma, \theta$ are not adjacent in the graph if and only if $\sigma \theta = \theta \sigma$. It is customary to write $\sigma - \theta$ to mean $m = 3$, $\sigma = \theta$ to mean $m = 4$ and $\sigma \equiv \theta$ to mean $m = 6$. The possible graphs for finite $W$ were determined by Coxeter (see [8]). Coxeter groups arise in much of algebra as...
Weyl groups related to root systems. The possible connected Coxeter graphs are then

\[ A_n : \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \]

\[ B_n \text{ or } C_n : \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \]

\[ D_n : \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \]

\[ E_6 : \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \]

\[ E_7 : \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \]

\[ E_8 : \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \]

\[ F_4 : \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \]

\[ G_2 : \bullet \text{---} \bullet \]

These graphs are closely related to the Dynkin diagrams of root systems.
Let $W$ be a Coxeter group, $\sigma \in W$. Then $\sigma = \theta_1 \cdots \theta_k$ for some $\theta_1, \ldots, \theta_k \in \Gamma$. If $k$ is minimal, then the length $l(\sigma)$ is defined to be $k$. Coxeter groups are characterized by Matsumoto's exchange condition [8, Theorem 4.4].

**Theorem (Exchange condition).** Let $\theta_1, \ldots, \theta_k \in \Gamma$, $\sigma = \theta_1 \cdots \theta_k$, $l(\sigma) = k$. If $\theta \in \Gamma$, then either $l(\theta \sigma) = k + 1$ or else $l(\theta \sigma) = k - 1$ and $\theta \sigma = \theta_1 \cdots \hat{\theta}_i \cdots \theta_k$ for some $i = 1, \ldots, k$.

**Remark.** The exchange condition implies the following.

(i) If $\theta_1, \ldots, \theta_n \in \Gamma$, $\sigma = \theta_1 \cdots \theta_n$, $l(\sigma) = k$, then $\sigma = \theta_{i_1} \cdots \theta_{i_k}$ for some $i_1 < \cdots < i_k$.

(ii) If $\theta_1, \ldots, \theta_n, \theta'_1, \ldots, \theta'_n \in \Gamma$, $\sigma = \theta_1 \cdots \theta_n = \theta'_1 \cdots \theta'_n$ and $l(\sigma) = n$, then $\{\theta_1, \ldots, \theta_n\} = \{\theta'_1, \ldots, \theta'_n\}$.

If $I \subseteq \Gamma$, let $W_I = \langle I \rangle$ denote the subgroup of $W$ generated by $I$. If $I, I' \subseteq \Gamma$, then $W_{I \cap I'} = W_I \cap W_I'$ and $W_I = W_{I'}$ if and only if $I = I'$.

Let $\mathcal{Z}$ be a partially ordered set with maximum element $1$, $\lambda: \mathcal{Z} \to 2^\Gamma$ such that $\lambda(1) = \Gamma$. If $J \in \mathcal{Z}$, we write $W_J$ for $W_{\lambda(J)}$. Let $\mathcal{W}(\lambda) = \{(J, W_J \sigma) | J \in \mathcal{Z}, \sigma \in W\}$. Define $(J_1, W_{J_1} \sigma) \leq (J_2, W_{J_2} \alpha)$ if $J_1 \subseteq J_2$ and $W_{J_1} \sigma \cap W_{J_2} \alpha \neq \emptyset$. Define $\lambda$ to be transitive if $\leq$ is transitive on $\mathcal{W}(\lambda)$. Define $\lambda$ to be regular if $(\mathcal{W}(\lambda), \leq)$ is a $\lambda$-semilattice. Then it can be seen [5] that $\lambda$ is transitive if and only if for all $J_1, J_2, J_3 \in \mathcal{Z}$ with $J_1 \geq J_2 \geq J_3$, any connected component of $\lambda(J_2)$ is contained in either $\lambda(J_1)$ or $\lambda(J_3)$. The main goal of this paper is to obtain a similarly usable characterization of regularity.

