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METASTABLE IMMERSION, SPAN AND THE TWO-TYPE OF A MANIFOLD

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ABSTRACT. The existence of metastable immersion or span for space forms and homogeneous spaces is shown to depend only on the two-type of the space.

1. **Introduction**. For many smooth n-manifolds, if k is in the metastable range $k > n/2$, then each of the properties of immersing with codimension k and having $n - k$ linearly independent vector fields depends only on the 2-type of the manifold. The following results make this statement more precise for two interesting classes of manifolds: homogeneous spaces and space forms. The general theorems which imply these results are given in the next section.

We use $\subseteq$ for immersion, $\simeq$ for homotopy equivalence, $\mathbb{Z}_p$ for $\mathbb{Z}[1/q;\ q$ relatively prime to $p]$, a subscript 2 for localization with respect to $H_*(\ ;\mathbb{Z}_2)$, a subscript 1/2 to denote localization with respect to $H_*(\ ;\mathbb{Z}[1/2])$ and a subscript 0 to denote localization with respect to $H_*(\ ;\mathbb{Q})$ (cf. [4]).

THEOREM 1.1. *If*

(i) V^{n+k} *is an arbitrary smooth manifold and one of* M^n and N^n *is a homogeneous space or space form, and* W^{n+k} *is a nilpotent homogeneous space or nilpotent space form,*

(ii) $M \subseteq V$,

(iii) $M_2 \simeq N_2$, $V_2 \simeq W_2$, *and*

(iv) $k > n/2$,

then $N \subseteq W$.

EXAMPLE 1.2. *If* $\mathbb{R}P^n \subseteq \mathbb{R}^{n+k}$, $N_2 \simeq \mathbb{R}P^n_2$ *and* $n \geq 16$, *then* $N \subseteq \mathbb{R}^{n+k}$.

In particular, all standard lens spaces $L^{2m+1}(4q + 2)$ immerse in the same euclidean space as $\mathbb{R}P^{2m+1}$, provided $m > 7$.

REMARK 1.3. A real flag manifold $0(n_1 + \ldots + n_k)/(0(n_1) \times \ldots \times 0(n_k))$ is nilpotent if and only if all the n_i are odd (cf. [12]).

COROLLARY 1.4. *If F is a finite group of odd order acting smoothly and freely on the*

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π-manifold space form or homogeneous π-manifold M^n, then $M/F \subseteq \mathbb{R}^{[3n/2]+1}$, where $[\,]$ denotes the integer part.

EXAMPLE 1.5. (cf. [24]) *The standard lens space $L^{2m+1}(2q + 1) \subseteq \mathbb{R}^{3m+2}$, and this result is sharp for m even provided $2q + 1$ does not divide $\binom{3m/2}{m/2}$, $3m/2$ choose $m/2$.*

COROLLARY 1.6. (cf. [17]) *If F is a finite group of odd order acting smoothly and freely on the homogeneous space G/H, and* $\dim H > (\dim G)/3$, *then $(G/H)/F \subseteq \mathbb{R}^{\dim G}$.*

EXAMPLE 1.7. (cf. [23] and [18])

$$(0(4m + 2)/(0(2m + 1) \times 0(2m + 1))/(\mathbb{Z}/(2q + 1)) \subseteq \mathbb{R}^{\binom{4m+2}{2}}$$

and this result is always sharp. The action of $\mathbb{Z}/(2q + 1)$ is induced by the diagonal action of $\mathbb{Z}/(2q + 1)$ in $\mathbb{C}^{2m+1}$, which can be seen to be free by considering eigenvalues.

THEOREM 1.8 (cf. [2]) *If*
(i) *M^n or N^n is a homogeneous or a space form,*
(ii) *M^n has an $(n - k)$-field ($(n - k)$ linearly independent vector fields),*
(iii) *$M_2 \simeq N_2$ and*
(iv) *$k > n/2$,*
then N has an $(n - k)$-field.

REMARK 1.9. Recall that if M^n has an $(n - k)$-field and does not have an $(n - k + 1)$-field we say that span $M = n - k$.

