Proceedings of the Royal Society of Edinburgh, **154**, 793–809, *2024*
DOI:10.1017/prm.2023.37

A Jordan–Hölder theorem for skew left braces and their applications to multipermutation solutions of the Yang–Baxter equation

A. Ballester-Bolinches
Department of Mathematics, Guangdong University of Education, Guangzhou, Guangdong 510303, People's Republic of China (Adolfo.Ballester@uv.es)
Departament de Matemàtiques, Universitat de València, Dr. Moliner, 50, 46100 Burjassot, València, Spain (Ramon.Esteban@uv.es, Vicent.Perez-Calabuig@uv.es)

R. Esteban-Romero and V. Pérez-Calabuig
Departament de Matemàtiques, Universitat de València, Dr. Moliner, 50, 46100 Burjassot, València, Spain (Ramon.Esteban@uv.es, Vicent.Perez-Calabuig@uv.es)

(Received 6 October 2022; accepted 17 March 2023)

Skew left braces arise naturally from the study of non-degenerate set-theoretic solutions of the Yang–Baxter equation. To understand the algebraic structure of skew left braces, a study of the decomposition into minimal substructures is relevant. We introduce chief series and prove a strengthened form of the Jordan–Hölder theorem for finite skew left braces. A characterization of right nilpotency and an application to multipermutation solutions are also given.

Keywords: Skew left brace; Yang–Baxter equation; Jordan–Hölder theorem; nilpotency; multipermutation solution

2020 *Mathematics Subject Classification:* 81R50; 20F29; 20B35; 20F16; 20C05; 16S34; 16T25

1. Introduction

The *Yang–Baxter equation* (YBE, for short), introduced in seminal works of Yang [**17**] and Baxter [**2**], is one of the basic equations in mathematical physics which led to the foundation of the theory of quantum groups. The set-theoretic point of view proposed by Drinfeld in [**6**] attracted great attention due to its links with other areas such as knot theory and Hopf algebras. Given a non-empty set X, a *set-theoretic solution* (a solution, for short) (X, r) of the YBE is a map $r\colon X \times X \to X \times X$

such that

$$r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23}$$

where the maps $r_{12}, r_{23} \colon X \times X \times X \to X \times X \times X$ are defined as $r_{12} = r \times \mathrm{id}_X$ and $r_{23} = \mathrm{id}_X \times r$.

Solutions satisfying additional conditions were intensively analysed in [**7**, **9**, **13**]. In particular, non-degenerate solutions, i.e. solutions (X, r) such that both projections of r are bijective, give rise to new algebraic structures. *Braces* were first introduced by Rump in [**14**] as a generalization of Jacobson radical rings in the context of involutive ($r^2 = \mathrm{id}_{X\times X}$) non-degenerate solutions. A generalization of this structure, the so-called *skew left brace*, was introduced by Guarnieri and Vendramin in [**10**] in order to study bijective (not necessarily involutive) non-degenerate solutions: every skew left brace provides a bijective non-degenerate solution and vice versa. However, there is no bijective correspondence nor categorical equivalence between skew braces and set-theoretic solutions.

These new algebraic structures brought a lot of connections with some mathematical topics of recent interest such as regular subgroups and Hopf–Galois extensions [**4**, **15**], trifactorized groups [**1**, **16**], braided groups [**8**], Bieberbach groups [**9**] or Garside theory [**5**].

Such wide ranging of links between diverse areas of mathematics shows that an in-depth study of the algebraic structure of skew left braces is essential. The more we know about skew left braces the more we know about their associated bijective non-degenerate solutions of the YBE. In this context, an algebraic structural study of skew left braces is imperative. Furthermore, as skew left braces are an interaction of two group structures on the same set and an extension of radical rings, it is natural to approach them with group and ring theoretical methods.

An important family of finite solutions of the Yang–Baxter equation is that of non-degenerate multipermutation solutions. Such solutions appeared in the paper [**7**] of Etingof, Schedler, and Soloviev as generalizations of permutation solutions and now these solutions appear in many different contexts.

In this paper, we study finite non-degenerate multipermutation solutions by means of the skew left brace structure of their permutation groups and via chief series of this skew left brace. We prove a sort of analogue of an important strengthened form of the Jordan–Hölder theorem on groups, but now in the context of skew left braces and introduced the notion of finite chief length. Although this result is interesting on their own, it can be used to characterize noetherian and artinian skew left braces introduced in [**11**]. The Jordan–Hölder theorem also allows us to study right nilpotency of skew left braces by means of their chief factors and relate the chief length of a skew left brace with multipermutation level of the associated solution of the Yang–Baxter equation. Some results in [**3**] on right nilpotency are improved for skew left braces with chief series.

2. Preliminaries

A *skew left brace* $(B, +, \cdot)$ is defined to be a set B endowed with two group structures $(B, +)$ (*the additive group*) and $(B, \cdot)$ (*the multiplicative group*) satisfying the following property:

$$a \cdot (b + c) = a \cdot b - a + a \cdot c, \quad \text{for every } a, b, c \in B. \tag{2.1}$$

From now on, *we use juxtaposition for the product of elements*. Recall that in skew left braces, the identity elements of both group structures coincide. *We will use* 1 *to denote both identity elements*.

Let $\mathfrak{X}$ be a class of groups. If $(B, +)$ belongs to $\mathfrak{X}$, then B is called a skew left brace *of* $\mathfrak{X}$*-type*. Rump's braces introduced in [**14**] are, in fact, the skew left braces of abelian type.