Before proceeding, we explain the motivation for the above considerations. The basic motivation comes from the theory of linear algebraic monoids ([3], [4], [6], [7]). It has been shown by L. Renner and one of the authors [8] that for a connected regular linear algebraic monoid $M$ with zero, the system of idempotents (biordered set in the sense of Nambooripad [2]) is determined by a 'type map' $\lambda$ from the finite lattice $\mathcal{Z}$ of principal ideals of $M$ into $2^\Gamma$, where $\Gamma$ is the Dynkin diagram of the group of units of $M$. One of the authors [5] considered the more general situation of monoids on a group $G$ with a $BN$-pair. Again the system of idempotents is characterized by a type map $\lambda: \mathcal{Z} \to 2^\Gamma$. Moreover it was shown in [5] that an abstract map $\lambda: \mathcal{Z} \to 2^\Gamma$ arises if and only if it is transitive. It was further shown in [5], that $\lambda$ comes from a regular monoid on $G$ if and only if $\lambda$ is regular.

For monoids $M$ on a group $G$ of Lie type, the partially ordered set $\mathcal{W}(\lambda)$ is isomorphic to the partially ordered set of 'diagonal idempotents' of $M$. We illustrate with an example. Let $G = GL(4, F)$ where $F$ is a field. Then one monoid on $G$ is $\mathcal{M}_4(F)$, the monoid of all $4 \times 4$ matrices over $F$. In this case the Weyl group of $G$ is the group $S_4$ of all $4 \times 4$ permutation matrices and $\Gamma$.
can be chosen to be
\[\Gamma = \left\{ \theta_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \theta_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \theta_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\} .\]

The graph structure for \(\Gamma\) is

\[\theta_1 \rightarrow \theta_2 \rightarrow \theta_3\]

The standard idempotent representatives for matrices in \(\mathcal{M}_4(F)\) of different ranks are given by the linearly ordered set

\[\mathcal{U} = \left\{ \emptyset, e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, 0 \right\} .\]

The corresponding regular map \(\lambda: \mathcal{U} \rightarrow 2^\Gamma\) is given by \(\lambda(e) = \{\theta \in \Gamma | e\theta = \theta e\}\), for all \(e \in \mathcal{U}\). Thus \(\lambda(\emptyset) = \Gamma, \lambda(e_1) = \{\theta_2, \theta_3\}, \lambda(e_2) = \{\theta_1, \theta_3\}, \lambda(e_3) = \{\theta_1, \theta_2\}\) and \(\lambda(0) = \Gamma\). The lattice \(\mathcal{W}(\lambda)\) is a sixteen element Boolean lattice isomorphic to the lattice of diagonal idempotents of \(\mathcal{M}_4(F)\).

We now fix a Coxeter group \(W = W(\Gamma)\). Before stating the main theorem, we prove some lemmas.

**Lemma 1.** Let \(\sigma_1, \ldots, \sigma_k, \theta \in \Gamma, \theta \neq \sigma_i\), \(i = 1, \ldots, k\). Let \(\sigma = \sigma_1 \cdots \sigma_k\). Suppose \(l(\sigma) = k\) and \(\sigma \theta = \theta \sigma \) for some \(\sigma \in W\) with \(l(\sigma) = k\) and \(\theta\) not appearing in \(\sigma\). Then \(\sigma_i \theta = \theta \sigma_i\) for \(i = 1, \ldots, k\).

**Proof.** We prove this by induction on \(k\). By the exchange condition

\[\sigma_2 \cdots \sigma_k \theta = \sigma_1 \theta \sigma = \theta \sigma'\]

with \(l(\sigma') = k - 1\), and \(\theta\) does not appear in \(\sigma'\). So by the induction hypothesis, \(\theta \sigma_i = \sigma_i \theta\) for \(i = 2, \ldots, k\). So if \(\sigma_1 \in \{\sigma_2, \ldots, \sigma_k\}\) we are done. So assume \(\sigma_1 \notin \{\sigma_2, \ldots, \sigma_k\}\). Now \(\sigma = u \sigma_i v, \sigma' = uv\) for some \(i \in \{2, \ldots, k\}\). Since \(\sigma_1\) does not appear in the left side of (1), we see that \(\sigma_1\) does not appear in \(uv = \sigma'\). Hence \(\sigma_1 = \sigma_i\) and \(\sigma_1 \theta u \sigma_1 v = \theta uv\). So \(\sigma_1 \theta u = \theta u \sigma_1\); and hence \(u \sigma_1 = \theta \sigma_1 u \). Since \(\theta\) does not appear in \(u \sigma_1\), we see by the exchange condition that \(\theta \sigma_1 \theta u = \sigma_1 u\). So \(\theta \sigma_1 \theta = \sigma_1\) and \(\theta \sigma_1 = \sigma_1 \theta\). This completes the proof.

