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ISOMORPHISMS BETWEEN GENERALIZED CARTAN TYPE *W* LIE ALGEBRAS IN CHARACTERISTIC 0

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ABSTRACT. In this paper, we determine when two simple generalized Cartan type W Lie algebras $W_d(A, T, \varphi)$ are isomorphic, and discuss the relationship between the Jacobian conjecture and the generalized Cartan type W Lie algebras.

1. **Introduction.** This paper is a sequel to the papers [1] and [2] in which generalized Cartan type W Lie algebras $W_d(A, T, \varphi)$ over a field F of characteristic 0 were studied. We have tried to make this paper independent of [1] and [2], and so, in Section 2, we give a short description of general Lie algebras, generalized Witt algebras, generalized Cartan type W Lie algebras, and recall some basic facts about them. In Section 3, we show that every isomorphism between two simple generalized Cartan type W Lie algebras arises from an isomorphism between the two associative algebras corresponding to the two Lie algebras, and, determine when two of the simple Lie algebras $W_d(A, T, \varphi)$ are isomorphic. In Section 4, we pose a conjecture on the general Lie algebras, and prove that the validity of this conjecture implies the validity of the Jacobian conjecture.

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2. **Generalized Cartan type *W* Lie algebras.** In this section, for the convenience of the reader, we recall the definition of general Lie algebras, generalized Witt algebras, generalized Cartan type W Lie algebras and some basic facts concerning them. For more details we refer the reader to the papers [1] and [2].

Let n be a positive integer, and $t_1, \ldots, t_n$ independent and commuting indeterminates over F. Denote by P_n and Q_n the polynomial algebra $F[t_1, \ldots, t_n]$, and the Laurent polynomial algebra $F[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ respectively. By $W_n = W_n(F)$ we denote the *Witt algebra*, *i.e.*, the Lie algebra of all formal vector fields

$$\sum_{i=1}^{n} f_i \frac{\partial}{\partial t_i} \tag{2.1}$$

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with coefficients $f_i \in Q_n$. The bracket in W_n is

$$\Big[f\frac{\partial}{\partial t_i}, g\frac{\partial}{\partial t_j}\Big] = f\frac{\partial(g)}{\partial t_i}\frac{\partial}{\partial t_j} - g\frac{\partial(f)}{\partial t_j}\frac{\partial}{\partial t_i},$$

where $f, g \in Q_n$, and $i, j \in \{1, 2, \ldots, n\}$. The subalgebra $W_n^+ = W_n^+(F)$ of W_n consisting of all vector fields (2.1) with polynomial coefficients, *i.e.*, $f_i \in P_n$, is known as *the general Lie algebra*, or *the Lie algebra of Cartan type W*. (There are also topologized versions of W_n and W_n^+ where the coefficients f_i are formal Laurent and power series in $t_1, \ldots, t_n$, respectively, and F is the real or complex field. For more details, please refer [10]). It is well known that W_n and W_n^+ are simple Lie algebras.

For any ring R, we can similarly define the Lie algebra $W_n^+(R)$ over R.

Let A be an abelian group, F a field of characteristic 0, and T a vector space over F. We denote by FA the group algebra of A over F. The elements t^x, $x \in A$, form a basis of this algebra, and the multiplication is defined by $t^x \cdot t^y = t^{x+y}$. We shall write 1 instead of t^0. The tensor product $W = FA \otimes_F T$ is a free left FA-module. We denote an arbitrary element of T by ∂ (to remind us of differential operators). For the sake of simplicity, we shall write $t^x\partial$ instead of $t^x \otimes \partial$. We now choose a pairing $\varphi\colon T \times A \longrightarrow F$ which is F-linear in the first variable and additive in the second one. For convenience we shall also use the following notations:

$$\varphi(\partial, x) = \langle \partial, x \rangle = \partial(x)$$

for arbitrary $\partial \in T$ and $x \in A$.

There is a unique F-bilinear map $W \times W \longrightarrow W$ sending $(t^x\partial_1, t^y\partial_2)$ to

$$[t^x\partial_1, t^y\partial_2] := t^{x+y}\big(\partial_1(y)\partial_2 - \partial_2(x)\partial_1\big), \tag{2.2}$$

for arbitrary $x, y \in A$ and $\partial_1, \partial_2 \in T$. It is easy to verify that this map makes W into a Lie algebra. We refer to this algebra $W = W(A, T, \varphi)$ as a *generalized Witt algebra*.

The subspaces $W_x = t^x T$, $x \in A$, define an A-gradation of W, *i.e.*, W is the direct sum of the W_x's, and $[W_x, W_y] \subset W_{x+y}$ for all $x, y \in A$.

It follows from (2.2) that $\mathrm{ad}(\partial)$ acts on W_x as a scalar $\partial(x)$. Hence each $\partial \in T$ is ad-semisimple, and T is a torus (*i.e.*, an abelian subalgebra consisting of ad-semisimple elements). In fact T is the only maximal torus of W (see [1, Lemma 4.1]).

The following theorem is due to Kawamoto [5].

THEOREM 2.1. *Suppose that the characteristic of F is 0. The Lie algebra $W = W(A, T, \varphi)$ is simple if and only if $A \neq 0$ and φ is nondegenerate in the sense that the following conditions hold:*

$$\langle \partial, x \rangle = 0, \quad \forall \partial \in T \Rightarrow x = 0 \tag{2.3}$$

and

$$\langle \partial, x \rangle = 0, \quad \forall x \in A \Rightarrow \partial = 0 \tag{2.4}$$

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Note that (2.3) implies that A is torsion free. This implies that FA is an integral domain and it implies that the invertible elements of FA have the form at^x, where $a \in F^*$, $x \in A$.

As mentioned earlier, W is a free left FA-module. There is also a natural structure of a left W-module on FA, namely the structure is such that

$$t^x\partial \cdot t^y = \partial(y)t^{x+y} \tag{2.5}$$

for $x, y \in A$ and $\partial \in T$. These two module structures are related by the identity

$$[fu, gv] = f(u \cdot g)v - g(v \cdot f)u + fg[u, v] \tag{2.6}$$

where $f, g \in FA$ and $u, v \in W$ are arbitrary. The W-module structure on FA gives rise to a homomorphism

$$W \longrightarrow \mathrm{Der}(FA) \tag{2.7}$$

because each $w \in W$ acts on FA as a derivation. Clearly (2.7) is also a homomorphism of FA-modules.