A *subbrace* of $(B, +, \cdot)$ is a subgroup of the additive group which is also a subgroup of the multiplicative group. A *homomorphism* between two skew left braces A and B is a map $f\colon A \to B$ satisfying that $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all $a, b \in A$. The kernel of f is defined as the set $\operatorname{Ker}(f) = \{a \in A \mid f(a) = 1\}$. If f is bijective, f is called an isomorphism. We shall say that the braces A and B are *isomorphic*, if there is an isomorphism between A and B. In this case, we write $A \cong B$

If $(B, +, \cdot)$ is a skew left brace, the multiplicative group $(B, \cdot)$ acts on the additive group $(B, +)$ via automorphisms: for every $a \in B$, the map $\lambda_a\colon B \to B$, given by $\lambda_a(b) = -a + ab$, is an automorphism of $(B, +)$ and the map $\lambda\colon (B, \cdot) \to \operatorname{Aut}(B, +)$ which sends $a \mapsto \lambda_a$ is a group homomorphism (see [**10**, proposition 1.9]). This group action relates the two operations on a skew left brace. For every $a, b \in B$, it holds

$$ab = a + \lambda_a(b) \quad \text{and} \quad a + b = a\lambda_{a^{-1}}(b). \tag{2.2}$$

From now on, if $(B, +, \cdot)$ is a skew left brace, we will write simply B and the operations on B are understood.

Lemma 2.1. *For a skew left brace B, the kernel of the action λ, that is,* $\operatorname{Ker}\lambda = \{b \in B \mid \lambda_b = \mathrm{id}_B\}$*, is a normal subgroup of $(B, \cdot)$ and a subgroup of $(B, +)$.*

Proof. Only the second part of the statement is in doubt.

Clearly, $1 \in \operatorname{Ker}\lambda$. Then, for every $a, b \in \operatorname{Ker}\lambda$, $a + b = a\lambda_{a^{-1}}(b) = ab \in \operatorname{Ker}\lambda$. Moreover, if $b \in \operatorname{Ker}\lambda$, then $-b = b(b^{-1} + b^{-1}) \in \operatorname{Ker}\lambda$. □

Following [**1**], we can construct the semidirect product $G_B = [K]_\lambda C$ with respect to the action of $C = (B, \cdot)$ on $K = (B, +)$ by means of λ. We will refer to G_B as *the semidirect product associated with* $(B, +, \cdot)$. We use multiplicative notation for the group operation on G_B.

Skew left braces may be viewed as generalizations of radical rings. This idea is behind the definition of ideal which plays a central role in the structural study of a skew left brace. First, we define the *star operation*, which plays an analogous role to multiplication in an associative ring.

Let B be a skew left brace. Denote the operation:

$$a * b = \lambda_a(b) - b = -a + ab - b, \quad \text{for every } a, b \in B.$$

https://doi.org/10.1017/prm.2023.37 Published online by Cambridge University Press

In $G_B = [K]_\lambda C$, if a is regarded as an element of C and b as an element of K, the previous binary operation can be represented as a commutator,

$$a * b = aba^{-1}b^{-1} = [a^{-1}, b^{-1}] \in [C, K] \subseteq K,$$

since C normalizes K.

Given two subsets X and Y of B, we define $X * Y$ as the subgroup of $(B, +)$ generated by $\{x * y \mid x \in X, y \in Y\}$. Again, in $G_B = [K]_\lambda C$, if we identify X as a subgroup E of C and Y as a subgroup H of K, this subgroup can be regarded as $\langle [a^{-1}, b^{-1}] \mid a \in E, b \in H\rangle = [E, H] \leqslant K$.

We say that a non-empty subset I of a skew left brace B is a *left ideal*, if $(I, +)$ is a subgroup of $(B, +)$ and $B * I \subseteq I$, or equivalently $\lambda_b(I) \subseteq I$, for every $b \in B$. We say that a left ideal I is an *ideal* if $(I, +)$ is a normal subgroup of $(B, +)$ and $Ia = aI$ for all $a \in B$. By [**3**, lemma 1.9], a left ideal I is an ideal of B if, and only if, $(I, +)$ is a normal subgroup of $(B, +)$ and $I * B \subseteq I$.

Ideals of skew left braces can be considered as true analogues of normal subgroups in groups and ideals in rings.

PROPOSITION 2.2 [**10**, lemma 2.3]. *Let B be a skew left brace and let I be an ideal of B. Then,*

1. *$bI = b + I$, for every $b \in B$.*
2. *$(I, \cdot)$ is a normal subgroup of $(B, \cdot)$.*
3. *I is a subbrace of B and B/I is also a skew left brace.*

COROLLARY 2.3. *Let I, J be ideals of a skew left brace B. Then,*

1. *$I \cap J$ is an ideal of B.*
2. *$IJ = I + J$ is an ideal of B.*

Proof. It is clear that only the second statement is in doubt.

By proposition 2.2, we have that $IJ = I + J$. Therefore, $IJ = I + J$ is a normal subgroup of $(B, +)$ and a normal subgroup of $(B, \cdot)$.

Let $b \in B$, $x \in I$ and $y \in J$. Then, $\lambda_b(x + y) = \lambda_b(x) + \lambda_b(y) \in I + J$. Thus, $\lambda_b(I + J) \subseteq I + J$. Consequently, IJ is an ideal of B. □

REMARK 2.4. Let I be an ideal of B. Recall that B/I is a skew left brace and therefore, the action $\lambda_{B/I} \colon (B/I, \cdot) \to \mathrm{Aut}(B/I, +)$ satisfies that

$$\lambda_{bI}(aI) = -bI + (bI)(aI) = (-b + ba)I = \lambda_b(a)I, \quad \text{for every } a, b \in B.$$

The next proposition shows the behaviour of left ideals and ideals under the star product.