**Lemma 2.** Let \(J_1, J_2 \subseteq \Gamma, \sigma \in \mathcal{W}_{J_2}, \alpha \in \mathcal{W}_{J_1}, \alpha \notin \mathcal{W}_{J_2}\). Let \(l(\sigma) = k\), \(\sigma = \sigma_1 \cdots \sigma_k, \sigma_i \in \Gamma\). Suppose that \(\sigma\) is of minimal length in \(\mathcal{W}_{J_1} \sigma \mathcal{W}_{J_1}\) and
that \( W_j \sigma \cap W_2 \sigma^{-1} \neq \emptyset \). Then there exists \( \theta \in J_1 \setminus J_2 \) such that \( \theta \sigma_i = \sigma_i \theta, \)
i = 1, \ldots, k.

**Proof.** We prove this by induction of \( l(\alpha) \). By the exchange condition

\[
l(\sigma x) = l(\sigma) + l(x) = l(x \sigma) \quad \text{for all } x \in W_j.
\]

In particular, \( l(\sigma \alpha) = l(\sigma) + l(\alpha) \). Now \( \sigma \alpha \in W_j W_2 \). So by the exchange condition, \( \sigma \alpha = uv \) for some \( u \in W_j \) and \( v \in W_2 \) such that \( l(uv) = l(u) + l(v) \). Since \( \alpha \notin W_2, u \neq 1 \). So \( u = \theta u_1 \) for some \( u_1 \in W_1 \), \( \theta \in J_1 \), \( l(u_1) = l(u) - 1 \). So \( u_1 v = \theta \sigma \alpha \). By (2), \( l(\theta \sigma) = l(\sigma) + 1 \). So by the exchange condition, \( \theta \sigma \alpha = \sigma_1 \) for some \( \alpha_1 \in W_1 \) with \( l(\alpha_1) = l(\alpha) - 1 \). So \( \sigma_1 = u_1 v \in W_j W_2 \). If \( \alpha \notin W_2 \), we are done, by the induction hypothesis. So assume \( \alpha_1 \in W_2 \). Since \( \alpha \notin W_2 \), we see that \( \alpha = \alpha_2 \sigma_2 \), \( \sigma_1 = \alpha_2 \sigma_3 \) with \( \pi \in J_1 \setminus J_2 \). If \( \pi \neq \theta \) then \( \pi \) appears in \( u_1 v = \sigma \alpha_1 \), a contradiction. Hence \( \pi = \theta \) and \( \theta \sigma_2 \sigma_3 = \sigma_2 \sigma_3 \). Then \( \theta \sigma \sigma_2 = \sigma \sigma_2 \). So \( \sigma_2 \sigma = \sigma \sigma_2 \), \( \sigma \sigma_2 \) does not appear in \( \sigma_2 \). We are now done, by Lemma 1.

**Theorem 1.** Let \( \mathcal{U} \) be a partially ordered set with maximum element 1 and let \( \lambda : \mathcal{U} \to 2^\Gamma \) be a transitive map such that \( \lambda(1) = \Gamma \). Then \( \lambda \) is regular if and only if

1. \( \mathcal{U} \) is a \( \wedge \)-semilattice,
2. if \( J_1, J_2 \in \mathcal{U} \), then \( \lambda(J_1) \cap \lambda(J_2) \subseteq \lambda(J_1 \wedge J_2) \),
3. if \( J \in \mathcal{U}, \theta \in \Gamma \), then \( \max\{J_1 \in \mathcal{U} \mid J_1 \leq J, \theta \in \lambda(J_1)\} \) exists,
4. if \( J_1, J_2 \in \mathcal{U}, J_1 \geq J_2 \) and \( X \) is a two element discrete subset of \( \lambda(J_1) \cup \lambda(J_2) \), then \( X \subseteq \lambda(J) \) for some \( J \in \mathcal{U} \) with \( J_1 \geq J \geq J_2 \).