Suppose that $W = W(A, T, \varphi)$ denotes a simple generalized Witt algebra over a field F of characteristic 0. Let I be an index set, $d \colon I \longrightarrow T$ an injective map, and write $d_i = d(i)$ for $i \in I$. We say that d is *admissible* if the following two conditions hold:
(Ind) $d_i, i \in I$, are linearly independent;
(Int) $d_i(A) = \mathbf{Z}$ for all $i \in I$.

We assume throughout that an admissible d has been fixed. We set

$$\begin{aligned}
A_d^+ &= \{x \in A : d_i(x) \geq 0, \forall i \in I\},\\
A_d^0 &= \{x \in A : d_i(x) = 0, \forall i \in I\},\\
A_{d,i} &= \big\{x \in A : d_i(x) = -1; d_j(x) \geq 0, \forall j \in I \setminus \{i\}\big\},\\
A_{d,i}^{\#} &= \big\{x \in A : d_i(x) = -1; d_j(x) = 0, \forall j \in I \setminus \{i\}\big\},\\
A_d &= A_d^+ \cup \Big(\bigcup_{i \in I} A_{d,i}\Big).
\end{aligned}$$

We now introduce some subspaces of W:

$$W_d^+ = \sum_{x \in A_d^+} W_x;$$

$$W_{d,i} = \Big(\sum_{x \in A_{d,i}} Ft^x\Big)d_i, \quad i \in I;$$

and

$$W_d = W_d(A, T, \varphi) = W_d^+ + \sum_{i \in I} W_{d,i}.$$

In fact all of these subspaces are subalgebras of W.

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We also introduce the subalgebra FA_d^+ of FA, which is the span of all elements t^x with $x \in A_d^+$. Since W is a left FA-module, we can view W also as a left FA_d^+-module. Then it is easy to see that the subspaces W_d^+ and W_d are FA_d^+-submodules of W.

Let $x \in A_{d,i}$ and $y \in A_d^+$. Then either $x + y \in A_d^+$ or $x + y \in A_{d,i}$ and $d_i(y) = 0$. In both cases we have

$$t^x d_i \cdot t^y = d_i(y)t^{x+y} \in FA_d^+.$$

Hence, by restricting the action of W on FA, we can view FA as a left W_d-module, and then FA_d^+ is a W_d-submodule of FA.

When d is fixed, and there is no danger of confusion, we shall write

$$A^+, \quad A_i, \quad A_i^\#, \quad W^+, \quad W_i, FA^+$$

instead of

$$A_d^+, \quad A_{d,i}, \quad A_{d,i}^\#, \quad W_d^+, \quad W_{d,i}, \quad FA_d^+,$$

respectively.

The following theorem is proved in [2].

THEOREM 2.2. *The Lie algebra W_d is simple if and only if the following conditions hold:*

(i) *if $\partial \in T$ and $\partial(x) = 0$ for all $x \in A_d$, then $\partial = 0$;*

(ii) *if $x \in A_d$, then $d_i(x) = 0$ for almost all $i \in I$;*

(iii) *$A_i^\# \neq \emptyset$ for all $i \in I$.* ∎

From now on (throughout the paper) we shall assume that W_d is simple, and in that case W_d is called an algebra of *generalized Cartan type W*. For more details on the Lie algebra W_d, please refer to the papers [1] and [7].

We conclude this section by a known lemma (and one of its corollaries), which follows directly from [9, Theorem 5.8]. It would not be strange if this lemma was known before [9].

LEMMA 2.3. *Suppose F is a field of characteristic 0. Then $W_n^+(F) \simeq W_m^+(F)$ if and only if $m = n$.* ∎

Note that in this lemma m and n can be infinity.

COROLLARY 2.4. *Suppose R is a domain of characteristic 0. Then $W_n^+(R) \simeq W_m^+(R)$ if and only if $m = n$.*

PROOF. ($\Rightarrow$) Denote the quotient field of R by $\bar{R}$. Then $W_n^+(\bar{R}) = \bar{R} \otimes_R W_n^+(R)$ and $W_m^+(\bar{R}) = \bar{R} \otimes_R W_m^+(R)$. It follows from $W_n^+(R) \simeq W_m^+(R)$ that $W_n^+(\bar{R}) \simeq W_m^+(\bar{R})$. By the above lemma, we know that $m = n$.

($\Leftarrow$) This direction is clear. ∎

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3**. Isomorphisms between generalized Cartan type W Lie algebras.** In this section, we shall mainly determine the necessary and sufficient conditions under which two generalized Cartan type W Lie algebras are isomorphic.

Consider two simple generalized Cartan type W Lie algebras $L = W_d(A, T, \varphi)$ and $L' = W_{d'}(A', T', \varphi')$. From results in [2] we know that $I = \emptyset$ if and only if $I' = \emptyset$.

If $I = I' = \emptyset$, then $L = W(A, T, \varphi)$ and $L' = W(A', T', \varphi')$ are simple generalized Witt algebras. The isomorphisms between L and L' have been completely determined in [1]. Therefore, from now on in this section, we may assume that $I \neq \emptyset$, $I' \neq \emptyset$ and $\dim T \geq \dim T' \geq 1$.

Suppose $\theta\colon L \longrightarrow L'$ is an isomorphism of Lie algebras. If $\dim T = 1$, then $\dim T' = 1$. Hence $L \simeq W_1^+ \simeq L'$. Therefore we may assume that $\dim T > 1$.

For $x \in A$, let

$$\mathcal{F}_x = \{f \in F{A'}^+_{d'} : \theta(\partial)\cdot f = \partial(x)f, \forall \partial \in T\},$$

and let

$$P = \{x \in A : \mathcal{F}_x \neq 0\}.$$

Since $W_{d'} \cdot F = 0$, we have $F \subset \mathcal{F}_0$ and so $0 \in P$.

We mention here that the statements and the proofs of Lemma 3.1 to Lemma 3.4 are similar to that of Lemma 5.1 to Lemma 5.4 in [2], which were used to determine the automorphisms of the simple Lie algebra $W_d(A, T, \varphi)$.

LEMMA 3.1. $F{A'}^+_{d'} = \bigoplus_{x\in A} \mathcal{F}_x$.