PROPOSITION 2.5 [**1**, lemma 4.3]. *Let B be a skew left brace. Suppose that L is a left ideal of B and I is an ideal of B. Then, $I * L$ is a left ideal of B. Moreover, $I * B$ is an ideal of B.*

https://doi.org/10.1017/prm.2023.37 Published online by Cambridge University Press

Definition 2.6. *Let I be an ideal of a skew left brace B.*

1. *I is called a minimal ideal of B if $I \neq 1$ and 1 and I are just the ideals of B contained in I.*
2. *I is called a maximal ideal of B if I is the only proper ideal of B containing I.*

Note that every finite skew left brace has minimal (respectively maximal) ideals. However, not every brace has minimal (respectively maximal) ideals. Therefore, the following finiteness conditions introduced in [**11**] for skew left braces are interesting.

Definition 2.7. *A skew left brace B is artinian (respectively noetherian) if every non-empty set of ideals of B has a minimal (respectively maximal) element with respect to the inclusion.*

It is clear that a skew left brace B is artinian (respectively noetherian) if, and only if, every descending (respectively ascending) chain of ideals of B is eventually stationary. In addition, every non-trivial ideal of an artinian skew left brace contains a minimal ideal, and every proper ideal of a noetherian skew left brace is contained in a maximal ideal. It is also rather clear that every skew left brace B is noetherian if, and only if, each ideal of B is finitely generated as an ideal, i.e., each ideal has a finite weight in the sense of [**11**, definition 4.1].

A skew left brace B is called *simple* if $B \neq 1$ and B has not proper ideals. Note that if B is simple, then B is the only minimal ideal of B and 1 is the only maximal ideal of B.

The following brace-theoretic radical is introduced and studied in [**11**].

Definition 2.8 [**11**, definition 3.1]. *The radical* $\operatorname{Rad}(B)$ *of a skew left brace B is the intersection of all maximal ideals of B, if such exists, and B otherwise.*

Note that $\operatorname{Rad}(B)$ is an ideal of B. Furthermore, $\operatorname{Rad}(B)J/J$ is contained in $\operatorname{Rad}(B/J)$ for all ideals J of B and $\operatorname{Rad}(B)/J = \operatorname{Rad}(B/J)$ if $J \subseteq \operatorname{Rad}(B)$.

Note that if B is a noetherian skew left brace, then $\operatorname{Rad}(B)$ is a proper ideal of B if $B \neq 1$.

The radical of a skew left brace is the brace-theoretic version of the Jacobson and Brown–McCoy radicals in ring theory and the Baer and Frattini subgroups in group theory.

In [**11**] a brace-theoretic analogue of the celebrated Artin–Weddeburn decomposition theorem for semisimple rings is proved: if B is an artinian skew left brace and $\operatorname{Rad}(B) \neq B$, then $B/\operatorname{Rad}(B)$ is isomorphic to a direct product of finitely many simple skew left braces. The radical of a skew left brace is also used there to show a brace-theoretic version of a well-known theorem of Gaschütz in group theory.

We end the section by presenting two left ideals closely related to nilpotency of skew left braces (see [**3**]). For a skew left brace B, we can consider the set of fixed elements of B by the action λ,

$$\operatorname{Fix}(B) = \{a \in B \mid \lambda_b(a) = a, \text{ for every } b \in B\},$$

https://doi.org/10.1017/prm.2023.37 Published online by Cambridge University Press

which turns out to be a left ideal, and the so-called *socle* of B,

$$\begin{aligned}\operatorname{Soc}(B) &= \{a \in B \mid \lambda_a(b) = b,\ a+b = b+a,\ \text{for every } b \in B\} \\ &= \operatorname{Ker}\lambda \cap \mathrm{Z}(B,+).\end{aligned}$$

It is well-known that $\operatorname{Soc}(B)$ is an ideal of B (see [**10**, lemma 2.5], for example). We provide an alternative simpler proof.

PROPOSITION 2.9. *$\operatorname{Soc}(B)$ is an ideal of B.*

Proof. By lemma 2.1, we have that $\operatorname{Ker}\lambda$ is a subgroup of $(B,+)$ and then $\operatorname{Soc}(B) = \operatorname{Ker}\lambda \cap \mathrm{Z}(B,+)$ is a normal subgroup of $(B,+)$.

Since $\mathrm{Z}(B,+)$ is a characteristic subgroup of $(B,+)$, it is clear that $\lambda_b(\operatorname{Soc}(B)) \subseteq \mathrm{Z}(B,+)$, for every $b \in B$. Therefore, for every $a \in \operatorname{Soc}(B)$ and $b \in B$

$$\lambda_b(a) = b + \lambda_b(a) - b = ba - b = b(a + b^{-1}) = bab^{-1}.$$

Thus, $\lambda_b(a) \in \operatorname{Ker}\lambda$, as $\operatorname{Ker}\lambda$ is normal in $(B,\cdot)$.

Hence, $\operatorname{Soc}(B)$ is a left ideal of B and, by definition, $\operatorname{Soc}(B) * B = 1 \subseteq \operatorname{Soc}(B)$. Therefore, $\operatorname{Soc}(B)$ is an ideal of B. □

3. Finiteness properties of skew left braces: a Jordan–Hölder theorem

A possible approach to left and right nilpotency of skew left braces is through series of ideals. In this section we study the finiteness property of finite chief length by proving a Jordan–Hölder theorem for chief series of a skew left brace. It allows us to give some connections between finite chief length and artinian and noetherian properties.

Throughout the section, B will denote a skew left brace.

DEFINITION 3.1. *Let I and J be two ideals of B such that $J \subseteq I$. We say that the section I/J is:*

- *a* chief factor *of B, if I/J is a minimal ideal of B/J;*
- *an s-*factor *of B, if $I/J \subseteq \operatorname{Soc}(B/J)$;*
- *an f-*factor *of B, if $I/J \subseteq \operatorname{Fix}(B/J)$.*
- *an r-*factor *of B, if $I/J \subseteq \operatorname{Rad}(B/J)$.*

*If $\tau \in \{s,f,r\}$, we say that I/J is a τ-*chief factor *of B if I/J is a chief factor of B which is a τ-*factor.