COROLLARY 1.10. *If F is a finite group of odd order acting smoothly and freely on the parallelizable homogenous manifold M^n, then M/F has an $[(n - 1)/2]$-field.*

EXAMPLE 1.11. (i) *$L^{2m+1}(2q + 1) \times S^1$ always has an m-field and* (ii) *it does not have an $(m + 3)$-field if m is even and $2q + 1$ does not divide $\binom{m+1}{m/2}$.*

REMARKS.

1. Do the conclusions of Theorems 1.1 and 1.8 follow from hypotheses (ii)–(iv) alone? This is a tempting generalization because we do not have a counter example, but the approach used here requires some additional hypothesis. As indicated above, hypotheses (i) illustrate the applicability to some familiar manifolds of the more technical hypotheses in Theorems 2.1 and 2.3.
2. The results in this paper improve those in [11], [12], [13], [14] and [15] by removing the special hypotheses about the spaces M or N that occur in these papers due to the special theories of localization used. We are also able to remove the hypothesis of odd codimension but only by introducing hypotheses (iv) of Theorem 2.1 and (iii) of Theorem 2.3 about the integrality of the rational Euler classes. This suffices for the special spaces of this section.

The remainder of this paper is organized as follows. In section 2 we state the general theorems which imply the results in section 1. In section 3 we prove Theorems 2.1 and

2.3. These theorems immediately imply the results in section 1, using the results of Friedlander [10] (cf. [13]) in the case of homogeneous spaces. That $\mathbb{R}P^n \subseteq \mathbb{R}^{n+k}$ implies $k > n/2$ for $n \geq 16$ in example 1.2 follows from [8]. That the implication is often false for $n < 16$ is given in [22]. The non-immersion of Example 1.5 and the non-existence of vector fields in Example 1.11 are computations in Pontrjagin classes which we also give in section 3. The sharpness of the result in Example 1.7 is a computation in Stiefel-Whitney classes which was done by Hiller and Stong [18].

2. **General Theorems**

THEOREM 2.1. *Suppose M^n and N^n are compact smooth manifolds, V^{n+k} an arbitrary smooth manifold and W^{n+k} a nilpotent smooth manifold such that the following hold.*

(i) *There exists an immersion $f: M \to V$.*

(ii) *There exist homotopy equivalences $\lambda: N_2 \to M_2$ and $\mu: V_2 \to W_2$.*

(iii) *There exists a map $g_{1/2}: N \to W_{1/2}$ such that the induced diagram*

$$\begin{array}{ccc} M_0 & \xrightarrow{f_0} & V_0 \\ \lambda_0 \downarrow & & \downarrow \mu_0 \\ N_0 & \xrightarrow[(g_{1/2})_0]{} & W_0 \end{array}$$

is homotopy commutative.

(iv) *If k is even there exists a homotopy equivalence $\phi: M_2 \to M_2$ or a homotopy equivalence $\theta: N_2 \to N_2$ such that $(\lambda^*\phi^*(X_f))_0$ or $(\theta^*\lambda^*(X_f))_0$ is integral in $H^k(N; \mathbb{Q})$. Here X_f denotes the Euler class of the immersion f.*

(v) $k > n/2$.

Then $N \subseteq W$.

REMARK. R. Cohen used a result closely related to but independent of Theorem 2.1, due to Brown and Peterson [6], in his proof of the immersion conjecture [7].

COROLLARY 2.2. *If M^n and N^n are compact smooth manifolds such that*

(i) $M \subseteq \mathbb{R}^{n+k}$,

(ii) $N_2 \simeq M_2$,

(iii) *k is odd or $H^k(M; \mathbb{Q}) = 0$, and*

(iv) $k > n/2$,

then $N \subseteq \mathbb{R}^{n+k}$.

THEOREM 2.3. *Let M^n and N^n be compact smooth manifolds such that the following hold.*

(i) *M^n has an $(n - k)$-field.*

(ii) $M_2 \simeq N_2$.

(iii) *If k is even there exists a homotopy equivalence $\phi: M_2 \to M_2$ or a homotopy equivalence $\theta: N_2 \to N_2$ such that $(\lambda^*\phi^*(X_\xi))_0$ or $(\theta^*\lambda^*(X_\xi))_0$ is integral in $H^k(N; \mathbb{Q})$.*

Here X_ξ denotes the Euler class of ξ^k, where the tangent bundle to M is $\xi^k \oplus 1^{n-k}$.
(iv) $k > n/2$.
Then N has an $(n-k)$-field.