PROOF. It suffices to show that the union of all $\mathcal{F}_x$, $x \in A$, spans FA^+_d. Let $f \in F{A'}^+_{d'}$, $f \neq 0$, and choose $\partial_0 \in T$, $\partial_0 \neq 0$. Since $f\theta(\partial_0) \in L' = W_{d'}$, we have

$$\theta^{-1}\big(f\theta(\partial_0)\big) = \sum_{i=1}^{n} t^{x_i}\partial_i, \tag{3.1}$$

where $x_i \in A_d$ are distinct and $\partial_i \in T$ are nonzero. By applying $\operatorname{ad}\partial$, $\partial \in T$, to both sides of (3.1), we obtain that

$$\big(\theta(\partial)\cdot f\big)\theta(\partial_0) = \sum_{i=1}^{n} \partial(x_i)\theta(t^{x_i}\partial_i),$$

and similarly

$$\big(\theta(\partial)^k\cdot f\big)\theta(\partial_0) = \sum_{i=1}^{n} \partial(x_i)^k\theta(t^{x_i}\partial_i), \quad k \geq 0. \tag{3.2}$$

By choosing ∂ such that $\partial(x_i)$ are distinct for $i = 1, \ldots, n$ and by taking $k = 0, 1, \ldots, n-1$ in (3.2), we obtain a system of linear equations to which Cramer's rule can be applied. We conclude that there exist $f_1, \ldots, f_n \in F{A'}^+_{d'}$ Such that

$$\theta(t^{x_i}\partial_i) = f_i\theta(\partial_0), \quad i = 1, \ldots, n. \tag{3.3}$$

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From (3.1) and (3.3) we deduce that

$$f = f_1 + \cdots + f_n. \tag{3.4}$$

By applying $\operatorname{ad}\theta(\partial')$ to both sides of (3.3), we obtain that

$$\theta(\partial')\cdot f_i = \partial'(x_i)f_i, \quad \forall \partial' \in T,$$

i.e., $f_i \in \mathcal{F}_{x_i}$. Hence (3.4) shows that f belongs to the sum of the $\mathcal{F}_x$, $x \in A$. ∎

Since $W_{d'}$ is simple and $W_{d'} \cdot FA'^{+}_{d'} \neq 0$, it follows that $\theta(T)\cdot FA'^{+}_{d'} \neq 0$. By using Lemma 3.1, we conclude that $P \neq \{0\}$.

LEMMA 3.2. *We have $P \subset A_d^+$ and $\dim \mathcal{F}_x = 1$ for all $x \in P$. Furthermore, if a nonzero $f \in \mathcal{F}_x$ is fixed, then for every $\partial \in T$ there exists a unique $\tilde{\partial} \in T$ such that*

$$f\theta(\partial) = \theta(t^x\tilde{\partial}). \tag{3.5}$$

PROOF. Let $x \in P$ and let $f, g \in \mathcal{F}_x$ be both nonzero. For arbitrary $\partial, \partial' \in T$ we have

$$[\theta(\partial'), f\theta(\partial)] = \big(\theta(\partial')\cdot f\big)\theta(\partial) = \partial'(x)f\theta(\partial),$$

and so

$$\theta^{-1}\big(f\theta(\partial)\big) \in W_d \cap W_x = W_d \cap t^xT. \tag{3.6}$$

In particular $W_d \cap W_x \neq 0$, and so $x \in A_d$. Hence $P \subset A_d$. Since $P + P \subset P$, we must have $P \subset A_d^+$. Note that (3.6) implies (3.5).

It remains to show that $\dim \mathcal{F}_x = 1$. By replacing f with g, we see that for each $\partial \in T$ there exists a unique $\hat{\partial} \in T$ such that

$$g\theta(\partial) = \theta(t^x\hat{\partial}). \tag{3.7}$$

For arbitrary $\partial, \partial' \in T$ we have

$$\begin{aligned} \theta\Big(t^{2x}\big(\tilde{\partial}(x)\hat{\partial}' - \hat{\partial}'(x)\tilde{\partial}\big)\Big) &= \theta([t^x\tilde{\partial}, t^x\hat{\partial}']) \\ &= [\theta(t^x\tilde{\partial}), \theta(t^x\hat{\partial}')] \\ &= [f\theta(\partial), g\theta(\partial')] \\ &= f\big(\theta(\partial)\cdot g\big)\theta(\partial') - g\big(\theta(\partial')\cdot f\big)\theta(\partial) \\ &= fg\big(\partial(x)\theta(\partial') - \partial'(x)\theta(\partial)\big). \end{aligned} \tag{3.8}$$

Since $\dim T > 1$, we can choose $\partial, \partial' \in T$ such that $\partial(x) \neq 0$, $\partial'(x) = 0$, and $\partial' \neq 0$. Then the right hand side of (3.8) is not 0, and so $\tilde{\partial}(x) \neq 0$ or $\hat{\partial}'(x) \neq 0$. Hence we have shown that there exists $\partial_1 \in T$ such that $\tilde{\partial}_1(x) \neq 0$ or $\hat{\partial}_1(x) \neq 0$. By replacing ∂ and ∂' in (3.8) with ∂_1 we infer that $\tilde{\partial}_1$ and $\hat{\partial}_1$ are linearly dependent. Hence f and g are linearly dependent and so $\dim \mathcal{F}_x = 1$. ∎

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Note that Lemma 3.2 implies that $\mathcal{F}_0 = F$.

Assume that $x \in P$ and $f \in \mathcal{F}_x \setminus \{0\}$. For $\partial \in T$ let $\tilde{\partial} \in T$ be such that (3.5) holds. Then for $t^y\partial' \in W_d$ we have

$$\theta([t^y\partial', t^x\tilde{\partial}]) = \theta\Big(t^{x+y}\big(\partial'(x)\tilde{\partial} - \tilde{\partial}(y)\partial'\big)\Big)$$

and

$$[\theta(t^y\partial'), \theta(t^x\tilde{\partial})] = [\theta(t^y\partial'), f\theta(\partial)] = \big(\theta(t^y\partial')\cdot f\big)\theta(\partial) + f\theta([t^y\partial', \partial]).$$

It follows that

$$\theta\Big(t^{x+y}\big(\partial'(x)\tilde{\partial} - \tilde{\partial}(y)\partial'\big)\Big) = \big(\theta(t^y\partial')\cdot f\big)\theta(\partial) - \partial(y)f\theta(t^y\partial'). \tag{3.9}$$

LEMMA 3.3. *We have $P = A_d^+$.*

PROOF. In view of Lemma 3.2, it suffices to show that $A_d^+ \subset P$. We claim first that

$$x + A_d^+ \subset P, \tag{3.10}$$

for all $x \in P \setminus \{0\}$. We fix a nonzero $f \in \mathcal{F}_x$. Let $y \in A_d^+$. Since $\dim T > 1$, we can choose $\partial \neq 0$ such that $\partial(y) = 0$, and ∂' such that ∂' and $\tilde{\partial}$ are linearly independent and $\partial'(x) \neq 0$. Then (3.9) gives that

$$\theta\Big(t^{x+y}\big(\partial'(x)\tilde{\partial} - \tilde{\partial}(y)\partial'\big)\Big) = \big(\theta(t^y\partial')\cdot f\big)\theta(\partial) \neq 0,$$

and so $\theta(t^y\partial')\cdot f \in \mathcal{F}_{x+y} \setminus \{0\}$. Hence (3.10) holds.