DEFINITION 3.2. *Let*

$$\mathcal{I}\colon 1 = I_0 \subseteq I_1 \subseteq \ldots \subseteq I_n = B,$$

be an ideal series of B. We say that $\mathcal{I}$ is

- *a* chief series *of B, if I_i/I_{i-1} is a minimal ideal of B/I_{i-1};*

- *an* s-series *of* B*, if* $I_i/I_{i-1} \subseteq \mathrm{Soc}(B/I_{i-1})$*;*
- *an* f-series *of* B*, if* $I_i/I_{i-1} \subseteq \mathrm{Fix}(B/I_{i-1})$*,*
  for every $1 \leqslant i \leqslant n$*.*

LEMMA 3.3. *Let* I, J *be ideals of* B *such that* $J \subseteq I$*. The following statements hold:*

1. I/J *is an* f*-factor if, and only if,* $B * I \subseteq J$*.*
2. I/J *is an* s*-factor if, and only if,* $I * B \subseteq J$ *and* $[I, B]_+ \subseteq J$*.*

Proof. Let $x \in I$. Then,

$$\begin{aligned} xJ \in \mathrm{Fix}(B/J) &\Leftrightarrow \lambda_b(x)J = xJ, \text{ for every } b \in B \\ &\Leftrightarrow bJ * xJ = J, \text{ for every } b \in B \\ &\Leftrightarrow b * x \in J, \text{ for every } b \in B. \end{aligned}$$

Then, I/J is an f-factor of B if, and only if, $b * x \in J$, for every $b \in B$ and every $x \in I$, and the first statement holds.

Let $x \in I$. Then, $xJ \in \mathrm{Soc}(B/J)$ implies that $xJ * bJ = J$, for every $b \in B$. Thus, if I/J is an s-factor, it follows that $I * B \subseteq J$. Moreover, since $I/J \subseteq Z(B/J, +)$, we have that $[I, B]_+ \subseteq J$.

Conversely, suppose that $I * B \subseteq J$ and $[I, B]_+ \subseteq J$. Given $x \in I$ and $b \in B$, it follows that $x * b = \lambda_x(b) - b \in J$, i.e. $\lambda_x(b)J = bJ$. Moreover, $(x + b)J = (b + x)J$. Thus, $I/J \subseteq \mathrm{Soc}(B/J)$. □

The next example shows the commutator condition is essential to characterize s-factors.

EXAMPLE 3.4. Suppose that $(B, +, \cdot)$ is a trivial skew left brace, i.e. a skew left brace where the two group structures coincide, $(B, +) = (B, \cdot)$. Let $(B, \cdot)$ be a group with trivial centre. Then, $\mathrm{Soc}(B) = 1$ and $\mathrm{Ker}\,\lambda = B$. Thus, for every non-trivial normal subgroup $1 \neq N \trianglelefteq B$, N is an ideal such that $N * B = 1$, but $N \nsubseteq \mathrm{Soc}(B)$.

LEMMA 3.5. *Let* I *and* J *be ideals of* B *and set* $L = IJ$*. Then,*

(i) $I/(I \cap J) \subseteq \mathrm{Soc}(B/(I \cap J))$ $(I/(I \cap J) \subseteq \mathrm{Fix}(B/(I \cap J)))$ *if, and only if,* $L/J \subseteq \mathrm{Soc}(B/J)$ $(L/J \subseteq \mathrm{Fix}(B/J))$*.*

(ii) $J/(I \cap J) \subseteq \mathrm{Soc}(B/(I \cap J))$ $(J/(I \cap J) \subseteq \mathrm{Fix}(B/(I \cap J)))$ *if, and only if,* $L/I \subseteq \mathrm{Soc}(B/I)$ $(L/I \subseteq \mathrm{Fix}(B/I))$*.*

Proof. We only prove (3.5) as (3.5) is analogous. We can assume without loss of generality that $I \cap J = 1$.

Suppose that $I \subseteq \mathrm{Soc}(B)$. Let $zJ \in L/J$ and $bJ \in B/J$. Then, $zJ = xJ$ for some element $x \in I$. Since $\lambda_x(b) = b$ and $x + b = b + x$, it follows that $\lambda_{zJ}(bJ) = \lambda_{xJ}(bJ) = \lambda_x(b)J = bJ$ and $zJ + bJ = (x + b)J = (b + x)J = bJ + zJ$. Consequently, $L/J \subseteq \mathrm{Soc}(B/J)$.

https://doi.org/10.1017/prm.2023.37 Published online by Cambridge University Press

Conversely, suppose that $L/J \subseteq \mathrm{Soc}(B/J)$. We shall prove that $I \subseteq \mathrm{Soc}(B)$. Let $x \in I$ and $y \in B$. Since $L/J \subseteq \mathrm{Soc}(B/J)$, then $\lambda_{xJ}(yJ) = \lambda_x(y)J = yJ$. By proposition 2.2, $\lambda_x(y)J = \lambda_x(y) + J = y + J = yJ$. Thus, $\lambda_x(y) - y = x * y \in J \cap I = 1$. Therefore, $\lambda_x(y) = y$.

Furthermore, $(x + y) + J = (y + x) + J$. Note that I is a normal subgroup of $(B, +)$. Hence, $y - x + y \in I$ so that $x + y - x - y \in I \cap J = 1$ and $x + y = y + x$. Therefore, $I \subseteq \mathrm{Soc}(B)$.