COROLLARY 2.4. *If M^n and N^n are compact smooth manifolds such that*
(i) *M has an $(n-k)$-field,*
(ii) $M_2 \simeq N_2$,
(iii) *k is odd or $H^k(M;\mathbb{Q}) = 0$, and*
(iv) $k > n/2$,
then N has an $(n-k)$-field.

REMARK. Our results hold in the *PL* category by replacing Hirsch's thesis [20] by the corresponding theorem for *PL*-manifolds due to Haefliger and Poenaru [16].

3. **Proof of Theorems 2.1 and 2.3.**

PROOF OF 2.1. If k is odd the proof proceeds as in [11], observing that the localization of Bousfield [4] with respect to $H_*(;\mathbb{Z}_2)$ allows us to remove the hypothesis of nilpotency from the manifolds M or N. We include the proof for completeness. If we define $g_2 = \mu \circ f_2 \circ \lambda : N_2 \to W_2$, then by hypothesis (iii) and the fracture lemma of [5] and [19], there is a map $g: N \to W$ for which $g_{1/2}$ and g_2 are the localizations that the notation suggests. Since f: $M^n \to V^{n+k}$ is an immersion, $f^*(\tau^{st}V) - \tau^{st}M$ lifts to $BSO(k)$. By Atiyah [1] (cf. [14]), $J\tau^{st}V_2 = \mu^*(J\tau^{st}W)_2$ and $\lambda^*(J\tau^{st}M)_2 = (J\tau^{st}N)_2$, so $\lambda^*J(f^*\tau^{st}V - \tau^{st}M)_2 = J(g^*\tau^{st}W - \tau^{st}N)_2$ lifts to $BSF(k)_2$ (cf. [11]). (We use the simple connectivity of the Thom complex of $\tau^{st}V$, etc.) Using James [21] (cf. [14]) $g^*\tau^{st}W - \tau^{st}N$ lifts to $BSO(k)$. The result now follows by Hirsch's thesis [20].

If k is even we observe that $BSF(k)_0 \simeq K(\mathbb{Q}, k)$, so that the lift $N \to BSF(k)_2$ given above may not be compatible with that to $BSF(k)_{1/2}$ given by [21] (cf. [14]), i.e. they may not agree in $BSF(k)_0$. But hypothesis (iv) allows the lift to $BSF(k)_2$ to be chosen so that it is compatible with that to $BSF(k)_{1/2}$ by using the identification of lifts of $N \to BSF_{1/2}$ to $BSF(k)_{1/2}$ with elements of $\mathbb{Z}[1/2]$. The remainder of the proof is the same as for k odd.

PROOF OF 2.3. We proceed as for 2.1 except that we must lift τ: $N \to BO(n)$ to $BO(k)$. (The case k odd is done in [15].) Again the point is that for k even $BSF(k)_0 \simeq K(\mathbb{Q}, k)$ and the lifts of N in $BSF(k)_2$ and $BSF(k)_{1/2}$ may not agree. But hypothesis (iii) allows us to choose them compatibly. As in [15] it is necessary to use Benlian-Wagoner [3] and Dupont [9] to show that $J\tau M_p \simeq J\tau N_p$.

The computations for Example 1.5 and 1.11 are standard results for Pontrjagin classes for $L^{2m+1}(2q+1)$ and say that $p = (1+p_1)^{m+1}$ so that $\bar{p} = p^{-1} = \sum_{i\geq 0}(-1)^i \binom{m+i}{i} p_1^i$. Of course we use the fact that $S^{2m+1} \times S^1$ is parallelizable.

The Computation for the sharpness of Example 1.7 was done by Hiller and Stong [18]. The immersion $0(2n)/(0(n) \times 0(n)) \subseteq \mathbb{R}^{\binom{2n}{2}}$ was first done by K. Lam [23]. The immersion $G/H \subseteq \mathbb{R}^{\dim G}$ for example 1.6 was done by Hiller [17].

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