Fix an $i \in I$ and $a_i \in A_i^\#$. We claim that

$$x \in P \,\&\, d_i(x) > 0 \Rightarrow x + a_i \in P. \tag{3.11}$$

Choose $\partial \in T \setminus \{0\}$ such that $\partial(a_i) = 0$. Then $d_i(x)\tilde{\partial} - \tilde{\partial}(a_i)d_i \neq 0$. By setting $y = a_i$ and $\partial' = d_i$ in (3.9), we obtain that

$$\theta(t^{x+a_i}(d_i(x)\tilde{\partial} - \tilde{\partial}(a_i)d_i) = \big(\theta(t^{a_i}d_i)\cdot f\big)\theta(\partial) \neq 0,$$

and so $\theta(t^{a_i}d_i)\cdot f \in \mathcal{F}_{x+a_i} \setminus \{0\}$. This proves our second claim.

Now let $y \in A_d^+ \setminus \{0\}$ be arbitrary. Let $x \in P$ be chosen so that $d_i(x) \geq d_i(y)$ for all $i \in I$ and

$$n = \sum_{i\in I} d_i(x)$$

is minimal. (It follows from (3.10) that such x exists.) Assume that there exists an $i \in I$ such that $d_i(x) > d_i(y)$. By (3.11) we have $x + a_i \in P$, which contradicts the choice of x. So $d_i(x) = d_i(y)$ for all $i \in I$. Then $y - x \in A_d^*$. By (3.10) it follows that $y = x + (y - x) \in P$.

For $\partial \in T$ let $K(\partial) = \{x \in A_d^+ : \partial(x) = 0\}$.

LEMMA 3.4. *For each $x \in A_d^+$ there exists a unique $f_x \in \mathcal{F}_x$ such that $f_x\theta(\partial) = \theta(t^x\partial)$ holds for all $\partial \in T$.*

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PROOF. For any $z \in A_d^+$, we fix a nonzero element $\bar{f}_z \in F_z$. Since $F_x F_y = F_{x+y}$ for any $x, y \in A_d^+$, there exists $a_{x,y} \in F^*$ such that $\bar{f}_x \bar{f}_y = a_{x,y} \bar{f}_{x+y}$ for any $x, y \in A_d^+$. For any $z \in A_d^+$, we define a linear map

$$\alpha_z : T \longrightarrow T, \quad \partial \longrightarrow \alpha_z(\partial),$$

where $\alpha_z(\partial)$ is defined by $\bar{f}_z \theta(\partial) = \theta\big(t^z \alpha_z(\partial)\big)$ (see (3.5)). By Lemma 3.2 we know that each α_z is injective.

Since $\theta^{-1}\big(\bar{f}_x \theta(\partial)\big) = t^x \alpha_x(\partial)$ and $\theta^{-1}\big(\bar{f}_y \theta(\partial')\big) = t^y \alpha_y(\partial')$, we deduce that

$$[t^x \alpha_x(\partial), t^y \alpha_y(\partial')] = t^{x+y}\big(\langle \alpha_x(\partial), y\rangle \alpha_y(\partial') - \langle \alpha_y(\partial'), x\rangle \alpha_x(\partial)\big),$$

and,

$$\begin{aligned} [t^x \alpha_x(\partial), t^y \alpha_y(\partial')] &= \theta^{-1}[\bar{f}_x \theta(\partial), \bar{f}_y \theta(\partial')] \\ &= \theta^{-1}\Big(\bar{f}_x \bar{f}_y \big(\partial(y)\theta(\partial') - \partial'(x)\theta(\partial)\big)\Big) \\ &= a_{x,y} t^{x+y}\big(\partial(y)\alpha_{x+y}(\partial') - \partial'(x)\alpha_{x+y}(\partial)\big). \end{aligned}$$

So we obtain that

$$\langle \alpha_x(\partial), y\rangle \alpha_y(\partial') - \langle \alpha_y(\partial'), x\rangle \alpha_x(\partial) = a_{x,y}\big(\partial(y)\alpha_{x+y}(\partial') - \partial'(x)\alpha_{x+y}(\partial)\big). \tag{3.12}$$

We claim that, for any $x, y \in A_d^+$, $\partial(y) = 0$ if and only if $\alpha_x(\partial)(y) = 0$.

This claim is clear if $\dim(T) = 1$. Next we suppose that $\dim(T) > 1$. For contradiction, we suppose that there exists a $\partial \in T$ such that $\partial(y) \neq 0$ and $\alpha_x(\partial)(y) = 0$, or, $\partial(y) = 0$ and $\alpha_x(\partial)(y) \neq 0$.

CASE 1. Suppose $\partial(y) \neq 0$ and $\alpha_x(\partial)(y) = 0$. By replacing y with $2y$ if necessary, we may assume that $\partial(x) \neq \partial(y)$. From (3.12) we obtain that

$$-\langle \alpha_y(\partial'), x\rangle \alpha_x(\partial) = a_{x,y}\big(\partial(y)\alpha_{x+y}(\partial') - \partial'(x)\alpha_{x+y}(\partial)\big),$$

i.e.,

$$a_{x,y}\partial(y)\alpha_{x+y}(\partial') = a_{x,y}\partial'(x)\alpha_{x+y}(\partial) - \langle \alpha_y(\partial'), x\rangle \alpha_x(\partial), \quad \forall \partial' \in T. \tag{3.12$'$}$$

By setting $\partial' = \partial$ in the above equation, we deduce that

$$\langle \alpha_y(\partial), x\rangle \alpha_x(\partial) = a_{x,y}\partial(x - y)\alpha_{x+y}(\partial) \neq 0.$$

Thus we know that $\alpha_{x+y}(\partial) \in F\alpha_x(\partial)$. In (3.12′) we see that the right hand side is always in $F\alpha_x(\partial)$ whatever ∂' is, but the left hand side is not in $F\alpha_x(\partial)$ for some ∂' since α_{x+y} is injective and $\dim(T) > 1$. This gives a contradiction.