Suppose that $I \subseteq \mathrm{Fix}(B)$. Let $aJ \in B/J$ and $bJ \in B/J$. Then, $\lambda_{bJ}(aJ) = \lambda_b(a)J = aJ$. This yields $L/J \subseteq \mathrm{Fix}(B/J)$. Conversely, if $L/J \in \mathrm{Fix}(B/J)$, then for every $a \in I$ and $b \in B$, $aJ = \lambda_{bJ}(aJ) = \lambda_b(a)J$. By proposition 2.2, $a + J = \lambda_b(a) + J$. Thus, $\lambda_b(a) - a = b * a \in J \cap I = 1$. Therefore, $\lambda_b(a) = a$ and so $I \subseteq \mathrm{Fix}(B)$. □

The following result is quite useful for constructing new chief series from old ones.

LEMMA 3.6. *Suppose that U and V are ideals of B such that $U \subseteq V$ and U/V is a chief factor of B. Let I be a ideal of B. Then, either $UI \neq VI$ or $U \cap I \neq V \cap I$. Furthermore,*

1. *If $UI \neq VI$, then UI/VI is a chief factor of B isomorphic to U/V.*
2. *If $U \cap I \neq V \cap I$, then $(U \cap I)/(V \cap I)$ is a chief factor of B isomorphic to U/V.*

Proof. Note that UI and VI are ideals of B by corollary 2.3. The isomorphism theorem implies that

$$UI/VI = U(VI)/(VI) \cong U/(U \cap VI).$$

Since $U \cap VI$ is an ideal of B by corollary 2.3, $V \subseteq U \cap VI \subseteq U$ and U/V is simple, it follows that either $V = U \cap VI$ or $U = U \cap VI$. If $V = U \cap VI$, then $U \neq U \cap VI$. Hence, $UI \neq VI$ and UI/VI is a chief factor of B isomorphic to U/V. If $U = U \cap VI$, then $U = V(U \cap I)$ and, by the isomorphism theorem, we have

$$U/V = (U \cap I)V/V \cong (U \cap I)/(V \cap I).$$

Consequently, $U \cap I \neq V \cap I$ and $(U \cap I)/(V \cap I)$ is a chief factor of B isomorphic to U/V. □

PROPOSITION 3.7. *Assume that B has a chief series and let I be an ideal of B. Then, I has a chief series, I is a member of a chief series of B and B/I has a chief series.*

Proof. Let $1 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_n = B$ be the given chief series of B and write $U_i = J_i \cap I$, so that $1 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n = I$ is a series of ideals of B. By lemma 3.6, each of the factors U_i/U_{i-1} is either trivial or a chief factor of B. It follows that if we delete U_i from the above series whenever $U_i = U_{i-1}$, what remains is part of a chief series of B. In particular, it is a chief series of I.

For the second statement, we denote $V_i = J_i I$ for $0 \leqslant i \leqslant n$. Then, $I = V_0 \leqslant V_1 \subseteq \cdots \subseteq V_n = B$ is a series of ideals of B. By lemma 3.6, either $V_i = V_{i-1}$ or V_i/V_{i-1}

https://doi.org/10.1017/prm.2023.37 Published online by Cambridge University Press

is a chief factor of B. It follows that by deleting V_i if $V_i = V_{i-1}$ we obtain part of a chief series of B.

It is clear that $1 = U_0 \subseteq \cdots \subseteq U_n = I = V_0 \subseteq \cdots \subseteq V_n = B$ is a chief series of B passing through I. Furthermore, $I = V_0/I \subseteq \cdots \subseteq V_n/I = B/I$ is a chief series of B/I because, by the isomorphism theorem,

$$(V_i/I)/(V_{i-1}/I) = V_i/V_{i-1}, \quad i \in \{1, \ldots, n\}. \qquad \square$$

LEMMA 3.8. *Let I and J be ideals of a finite skew left brace, with $J \subseteq I$. If $I/J \subseteq \mathrm{Rad}(B/J)$, then $I \subseteq \mathrm{Rad}(B)J$.*

Proof. We use induction on the order of B. If $J = 1$, the result is obviously true. Hence, we suppose $J \neq 1$ and let L be a minimal ideal of B contained in J. Since the hypothesis carry over to B/L, we conclude by induction that $I/L \subseteq \mathrm{Rad}(B/L)J/L$.

If $L \subseteq \mathrm{Rad}(B)$ then $\mathrm{Rad}(B/L) = \mathrm{Rad}(B)/L$ and thus, $I \subseteq \mathrm{Rad}(B)J$, as desired.

Therefore, suppose that L is not contained in $\mathrm{Rad}(B)$. Let M be a maximal ideal of B such that $L \nsubseteq M$. Then, $B = L + M = LM$ and $L \cap M = 1$. Hence, $I = L(I \cap M)$.

The isomorphism $x \mapsto xL$ from M onto B/L yields $\mathrm{Rad}(M)L/L = \mathrm{Rad}(B/L)$ and therefore, $I \subseteq \mathrm{Rad}(M)LJ = \mathrm{Rad}(M)J$. By [**11**, proposition 3.9], $\mathrm{Rad}(M) \subseteq \mathrm{Rad}(B)$. Consequently, we obtain $I \subseteq \mathrm{Rad}(B)J$. $\square$

LEMMA 3.9. *Let I and J be distinct minimal ideals of a finite skew left brace B. Then, there exists a bijection*

$$f \colon \{I, IJ/I\} \to \{J, IJ/J\}$$

such that, corresponding chief factors are isomorphic and r-chief factors correspond to one another.

Proof. Put $L = IJ$, and assume that $I \subseteq \mathrm{Rad}(B)$. Then, $L/J \subseteq \mathrm{Rad}(B)J/J \subseteq \mathrm{Rad}(B/J)$.