**Proof.** First we prove necessity. So assume that \( \lambda \) is regular. So \( \mathcal{U}(\lambda), (\leq) \) is a \( \wedge \)-semilattice. Let \( J_1, J_2 \in \mathcal{U} \), such that \( (J_1, W_{J_1}) \wedge (J_2, W_{J_2}) = (J, W_J) \). Let \( J' \in \mathcal{U} \) with \( J_1 \geq J', J_2 \geq J' \). Then \( (J_1, W_{J_1}) \geq (J', W_{J'}) \), \( i = 1, 2 \). So \( (J, W_J) \geq (J', W_{J'}) \). So \( J \geq J' \). Hence \( J = J_1 \wedge J_2 \). Also \( (J, W_J) \geq (J, W_J) \) whereby \( W_J \alpha = W_J \). If \( \theta \in \lambda(J_1) \cap \lambda(J_2) \), then \( \theta \in W_{J_1} \cap W_{J_2} \). So \( (J, W_J) = (J_1, W_{J_1}) \wedge (J_2, W_{J_2}) = (J, W_J) \wedge (J_2, W_{J_2}) = (J, W_J) \). So \( \theta \in W_J \) and hence \( \theta \in \lambda(J) \). This proves (i) and (ii).

Next let \( \theta \in \Gamma \lambda(J) \), \( (J, W_J) \wedge (J, W_J) = (J_1, W_{J_1}) \). So \( J_1 \leq J, \theta \in W_J \). So \( \theta \in W_J \). Since \( \theta \notin \lambda(J) \), \( \theta \in \lambda(J_1) \). So \( (J, W_J) \geq (J_1, W_{J_1}) \), \( (J, W_J) \geq (J_1, W_{J_1}) \). So \( (J_1, W_{J_1}) \geq (J_1, W_{J_1}) \) whereby \( W_{J_1} = W_{J_1} \). Let \( J_2 \in \mathcal{U} \) with \( \theta \in \lambda(J_2) \). Then \( (J, W_J) \geq (J_2, W_{J_2}) \). So \( (J_1, W_{J_1}) \geq (J_2, W_{J_2}) \). This proves (iii).

Finally we prove (iv). We can assume that \( X \notin \lambda(J_1), X \notin \lambda(J_2) \). So \( X = \{\theta, \pi\} \) with \( \pi \in \lambda(J_1) \setminus \lambda(J_2) \). Then \( \theta \pi = \pi \theta \). Let \( J = \max\{J_3\} J_3 \leq
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Then as above \((J_1, W_{J_1}) \land (J_1, W_{J_1} \theta) = (J_3, W_{J_3})\). Since \(\pi \in \lambda(J_1)\), \((J_1, W_{J_1}) \geq (J_2, W_{J_2} \pi)\). Since \(\theta \pi = \pi \theta\), \(W_{J_1} \theta \cap W_{J_2} \pi \neq \emptyset\). So \((J_1, W_{J_1} \theta) \geq (J_2, W_{J_2} \pi)\), whence \((J, W_J) \geq (J_2, W_{J_2} \pi)\). Hence \(J \geq J_2\) and \(\pi \in W_{J_2} W_J\). Since \(\pi \notin \lambda(J_2)\), \(\pi \in \lambda(J)\). So \(\theta, \pi \in \lambda(J)\).

Conversely assume that (i), (ii), (iii) and (iv) are valid. First we claim that for any \(J \in \mathcal{L}, X \subseteq \Gamma,\)

\[
\text{max}\{J_1 \in \mathcal{L} | J_1 \leq J, X \subseteq \lambda(J_1)\} \text{ exists.}
\]

We prove this by induction on \(|X|\). If \(X \subseteq \lambda(J)\), there is nothing to prove. Otherwise there exists \(\theta \in X \setminus \lambda(J)\). By (iii), \(J_0 = \max\{J_1 \leq J | \theta \in \lambda(J_1)\}\) exists. By the induction hypothesis, \(J_2 = \max\{J_1 \leq J_0 | X \setminus \{\theta\} \subseteq \lambda(J_1)\}\) exists. Now \(J_2 \leq J_0 \leq J, \theta \in \lambda(J_0), \theta \notin \lambda(J)\). So by transitivity \(\theta \in \lambda(J_2)\). So \(X \subseteq \lambda(J_2)\). Now let \(J_1 \leq J\) such that \(X \subseteq \lambda(J_1)\). Then \(\theta \in \lambda(J_1)\). So \(J_1 \leq J_0\) and then \(J_1 \leq J_2\). Hence (3) holds.