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CASE 2. Suppose $\partial(y) = 0$ and $\alpha_x(\partial)(y) \neq 0$. From (3.12) we obtain

$$a_{x,y}\partial'(x)\alpha_{x+y}(\partial) = \langle \alpha_y(\partial'), x\rangle \alpha_x(\partial) - \langle \alpha_x(\partial), y\rangle \alpha_y(\partial').$$

Since $\dim(T) > 1$ we can choose nonzero element $\partial' \in T$ such that $\langle \alpha_y(\partial'), x\rangle = 0$. By Case 1 we know that $\partial'(x) = 0$. So it follows from the above equation that $\alpha_x(\partial)(y) = 0$, contrary to our hypothesis.

Hence our first claim is proved.

Let us fix $x, y \in A_d^+$ and $f \in F_x \setminus \{0\}$. By the above claim, we know that

$$K(\partial) = K\big(\alpha_x(\partial)\big), \quad \forall \partial \in T. \tag{3.13}$$

Let $\partial, \partial' \in T$ be arbitrary. Choose $a, b \in F$, not both zero, such that $(a\partial + b\partial')(y) = 0$. By applying (3.13) to $a\partial + b\partial'$ instead of ∂, we conclude that $\big(a\alpha_x(\partial) + b\alpha_x(\partial')\big)(y) = 0$. Hence

$$\begin{vmatrix} \partial(y) & \alpha_x(\partial)(y) \\ \partial'(y) & \alpha_x(\partial')(y) \end{vmatrix} = 0,$$

and consequently there exists $c(x, y, f) \in F^*$ such that

$$\alpha_x(\partial)(y) = c(x, y, f)\partial(y), \quad \forall \partial \in T. \tag{3.14}$$

We claim that $c(x, y, f)$ is independent of $y \in A_d^+ \setminus \{0\}$.

For any $z \in A$ let $\hat{z}: T \longrightarrow F$ be the linear function defined by $\hat{z}(\partial) = \partial(z)$. Since $\dim T > 1$, we can choose $z \in A_d^+$ such that $\hat{y}$ and $\hat{z}$ are linearly independent. In order to prove our claim, it suffices to show that $c(x, y, f) = c(x, z, f)$ when $\hat{y}$ and $\hat{z}$ are linearly independent. In that case we can choose $\partial_1, \partial_2 \in T$ such that

$$\partial_1(y) = \partial_2(z) = 0, \quad \partial_1(z) = \partial_2(y) = 1.$$

By (3.13) and (3.14), we have

$$\begin{aligned} c(x, y, f) &= \alpha_x(\partial_2)(y) = \alpha_x(\partial_2)(y + z) = c(x, y + z, f), \\ c(x, z, f) &= \alpha_x(\partial_1)(z) = \alpha_x(\partial_1)(y + z) = c(x, y + z, f), \end{aligned}$$

and so our second claim is proved.

We conclude that there is a constant $c(x, f) \in F^*$ such that

$$\alpha_x(\partial)(y) = c(x, f)\partial(y), \quad \forall \partial \in T, \ \forall y \in A_d^+.$$

Further, we can deduce that $\alpha_x(\partial)(y) = c(x, f)\partial(y)$, $\forall \partial \in T$, $\forall y \in A_d$. Then from Theorem 2.2(i) it follows that $\alpha_x(\partial) = c(x, f)\partial$ for all $\partial \in T$. If $f_x = c(x, f)^{-1} f$, then (3.5) implies that $f_x\theta(\partial) = \theta\big(t^x\partial\big)$ holds for all $\partial \in T$.

The uniqueness of f_x is obvious. ∎

The following theorem is one of our main results in this paper. In this theorem we do not need the restrictions on I and T.

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THEOREM 3.5. *Suppose that $L = W_d(A, T, \varphi)$ and $L' = W_{d'}(A', T', \varphi')$ are simple generalized Cartan type W Lie algebras. Then for any Lie algebra isomorphism $\theta\colon L \longrightarrow L'$, there exists a unique associative algebra isomorphism $\Psi_\theta\colon FA_d^+ \longrightarrow FA'^{+}_{d'}$ such that*

$$\theta(fw) = \Psi_\theta(f)\theta(w), \tag{3.15}$$

and

$$\Psi_\theta(w \cdot f) = \theta(w) \cdot \Psi_\theta(f) \tag{3.16}$$

hold for all $f \in FA_d^+$, and $w \in W_d$.

PROOF. Remember that we have assumed that $\dim T \geq \dim T'$.

CASE 1. $I = \emptyset$. We know that $I' = \emptyset$ also. Then $L = W(A, T, \varphi)$ and $L' = W(A', T', \varphi')$ are simple generalized Witt algebras. It follows from [1, Theorem 4.2] that there exists $\chi \in \mathrm{Hom}(A, F^*)$, an isomorphisms $\mu\colon A \longrightarrow A'$ and $\tau\colon T \longrightarrow T'$ satisfying

$$\partial(x) = \langle \tau\partial, \mu(x)\rangle, \quad \forall \partial \in T,\ x \in A$$

such that $\theta(t^x\partial) = \chi(x)t^{\mu(x)}\tau\partial$. Set $\Psi_\theta(t^x) = \chi(x)t^{\mu(x)}$. It is easy to verify that (3.15) and (3.16) are satisfied.

CASE 2. $I \neq \emptyset$ and $\dim T > 1$. Define the linear map $\sigma\colon FA_d^+ \longrightarrow FA'^{+}_{d'}$ by setting $\sigma(t^x) = f_x$, $x \in A_d^+$, where $f_x \in \mathcal{F}_x$ is defined as in Lemma 3.4. Hence we have

$$\theta(t^x\partial) = f_x\theta(\partial), \quad x \in A_d^+,\ \partial \in T. \tag{3.17}$$

As $f_x \neq 0$ for $x \in A_d^+$, Lemmas 3.1 and 3.3 imply that σ is bijective.