If $L/I \subseteq \mathrm{Rad}(B/I)$ then $L \subseteq \mathrm{Rad}(B)$, because $\mathrm{Rad}(B/I) = \mathrm{Rad}(B)/I$. In this case, the map $f(I) = L/J$ and $f(L/I) = J$ satisfies the requirements. If $L/I \nsubseteq \mathrm{Rad}(B/I) = \mathrm{Rad}(B)/I$, then J is not an r-chief factor of B and the same choice of f will suffice. It only remains to consider the case where $I \cap \mathrm{Rad}(B) = J \cap \mathrm{Rad}(B) = 1$ and $L/J \subseteq \mathrm{Rad}(B/J)$.

Let M be a maximal ideal of B such that $B = I + M = IM$ and $I \cap M = 1$. Write $K = L \cap M$. Then, K is an ideal of B and $KI = L$. Hence, $K \cong L/I \cong J$.

If $K = J$, then $L \subseteq M$, which is a contradiction to the fact $I \cap M = 1$. Therefore, $K \neq J$, $L = KJ$ and $I \cong K \cong J$. By lemma 3.8, $L \subseteq \mathrm{Rad}(B)J$. Hence, $L = J(L \cap \mathrm{Rad}(B))$.

Note that $J \cap (L \cap \mathrm{Rad}(B)) = 1$ and $L \cap \mathrm{Rad}(B) \subseteq K$. Thus, $K = L \cap \mathrm{Rad}(B)$ and K is an r-chief factor of B which is isomorphic to L/J. Since $L/I = KI/I \subseteq \mathrm{Rad}(B)I/I = \mathrm{Rad}(B)/I$, it follows that $f(I) = J$ and $f(L/I) = L/J$ satisfies the requirements of the lemma. $\square$

https://doi.org/10.1017/prm.2023.37 Published online by Cambridge University Press

THEOREM 3.10. *Assume that B has chief series. Then, the chief factors in a chief series are unique, namely, if*

$$\mathcal{I}\colon 1 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = B,$$
$$\mathcal{J}\colon 1 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_m = B$$

are two chief series of B, then $n = m$ and there is some permutation π of $\{1, \ldots, n\}$ such that

$$J_{\pi(i)}/J_{\pi(i)-1} \cong I_i/I_{i-1}, \quad 1 \leqslant i \leqslant n.$$

Furthermore, I_i/I_{i-1} is an τ-chief factor of B if, and only if, $J_{\pi(i)}/J_{\pi(i)-1}$ is an τ-chief factor of B, where $\tau \in \{s, f\}$. If, in addition, B is finite, r-chief factors of $\mathcal{I}$ correspond to r-chief factors of $\mathcal{J}$.

Proof. We call two chief series $\mathcal{X}$ and $\mathcal{Y}$ of B *equivalent* if $\mathcal{X}$ and $\mathcal{Y}$ have the same length and isomorphic factors (up to rearrangement) such that the τ-factors of $\mathcal{I}$ correspond to the τ-factors of $\mathcal{J}$. We write in this case $\mathcal{X} \sim \mathcal{Y}$. It is clear that $\sim$ is an equivalence relation on the set of all chief series of B.

We use induction on the length n of $\mathcal{I}$ to show that $\mathcal{I} \sim \mathcal{J}$.

Assume that $n = 1$. Then, B is simple, and in that case $\mathcal{I} = \mathcal{J}$. Let $n > 1$ and suppose that the theorem is true for all skew left braces with chief series of length $\leqslant n - 1$. Write $I = I_1$ and $J = J_1$. If $I = J$, we can form the following chief series of B/I

$$\mathcal{I}/I\colon \quad 1 = I_1/I \subseteq I_2/I \subseteq \cdots \subseteq I_n/I = B/I,$$
$$\mathcal{J}/I\colon \quad 1 = J_1/I \subseteq J_2/I \subseteq \cdots \subseteq J_m/I = B/I.$$

Since the length of $\mathcal{I}/I$ is $n - 1$, the induction hypothesis applied to B/I yields $\mathcal{I}/I \sim \mathcal{J}/I$. Hence, $\mathcal{I} \sim \mathcal{J}$.

Now assume that $I \neq J$ and set $U = IJ$. By corollary 2.3, U is an ideal of B. In this case U/I and U/J are chief factors of B and, by proposition 3.7, there exist chief series $\mathcal{V}_1$ and $\mathcal{V}_2$ of B of the following form:

$$\mathcal{V}_1\colon \quad 1 = V_0 \subseteq V_1 = I \subseteq V_2 = U \subseteq V_3 \cdots \subseteq V_r = B,$$
$$\mathcal{V}_2\colon \quad 1 = W_0 \subseteq W_1 = J \subseteq W_2 = U \subseteq V_3 \cdots \subseteq V_r = B.$$

Since $\mathcal{I}$ and $\mathcal{V}_1$ have the minimal ideal I in common, the induction yields $\mathcal{I} \sim \mathcal{V}_1$. Similarly, $\mathcal{J} \sim \mathcal{V}_2$. Furthermore, as the chief series $\mathcal{V}_1$ and $\mathcal{V}_2$ coincide above U and $I \cap J = 1$, we can apply lemma 3.5 to conclude that $\mathcal{V}_1 \sim \mathcal{V}_2$. Consequently, $\mathcal{I} \sim \mathcal{J}$.

Assume now that B is finite. Then, arguing as above, it follows that the bijection stated in the theorem also respects r-chief factors by lemma 3.9. □

If B is a skew left brace that has a chief series $1 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = B$, we say that B has a *finite chief length* and we write $\mathrm{fcl}(B) = n$. We suppose that $\mathrm{fcl}(B) = 0$ if $B = 1$. Note that by Theorem 3.10, the chief length of B is uniquely determined. If B has no chief series, we say that it has *infinite chief length*. The chief length of a skew left brace provides a handle we can use to prove interesting results for finite chief length skew left braces, even if the skew left brace happens to be infinite.