Next we claim that if \(J_1, J_2 \in \mathcal{L}, J_2 \leq J_1, \sigma_1, \ldots, \sigma_k \in \lambda(J_2), \pi \in \lambda(J_1),\) then \(\pi \sigma_i = \sigma_i \pi, i = 1, \ldots, k,\) implies that there exists \(J \in \mathcal{L}\) with

\[
J_2 \leq J \leq J_1, \quad \pi, \sigma_1, \ldots, \sigma_k \in \lambda(J).
\]

We prove this by induction on \(k\). If \(\sigma_i \in \lambda(J_1)\) for all \(i\), there is nothing to prove. So assume \(\sigma_i \notin \lambda(J_1)\). By condition (iv), there exists \(J_3 \in \mathcal{L}, J_2 \leq J_3 \leq J_1\) such that \(\pi, \sigma_i \in \lambda(J_3)\). By the induction hypothesis, there exists \(J \in \mathcal{L}, J_2 \leq J \leq J_3\) such that \(\pi, \sigma_2, \ldots, \sigma_k \in \lambda(J)\). Now \(J \leq J_3 \leq J_1\), \(\sigma_1 \in \lambda(J_3), \sigma_1 \notin \lambda(J_1)\). So by transitivity \(\sigma_1 \in \lambda(J)\). So \(\pi, \sigma_1, \ldots, \sigma_k \in \lambda(J)\). This proves (4).

Let \((J_1, W_{J_1} \sigma_1), (J_2, W_{J_2} \sigma_2) \in \mathcal{W}(\lambda)\). We need to show that \((J_1, W_{J_1} \sigma_1) \land (J_2, W_{J_2} \sigma_2)\) exists in \(\mathcal{W}(\lambda)\). If \(\pi \in W\), then \((J, W_J \sigma) \rightarrow (J, W_J \sigma \pi)\) is an automorphism of \(\mathcal{W}(\lambda)\). For this reason we need only show that \((J_1, W_{J_1}) \land (J_2, W_{J_2})\) exists where \(\sigma \in W\) is such that it is an element of minimum length in \(W_{J_2} \sigma W_{J_1}\). Then by the exchange condition \(l(\delta \sigma) = l(\delta) + l(\sigma), l(\pi \gamma) = l(\pi) + l(\gamma)\) for all \(\delta \in W_{J_2}, \gamma \in W_{J_1}\). There exists a maximum \(J_3 \leq J_1 \land J_2\) such that \(\sigma \in W_{J_1}\). We claim that \((J_1, W_{J_1}) \land (J_2, W_{J_2}) = (J_3, W_{J_3})\). So let \((J_1, W_{J_1}) \geq (J_4, W_{J_4}), (J_2, W_{J_2}) \geq (J_4, W_{J_4})\). We can assume that \(\alpha\) is of minimum length in \(W_{J_2} \alpha\). Now \(\alpha \in W_{J_2} W_{J_1}\) and hence \(\alpha \in W_{J_1}\). Also \(\sigma \in W_{J_2} W_{J_4} \alpha \subseteq W_{J_2} W_{J_4} W_{J_1}\), hence \(\sigma \in W_{J_4}\). Therefore \(J_4 \leq J_3\). There exists \(u \in W_{J_4}, v \in W_{J_2}\) such that \(u \alpha = v \sigma\). So \(u = v \sigma \alpha^{-1} \in W_{J_2} W_{J_1} W_{J_4}\) by the exchange condition \(u = abc\) for some \(a \in W_{J_1} \cap W_{J_4}, b \in W_{J_2} \cap W_{J_4}, c \in W_{J_1} \cap W_{J_4}\). Now

\[
(J_3, W_{J_3} \sigma \alpha) \geq (J_4, W_{J_4} \sigma \alpha) = (J_4, W_{J_4} \alpha).
\]
Also,

\[(J_3, W_{J_3}c\alpha) = (J_3, W_{J_3}bca) \leq (J_2, W_{J_2}bca)\]

\[(6) = (J_2, W_{J_2}abca) = (J_2, W_{J_2}u\alpha) = (J_2, W_{J_2}v\sigma)\]

\[(J_2, W_{J_2}\sigma).\]

Moreover

\[(7) = (J_3, W_{J_3}c\alpha) \leq (J_1, W_{J_1}c\alpha) = (J_1, W_{J_1})\]

By (5), (6) and (7), it is clear that without loss of generality we can assume that \(J_3 = J_4\).