We claim that σ is an isomorphism of associative algebras, or equivalently that

$$f_xf_y = f_{x+y}, \quad \forall x, y \in A_d^+. \tag{3.18}$$

If $\partial, \partial' \in T$ then

$$[f_x\theta(\partial), f_y\theta(\partial')] = f_xf_y\big(\partial(y)\theta(\partial') - \partial'(x)\theta(\partial)\big)$$

and

$$[t^x\partial, t^y\partial'] = t^{x+y}\big(\partial(y)\partial' - \partial'(x)\partial\big).$$

By applying θ to the last equation and by using (3.17), we conclude that

$$(f_xf_y - f_{x+y})\big(\partial(y)\theta(\partial') - \partial'(x)\theta(\partial)\big) = 0.$$

Since $f_0 = 1$, (3.18) holds if $x = 0$ or $y = 0$. If $x \neq 0$ then we can choose linearly independent $\partial, \partial' \in T$ such that $\partial'(x) \neq 0$. Hence the above equation implies that (3.18) is valid.

We now claim that if $t^y\partial \in W_d$ and $x \in A_d^+$ then

$$\theta(t^y\partial) \cdot f_x = \partial(x)f_{x+y}. \tag{3.19}$$

Assume that $x + y \notin A_d^+$. Then $y, x + y \in A_i$ for some $i \in I$ and consequently $d_i(x) = 0$. Since $t^y\partial \in W_d$, we have $\partial \in Fd_i$, and so $\partial(x) = 0$. Although f_{x+y} is not defined when $x + y \notin A_d^+$, we should interpret $\partial(x)f_{x+y}$ as 0.

In order to prove (3.19), we consider two cases.

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SUBCASE 1. $y \in A_d^+$. Then $\theta(t^y\partial) = f_y\theta(\partial)$ and so

$$\theta(t^y\partial)\cdot f_x = f_y\theta(\partial)\cdot f_x = \partial(x)f_xf_y = \partial(x)f_{x+y}.$$

SUBCASE 2. $y \in A_i$ for some $i \in I$ and $\partial = d_i$. We apply formula (3.9) with $f = f_x$ and $\partial' = d_i$. Then $\tilde{\partial} = \partial$ by Lemma 3.4 and we obtain

$$\theta\Big(t^{x+y}\big(d_i(x)\partial - \partial(y)d_i\big)\Big) = \big(\theta(t^yd_i)\cdot f_x\big)\theta(\partial) - \partial(y)f_x\theta(t^yd_i).$$

We choose $\partial \in T\setminus\{0\}$ such that $\partial(y) = 0$ and obtain

$$\big(\theta(t^yd_i)\cdot f_x\big)\theta(\partial) = d_i(x)\theta(t^{x+y}\partial). \tag{3.20}$$

Hence if $d_i(x) = 0$, then (3.19) holds. Assume now that $d_i(x) \neq 0$. Then $d_i(x) > 0$ and so $x + y \in A_d^+$. By (3.17) we have $\theta(t^{x+y}\partial) = f_{x+y}\theta(\partial)$ and so (3.19) follows from (3.20).

Hence our second claim is proved.

We now define $\Psi_\theta = \sigma$. In order to verify (3.15) and (3.16), we may assume that $f = t^x$, $x \in A_d^+$, and $w = t^y\partial$. Then

$$\sigma(w\cdot f) = \sigma(t^y\partial\cdot t^x) = \partial(x)\sigma(t^{x+y}) = \partial(x)f_{x+y},$$

and, by using (3.19),

$$\theta(w)\cdot\sigma(f) = \theta(t^y\partial)\cdot f_x = \partial(x)f_{x+y}.$$

Hence (3.16) holds.

In order to prove (3.15), it suffices to check that

$$\theta(fw)\cdot f_z = \sigma(f)\theta(w)\cdot f_z$$

holds for all $z \in A_d^+$. By using (3.19) we obtain that

$$\theta(fw)\cdot f_z = \theta(t^{x+y}\partial)\cdot f_z = \partial(z)f_{x+y+z},$$

and

$$\sigma(f)\theta(w)\cdot f_z = \sigma(t^x)\theta(t^y\partial)\cdot f_z = f_x\partial(z)f_{y+z} = \partial(z)f_{x+y+z}.$$

Hence (3.15) holds.

The condition (3.15) uniquely determines Ψ_θ. Indeed if we take $f = t^x$, $x \in A_d^+$, and $w = \partial \in T$, then (3.15) becomes

$$\theta(t^x\partial) = \Psi_\theta(t^x)\theta(\partial).$$

Hence Lemma 3.4 implies that $\Psi_\theta(t^x) = f_x$ for all $x \in A_d^+$, *i.e.*, $\Psi_\theta = \sigma$.

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CASE 3. $I \neq \emptyset$ and $\dim T = 1$. It follows that $I' \neq \emptyset$ and $\dim T' = 1$ also. Denote $I = I' = \{1\}$. Then $d_1 : A \longrightarrow \mathbf{Z}$ and $d_1' : A' \longrightarrow \mathbf{Z}$ are isomorphisms. We can identify W_d and $W_{d'}$ with W_1^+, the Lie algebra of polynomial vector fields $P(t)\frac{d}{dt}$, $P(t) \in F[t]$. Under this identification $d_1 = t\frac{d}{dt}$. The elements $e_i = t^{i+1}\frac{d}{dt}$, $i \geq -1$, form a basis of W_d. Note that $FA_d^+ = F[t]$.

The set of $w \in W_d$ such that $\mathrm{ad}(w)$ is locally nilpotent (resp. locally finite) is Fe_{-1} (resp. $Fe_{-1} + Fe_0$). Furthermore, for $w \in W_d \setminus \{0\}$, $\mathrm{ad}(w)$ is semisimple if and only if $w = ae_0 + be_{-1}$ with $a \neq 0$. Each $\mu \in F$ determines an automorphism (or an isomorphism from $L = W_1^+$ to $L' = W_1^+$) $\theta_\mu = \exp\big(\mu\, \mathrm{ad}(e_{-1})\big)$ of W_d. Since $\theta_\mu(e_0) = e_0 + \mu e_{-1}$, we see that each nonzero ad-semisimple element of W_d is conjugate under $\mathrm{Aut}(W_d)$ to some ae_0, $a \in F^*$.