Applying proposition 3.7, every ideal I of a skew left brace B of finite chief length is part of a chief series of B and we can get a chief series of the quotient brace B/I just considering the ideals of that series lying above I. Therefore, we have

PROPOSITION 3.11. *Let B a skew left brace of finite chief length and suppose that I is an ideal of B. If $I > 1$, then* $\mathrm{fcl}(B/I) < \mathrm{fcl}(B)$.

The finite chief length is a finiteness condition that can be viewed as decomposing into the artinian and noetherian conditions. This is the content of our next result.

THEOREM 3.12. *A skew left brace B has finite chief length if, and only if, B is artinian and noetherian.*

Proof. Assume that $B \neq 1$ has finite chief length. Let $I_1 \subseteq I_2 \dots$ be any ascending chain of ideals of B. We may assume that $I = I_1 \neq 1$. Then, $I_1/I \subseteq I_2/I \dots$ is an ascending chain of ideals of B/I. By proposition 3.11, $\mathrm{fcl}(B/I) < \mathrm{fcl}(B)$. If we argue by induction on the chief length of G, we have that $I_n = I_{n+1} = \dots$, and hence B is noetherian. Assume that $I_1 \supseteq I_2 \dots$ is any descending chain of ideals of B. Let L be a minimal ideal of B. If L is contained in I_k for all $k \geqslant 1$, then we can argue as above. Otherwise, there exists a $t \geqslant 1$ such that $I_t \cap L = 1$. Since $(I_1 + L)/L \supseteq (I_2 + L)/L \dots$ is a descending chain of ideals of B/L, we can apply induction to conclude that $I_m + L = I_{m+e} + L$ for some $m \geqslant t$ and all $e \geqslant 0$. Since $I_m \cap I = I_{m+e} \cap I = 0$, we conclude that $I_m = I_{m+e}$ for all $e \geqslant 0$, and B is artinian.

Now assume that B is artinian and noetherian but B has no finite length. Apply the noetherian condition to the set $\mathcal{T}$ of proper ideals I of B that have a chain of ideals of B: $1 = I_0 \subseteq I_1 \dots \subseteq I_n = I$ such that I_t/I_{t-1} is a chief factor of B for all $1 \leqslant t \leqslant n$ (note that $B \neq 1$), and select a maximal member A. Since B has no finite length, A is proper in B. Apply now the artinian condition to the set of all ideals of B that properly contain A and select a minimal element $C \leqslant B$. It is clear that C/A is a chief factor of B, and so we can append C to the end of part of a chief series of B: $1 = V_0 \subseteq V_1 \dots \subseteq V_n = A$ to conclude that $C \in \mathcal{T}$. This contradicts the maximality of A in $\mathcal{T}$. □

4. Applications of the Jordan–Hölder theorem to nilpotency of skew left braces

As in the case of groups, nilpotency on skew left braces can be defined in terms of iterated series. Let X, Y be subsets of a skew left brace B. We define

$$L_1(X,Y) = Y; \quad L_{n+1} = X * L_n(X,Y); \quad \text{for every } n \geqslant 1.$$
$$R_1(X,Y) = X; \quad R_{n+1} = R_n(X,Y) * Y; \quad \text{for every } n \geqslant 1.$$

LEMMA 4.1. *If $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$, then $L_n(X_1,Y_1) \subseteq L_n(X_2,Y_2)$ and $R_n(X_1,Y_1) \subseteq R_n(X_2,Y_2)$, for every $n \in \mathbb{N}$.*

Proof. It is straightforward from the above definition. □

Following [14], for the case $X = Y = B$, we denote $B^n := L_n(B,B)$ and $B^{(n)} := R_n(B,B)$. Proposition 2.5 yields that B^i and $B^{(i)}$ are, respectively, a left ideal and an ideal of B. The descendent series $\{B^n\}_{n\in\mathbb{N}}$ and $\{B^{(n)}\}_{n\in\mathbb{N}}$ are called, respectively, the *left and right series* of B.

Definition 4.2. *B is said to be left (right) nilpotent, if the left (right) series reaches the trivial subbrace* 1*. Moreover, we say that B has left (right) nilpotent of class m, if m is the smallest natural such that $B^m = 1$ ($B^{(m)} = 1$).*

Remark 4.3. In [3] it is proved that left (right) nilpotency is closed under taking quotients, subbraces and finite direct products.

Definition 4.4. *We call a sequence of left ideals $1 = L_0 \subseteq L_1 \subseteq \ldots \subseteq L_n = B$ an f-series of left ideals if, $B * L_i \subseteq L_{i-1}$, for all $1 \leqslant i \leqslant n$.*

It is proved in [3, proposition 2.26] that every non-zero ideal of a left nilpotent skew left brace B has non-zero intersection with $\mathrm{Fix}(B)$. This result is a direct consequence of the following characterization of left nilpotency.

Proposition 4.5. *B is left nilpotent if, and only if, it admits an f-series of left ideals.*

Proof. If B is left nilpotent, then left series of B clearly is an f-series of left ideals.

Conversely, suppose that B admits an f-series of left ideals

$$1 = L_0 \subseteq L_1 \subseteq \ldots \subseteq L_n = B.$$

We shall prove by induction that $B^i \subseteq L_{n-i+1}$, for every $1 \leqslant i \leqslant n+1$. It is clear that $B^1 = B = L_n$. Suppose that $B^i \subseteq L_{n-i+1}$, for some $1 \leqslant i < n+1$. Then, by definition of f-series of left ideals

$$B^{i+1} = B * B^i \subseteq B * L_{n-i+1} \subseteq L_{n-i+1} = L_{n-i}.$$

Hence, $B^{n+1} = 1$ and then, B is left nilpotent. □

On the other hand, in [3] it is proved that skew left braces of nilpotent type are right nilpotent if, and only if, they admit an s-series. Theorem 3.10 allows us to improve this result giving a characterization of the right nilpotency of a skew left brace B with chief series. In this case, the right nilpotency class of B is bounded by its chief length.