First we consider the case \(J_1 = J_2\). We assume \(\alpha \notin W_{J_3}\) and obtain a contradiction. Let \(l(\sigma) = k, \sigma = \sigma_1 \cdots \sigma_k, \sigma_i \in \Gamma\). Then \(\sigma_1, \ldots, \sigma_k \in \lambda(J_3), \alpha \in W_{J_1}\). By Lemma 1 there exists \(\theta \in \lambda(J_1) \lambda(J_3)\) such that \(\theta \sigma_i = \sigma_i \theta, i = 1, \ldots, k\). By (4), there exists \(J \in \mathcal{U}, J_3 \leq J \leq J_1\) such that \(\sigma_1, \ldots, \sigma_k, \theta \in \lambda(J)\). So \(\sigma \in W_J\) and \(J = J_3\). So \(\theta \in \lambda(J_3)\), a contradiction.

Next we consider the case where \(J_1 \geq J_2\). Since \(\alpha \in W_{J_1}\), we have \((J_3, W_{J_3}\alpha) \leq (J_2, W_{J_2}\alpha) \leq (J_1, W_{J_1})\). Also \(\alpha \in W_{J_3} W_{J_2} \sigma \subseteq W_{J_1} W_{J_2} W_{J_3}\). Since \(\alpha\) is of minimum length in \(W_{J_3}\alpha, \alpha \in W_{J_2} W_{J_3}\). So \((J_3, W_{J_3}) \leq (J_2, W_{J_2}\alpha)\).

By the above, \((J_2, W_{J_2}\sigma) \land (J_2, W_{J_2}\alpha) = (J_0, W_{J_0}\beta)\) exists. Then \((J_0, W_{J_0}\beta) \leq (J_2, W_{J_2}\sigma), (J_0, W_{J_0}\beta) \leq (J_1, W_{J_1})\). So as before \(\sigma \in W_{J_0}\). Hence \(J_3 \geq J_0\). But \((J_0, W_{J_0}\beta) \geq (J_3, W_{J_3})\) and \((J_0, W_{J_0}\beta) \geq (J_3, W_{J_3}\alpha)\) so \(J_3 = J_0\) and \(W_{J_3} = W_{J_3}\beta = W_{J_3}\alpha\).

Finally we consider the general case. Now \(\alpha \in W_{J_3} W_{J_2} W_{J_3}\). Since \(\alpha\) is of minimum length in \(W_{J_3}\alpha, \alpha \in W_{J_2} W_{J_3}\). Since also \(\alpha \in W_{J_1}\) we see by the exchange condition that \(\alpha = ab\) for some \(a \in W_{J_1} \cap W_{J_2}, b \in W_{J_1} \cap W_{J_3}\). Let \(J = J_1 \land J_2\). Then \(a \in W_J\) by (ii). Now \(W_{J_1} \cap W_{J_2} \subseteq W_J\). So \((J, W_{J_1}\alpha) = (J, W_{J_2} b) \leq (J_1, W_{J_1})\). Also \((J, W_{J_2}\alpha) = (J, W_{J_2} b) \geq (J_3, W_{J_3}), (J_3, W_{J_3}\alpha)\).

Since \(J_2 \geq J\), we see by the above that \((J_2, W_{J_2}\sigma) \land (J, W_{J_2}\alpha) = (J_0, W_{J_0}\beta)\) exists. Then \((J_0, W_{J_0}\beta) \leq (J_1, W_{J_1}), (J_2, W_{J_2}\sigma)\). So as above \(\sigma \in W_{J_0}\). Hence \(J_0 \leq J_3\). But \((J_0, W_{J_0}\beta) \geq (J_3, W_{J_3}), (J_3, W_{J_3}\alpha)\). So \(J_0 = J_3\) and \(W_{J_3} = W_{J_3}\beta = W_{J_3} = W_{J_3}\alpha\).