Each $l \in F^*$ defines another automorphism θ^l of W_d such that $\theta^l(e_i) = l^i e_i$, $i \geq -1$. By using the above facts, it is not hard to show that every $\theta \in \mathrm{Aut}(W_d)$ has the form $\theta = \theta_\mu \theta^l$. We now define $\Psi_\theta = \sigma$ by

$$\sigma(t^i) = l^i (t + \mu)^i, \quad i \geq 0.$$

Then Ψ_θ satisfies (3.15) and (3.16). ∎

For the Lie algebra isomorphism $\theta : L \longrightarrow L'$ we denote $\Psi(\theta)$ by σ. Since $\sigma : FA_d^+ \longrightarrow FA'^+_{d'}$ is an isomorphism of associative algebras, and FA_d^0, $FA'^0_{d'}$ are the subalgebras of FA_d^+, $FA'^+_{d'}$, respectively, generated by invertible elements, then $\sigma(FA_d^0) = FA'^0_{d'}$. Let $T_d = \oplus_{i \in I} Fd_i$. Then $W_{[d]} = FA \otimes T_d$ is a subalgebra of $W(A, T, \varphi)$. It follows that $W_{[d]}^+ = W_{[d]} \cap W_d$ is a subalgebra of $W_d(A, T, \varphi)$. Similarly we can define T_d', $W_{[d']}$ and $W_{[d']}^+$. If we assume $|I| < \infty$, $|I'| < \infty$, fix $x_i \in A_i^\#$ for each $i \in I$ and fix $x'_{i'} \in A'^\#_{i}$ for each $i' \in I'$, then from the following well known lemma it follows that $W_{[d]}^+ = \mathrm{Der}_{FA_d^0}(FA_d^0[t^{x_i}; i \in I])$ and $W_{[d']}^+ = \mathrm{Der}_{FA'^0_{d'}}(FA'^0_{d'}[t^{x_i'}; i \in I'])$.

LEMMA 3.6. *Suppose R is a domain, $x_1, x_2, \ldots, x_n$ are independent and commuting indeterminates over R. Then $\mathrm{Der}_R(R[x_1, x_2, \ldots, x_n])$ is spanned by all the derivations $f\frac{\partial}{\partial x_i}$, where $f \in R[x_1, x_2, \ldots, x_n]$ and $i \in \{1, 2, \ldots, n\}$.* ∎

LEMMA 3.7. *Suppose that I is finite and that $w \in W_d$. Then $w \in W_{[d]}^+$ if and only if $w \cdot FA_d^0 = 0$.* ∎

Since $\sigma(FA_d^0) = FA'^0_{d'}$, using Lemma 3.6, by the identities (3.15) and (3.16) we deduce that $\theta(W_{[d]}^+) = W_{[d']}^+$. Now we can prove our Isomorphism Theorem.

THEOREM 3.8. *Suppose that $L = W_d(A, T, \varphi)$ and $L' = W_{d'}(A', T', \varphi')$ are simple generalized Cartan type W Lie algebras with $|I| < \infty$. Then $W_d(A, T, \varphi) \simeq W_{d'}(A', T', \varphi')$ if and only if, $|I| = |I'|$ and there exist a group isomorphism $\tilde{\sigma} : A \longrightarrow A'$ and a vector space isomorphism $\tilde{\tau} : T \longrightarrow T'$ such that*

(a) $\{d_i' \mid i \in I'\} = \{\tilde{\tau}(d_i) \mid i \in I\}$;

(b) $\langle \tilde{\tau}(\partial), \tilde{\sigma}(x) \rangle = \langle \partial, x \rangle$, $\forall \partial \in T, x \in A$.

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PROOF. ($\Leftarrow$) If (a) and (b) hold for $\tilde{\sigma}: A \longrightarrow A'$ and $\tilde{\tau}: T \longrightarrow T'$, suppose $\tilde{\tau}(d_i) = d'_{i'}$ for all $i \in I$, where $i \longrightarrow i'$ is a bijection from I to I'. Then $\tilde{\sigma}(A_i) = A'_{i'}$ and $\tilde{\sigma}(A_i^{\#}) = A'^{\#}_{i'}$. It is easy to verify that the following linear map

$$\begin{aligned} W_d(A, T, \varphi) &\longrightarrow W_{d'}(A', T', \varphi') \\ t^x \partial &\longrightarrow t^{\tilde{\sigma}(x)} \tilde{\tau}(\partial) \end{aligned}$$

is an isomorphism of Lie algebras.

($\Rightarrow$) CASE 1. $I = \emptyset$. In this case the statement of this theorem follows from [2, Theorem 4.2].

CASE 2. $I \neq \emptyset$. Suppose $\theta: W_d(A, T, \varphi) \longrightarrow W_{d'}(A', T', \varphi')$ is an isomorphism of Lie algebras and $\sigma = \Psi_\theta: FA_d^+ \longrightarrow FA'^+_{d'}$ is the associative algebra isomorphism in Theorem 3.5. We know that $\sigma(FA_d^0) = FA'^0_{d'}$, $\theta(W_{[d]}^+) = W_{[d']}^+$, $W_{[d]}^+ = \mathrm{Der}_{FA_d^0}(FA_d^0[t^{x_i}; i \in I])$ and $W_{[d']}^+ = \mathrm{Der}_{FA'^0_{d'}}(FA'^0_{d'}[t^{x'_i}; i \in I'])$. By Lemma 2.4 we have $|I| = |I'|$. We may assume that $I = I'$ and $i' = i$ for $i \in I$. Fix subspaces $\overline{T} \subseteq T$ and $\overline{T}' \subseteq T'$ such that $\overline{T} \oplus T_d = T$ and $\overline{T}' \oplus T'_{d'} = T$. Since the normalizers of $W_{[d]}^+$ and $W_{[d']}^+$ are $V = W_{[d]}^+ \oplus FA_d^0 \cdot \overline{T}$ and $V' = W_{[d']}^+ \oplus FA'^0_{d'} \cdot \overline{T}'$ respectively, it follows that $\theta(V) = V'$. Since $\sigma(FA_d^0) = FA'^0_{d'}$ we have a group isomorphism $\sigma': A_d^0 \longrightarrow A'^0_{d'}$ and a $\chi \in \mathrm{Hom}(A_d^0, F^*)$ such that $\sigma(t^x) = \chi(x) t^{\sigma'(x)}$.

For any $\partial \in \overline{T}$ and $x \in A_d^0$, we have $\theta(\partial) \cdot \sigma(t^x) = \sigma(\partial \cdot t^x) = \sigma\big(\partial(x) t^x\big) = \partial(x)\chi(x) t^{\sigma'(x)}$, and $\theta(\partial) \cdot \sigma(t^x) = \chi(x)\theta(\partial) \cdot t^{\sigma'(x)}$. Then $\partial(x) t^{\sigma'(x)} = \theta(\partial) \cdot t^{\sigma'(x)}$.