This completes the proof of sufficiency.

**Corollary 1.** If \(\mathcal{U}\) is a finite linearly ordered set, then a transitive map \(\lambda\) is regular if and only if for all \(J_1, J_2 \in \mathcal{U}, X\) a two element discrete subset of \(\lambda(J_1) \cup \lambda(J_2), X \subseteq \lambda(J)\) for some \(J\) between \(J_1\) and \(J_2\).

If \(\lambda: \mathcal{U} \to 2^\Gamma, X \subseteq \Gamma\), then let \(\lambda_X: \mathcal{U} \to 2^X\) where for \(J \in \mathcal{U}, \lambda_X(J) = \lambda(J) \cap X\).

**Corollary 2.** Let \(\mathcal{U}\) be a partially ordered set with a maximum element 1 and let \(\lambda: \mathcal{U} \to 2^\Gamma\) be such that \(\lambda(1) = \Gamma\). Then \(\lambda\) is transitive (respectively
regular) if and only if \( \lambda_X \) is transitive (respectively regular) for all rank \( \leq 2 \) subgraphs \( X \) of \( \Gamma \).

In [1] a universal transitive map \( u: U(\Gamma) \rightarrow 2^\Gamma \) was constructed. It has the property that for any transitive map \( \lambda: \mathcal{Z} \rightarrow 2^\Gamma \), there is an order preserving map \( \gamma: \mathcal{Z} \rightarrow U(\Gamma) \) such that \( \lambda = u \circ \gamma \). The partially ordered set \( U = U(\Gamma) \) was constructed as follows:

\[
U = U(\Gamma) = \{(A, B)|A, B \in 2^\Gamma, A \cap B = \emptyset \}
\]

and each connected component of \( A \cup B \) is either contained in \( A \) or contained in \( B \).

For \((A, B), (A', B') \in U\) we define \((A, B) \leq (A', B')\) if \( A \subseteq A' \) and \( B' \subseteq B \). Then \((U, \leq)\) is a distributive lattice with \((A, B) \vee (A', B') = (A \cup B, B \cap B')\) and \((A, B) \wedge (A', B') = (A \cap A', B \cup B')\).

**Corollary 3.** The map \( u: U(\Gamma) \rightarrow 2^\Gamma \), where \( u(A, B) = A \cup B \), is regular.

**Proof.** Clearly \( U(\Gamma) \) is a \( \wedge \)-semilattice. Let \( J_1 = (A_1, B_1), J_2 = (A_2, B_2) \in U(\Gamma) \). Then

\[
u(J_1) \cap u(J_2) = (A_1 \cup B_1) \cap (A_2 \cup B_2)
= (A_1 \cap A_2) \cup (A_1 \cap B_2) \cup (B_1 \cap A_2) \cup (B_1 \cap B_2)
\subseteq (A_1 \cap A_2) \cup B_1 \cup B_2
= u(J_1 \wedge J_2).
\]

Take any \( J = (A, B) \in U(\Gamma) \) and \( \theta \in \Gamma \). Then \( \max\{J' \in U(\Gamma)|J' \leq J, \theta \in u(J')\} = \bigvee\{J' \in U(\Gamma)|J' \leq J, \theta \in u(J')\} \) exists since \( U(\Gamma) \) is a finite lattice.

Let \( J_1 = (A_1, B_1) \geq J_2 = (A_2, B_2) \) and \( X \) be a 2-element discrete subset of \( \Gamma \) such that

\[
X \subseteq u(J_1) \cup u(J_2) = (A_1 \cup B_1) \cup (A_2 \cup B_2) = A_1 \cup B_2.
\]

Then \( X = (X \cap A_1) \cup (X \setminus A_1) \) with \( X \setminus A_1 \subseteq B_2 \). Take \( J = (C, D) \) where \( C = A_2 \cup (X \setminus A_1), D = B_1 \cup (X \setminus A_1) \). Then \( C \cap D = \emptyset \). Now \( B_1 \subseteq B_2 \) and \( X \) is discrete. So every connected component of \( C \cup D \) is contained in \( C \) or in \( D \). Thus \( J \in U(\Gamma) \). Also \( J_1 \geq J \geq J_2 \) and \( X \subseteq u(J) \). This completes the proof.
References


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