Since $\partial \in \overline{T} \subset V$ and $\theta(V) = V'$, we may assume that $\theta(\partial) = \tau(\partial) + w_\partial$ where $\tau(\partial) \in FA'^0_{d'}\overline{T}'$, $w_\partial \in W_{[d']}^+$. We have $\partial(x) t^{\sigma'(x)} = \tau(\partial) t^{\sigma'(x)}$, for all $x \in A_d^0$. By Lemma 3.7 we know that, for any $\bar{\partial} \in \overline{T}'$, $\bar{\partial}(FA'^0_{d'}) = 0$ implies $\bar{\partial} = 0$. So it follows that $\tau(\partial) \in \overline{T}'$, and, $\langle \tau(\partial), \sigma'(x) \rangle = \partial(x)$. Thus we have got a vector space isomorphism $\tau: \overline{T} \longrightarrow \overline{T}'$ and a group isomorphism $\sigma': A_d^0 \longrightarrow A'^0_{d'}$ such that

$$\langle \tau(\partial), \sigma'(x) \rangle = \partial(x), \quad \forall \partial \in \overline{T},\ x \in A_d^0.$$

Now for each $i \in I$, we fix a $-x_i \in A_i^{\#}$, and a $-x'_i \in A'^{\#}_i$. Note that $A^+ = A_d^0 + \sum_{i \in I} \mathbf{Z}_+ x_i$ and $A'^+ = A'^0_{d'} + \sum_{i \in I} \mathbf{Z}_+ x'_i$. Define the map $\tilde{\sigma}: A_d^+ \longrightarrow A'^+_{d'}$ by sending $x + \sum_{i \in I} k_i x_i$ to $\sigma'(x) + \sum_{i \in I} k_i x'_i$, where $x \in A_d^0$. It is clear that $\tilde{\sigma}$ can be uniquely extended to a group isomorphism from A to A'.

Now choose a basis $\{\partial_j j \in J\}$ of $\overline{T}$. Then $\{\partial'_j = \tau(\partial_j) \mid j \in J\}$ is a basis of $\overline{T}'$. For each ∂_j let $\partial_j(x_i) = a_{ji}$ and $\partial'_j(x'_i) = a'_{ji}$. Set $\bar{\partial}_j = \partial'_j + \sum_{i \in I}(a_{ji} - a'_{ji}) d'_i$. Now we define the linear map

$$\begin{aligned} \tilde{\tau}: T &\longrightarrow T' \\ d_i &\longrightarrow d'_i, \quad \forall i \in I \\ \partial_j &\longrightarrow \bar{\partial}_j, \quad j \in J. \end{aligned}$$

It is not difficult to verify that $\tilde{\tau}$ and $\tilde{\sigma}$ satisfy the conditions (a) and (b) in this theorem. This completes the proof of this theorem. ∎

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4. **The Jacobian conjecture and the general Lie algebras.** In this section, we shall give a conjecture on the general Lie algebras W_n^+, and show that the validity of this conjecture implies the validity of the Jacobian conjecture.

CONJECTURE 1. *Every nonzero endomorphism of the general Lie algebra $W_n^+(F)$ is an automorphism of the Lie algebra.*

Let us first recall the Jacobian conjecture.

Let F be a field of characteristic 0, n a positive integer, and $t_1, \ldots, t_n$ independent and commuting indeterminates over F. Denote by $P_n = F[t_1, \ldots, t_n]$. Let $f_1, \ldots, f_n \in P_n$. We know that the Jacobian matrix of $f_1, \ldots, f_n$ is defined as

$$J(f_1, \ldots, f_n) = \begin{bmatrix} \frac{\partial(f_1)}{\partial x_1} & \frac{\partial(f_1)}{\partial x_2} & \ldots & \frac{\partial(f_1)}{\partial x_n} \\ & \ldots & \ldots & \\ \frac{\partial(f_n)}{\partial x_1} & \frac{\partial(f_n)}{\partial x_2} & \ldots & \frac{\partial(f_n)}{\partial x_n} \end{bmatrix}.$$

It is well known that $F[f_1, \ldots, f_n] = F[x_1, \ldots, x_n]$ implies $\det J(f_1, \ldots, f_n) \in F^*$. The converse of this statement is the Jacobian conjecture, *i.e.*,

JACOBIAN CONJECTURE. Suppose $f_1, \ldots, f_n \in P_n$. If $\det J(f_1, \ldots, f_n) \in F^*$, then $F[f_1, \ldots, f_n] = F[x_1, \ldots, x_n]$.

The Jacobian conjecture is still open for $n > 1$ to the knowledge of the author. For more details, please refer to the paper [3].

THEOREM 4.1. *The validity of Conjecture 1 implies the validity of the Jacobian conjecture.*

PROOF. Suppose $f_1, \ldots, f_n \in P_n$ with $\det J(f_1, \ldots, f_n) \in F^*$. We may assume that $\det J(f_1, \ldots, f_n) = 1$. Denote the (i,j)-cofactor of $J(f_1, \ldots, f_n)$ by M_{ij}. Let

$$D_i = \sum_{j \in I} M_{ij} \frac{\partial}{\partial x_j}, \quad \forall i = 1, 2, \ldots, n.$$

Then we have $D_i(f_j) = \delta_{ij}$. It follows that D_i's commute in $F[f_1, \ldots, f_n]$. From $\det J(f_1, \ldots, f_n) = 1$, we deduce that $\Omega_{\frac{F[x_1,\ldots,x_n]}{F[f_1,\ldots,f_n]}} = 0$ (see Section I.2(1) in [3]). It follows that D_i's commute in $F[x_1, \ldots, x_n]$, *i.e.*, $[D_i, D_j] = 0$ for all $i, j \in \{1, 2, \ldots, n\}$. It is clear that the following linear map

$$\theta \colon W_n^+ \longrightarrow W_n^+$$
$$x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \frac{\partial}{\partial x_i} \longrightarrow f_1^{k_1} f_2^{k_2} \cdots f_n^{k_n} D_i, \quad \forall i = 1, 2, \ldots, n,$$

is an endomorphism of the Lie algebra W_n^+. It follows from the invalidity of Conjecture 1 that θ is an automorphism of W_n^+. Then by Theorem 3.5, or [2, Theorem 5.5], or [9, Theorem 3.1], the corresponding associative algebra automorphism $\Psi_\theta \colon F[x_1, \ldots, x_n] \longrightarrow F[x_1, \ldots, x_n]$ is defined by

$$\Psi_\theta(x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}) = f_1^{k_1} f_2^{k_2} \cdots f_n^{k_n}.$$

Therefore $F[f_1, \ldots, f_n] = F[x_1, \ldots, x_n]$. ∎

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