Theorem 4.6. *Assume that B is a skew left brace with chief series. Then, every chief factor of B is an s-factor if, and only if, B is of nilpotent type and right nilpotent. In this case, $\mathrm{nil\,class}_r(B) \leqslant \mathrm{fcl}(B)$.*

Proof. Suppose that every chief factor of B is an s-factor. Let

$$1 = I_0 \subseteq I_1 \subseteq \ldots \subseteq I_n = B.$$

be a chief series of B, $n = \mathrm{fcl}(B)$. Then,

$$I_i/I_{i-1} \subseteq \mathrm{Soc}(B/I_{i-1}) \subseteq Z(B/I_{i-1}, +),$$

for every $1 \leqslant i \leqslant n$. Therefore, B is of nilpotent type. Moreover, we shall see by induction that $B^{(i)} \subseteq I_{n-i+1}$, for every $1 \leqslant i \leqslant n+1$. The case $i = 1$ is obvious;

suppose that $B^{(i)} \subseteq I_{n-i+1}$, for some $1 \leqslant i < n$. Since $I_{n-i+1}/I_{n-i} \subseteq \operatorname{Soc}(B/I_{n-i})$ we can apply Lemma 3.3 to conclude that $B^{(i+1)} = B^{(i)} * B \subseteq I_{n-i+1} * B \subseteq I_{n-i}$. Hence, $B^{(n)} = 1$ and, therefore, B is right nilpotent. Furthermore, $\operatorname{nil\,class}_r(B) \leqslant n$.

Conversely, suppose that B is of nilpotent type and right nilpotent. Assume that there exists a chief factor I/J of B such that $I/J \nsubseteq \operatorname{Soc}(B/J)$. Since I/J is a minimal ideal of B/J and $\operatorname{Soc}(B/J)$ is an ideal of B/J, it follows that $I/J \cap \operatorname{Soc}(B/J) = 1$. Furthermore, as $(B/J, +)$ is nilpotent, there exists $1 \neq b \in I/J \cap Z(B/J, +)$.

By the minimality of I/J, we have that either $I/J * B/J = 1$ or $I/J * B/J = I/J$. Assume that $I/J * B/J = I/J$. Then, $R_t(I/J, B/J) = I/J \subseteq R_t(B/J, B/J) = (B/J)^{(t)}$ for every $t \in \mathbb{N}$ by lemma 4.1. But B/J is also right nilpotent and so $(B/J)^{(m)} = 1$ for some $m \in \mathbb{N}$. In particular, $I/J = 1$. This contradiction yields $I/J * B/J = 1$. Then, $1 \neq b \in I/J \cap \operatorname{Soc}(B/J)$, which is a contradiction. Consequently, $I/J \subseteq \operatorname{Soc}(B/J)$. □

The following results in [**3**] for skew left braces with chief series are consequences of theorem 4.6.

Corollary 4.7. *Let B a skew left brace with chief series. then,*

1. *[**3**, theorem 2.8] Suppose that B is right nilpotent of nilpotent type and let I be a non-trivial ideal. Then, $I \cap \operatorname{Soc}(B) \neq 1$. In particular, $\operatorname{Soc}(B) \neq 1$.*
2. *[**3**, corollary 2.10] Suppose that B is right nilpotent of nilpotent type and let I be a minimal ideal of B. Then, $I \subseteq \operatorname{Soc}(B)$.*
3. *[**3**, proposition 2.17] If $B/\operatorname{Soc}(B)$ is right nilpotent, then B is right nilpotent.*

5. Application of the Jordan–Hölder theorem to multipermutation solutions of the Yang–Baxter equation

The main result of this section is a characterization of multipermutation solutions by means of the chief factors of a skew left brace.

Let (X, r) be a non-degenerate solution of the Yang–Baxter equation, non-degenerate solution of the YBE for short, given by $r(x, y) = (\sigma_x(y), \tau_y(x))$, for all $x, y \in X$. Following [**12**], we can consider the so-called retraction relation $\sim$ on X, defined by $x \sim y$ if $\sigma_x = \sigma_y$ and $\tau_x = \tau_y$.

If $[x]$ denotes the $\sim$-class of $x \in X$, then a natural induced solution $\operatorname{Ret}(X, r) = (X/\sim, \bar{r})$ called the *retraction* of (X, r) arises, where $\bar{r}$ is defined by

$$\bar{r}([x], [y]) = ([\sigma_x(y)], [\tau_y(x)]), \quad \text{for all } [x], [y] \in X/\sim.$$

We can iterate this process and define inductively

$$\operatorname{Ret}^1(X, r) = \operatorname{Ret}(X, r),$$
$$\operatorname{Ret}^{n+1}(X, r) = \operatorname{Ret}(\operatorname{Ret}^n(X, r)), \quad \text{for all } n \geqslant 1.$$

A solution (X, r) is said to be a *multipermutation solution of level m*, if m is the smallest natural such that $\operatorname{Ret}^m(X, r)$ has cardinality 1.

If B is a skew left brace, let $r_B\colon B \times B \to B \times B$ be the map given by

$$r_B(a,b) = (\lambda_a(b), \rho_b(a)) \quad \text{for every } a, b \in B.$$

where $\rho\colon (B,\cdot) \to \mathrm{Sym}(B)$ is defined by $b \mapsto \rho_b$, with $\rho_b(a) = (a^{-1}+b)^{-1}b$, for every $a \in B$. Then, ρ is an anti-homomorphism and r_B is a bijective non-degenerate solution of the YBE called the *solution associated* with B.

Let B be a skew left brace; then, we define inductively the ascending series: $\mathrm{Soc}_1(B) = \mathrm{Soc}(B)$ and for every $n \geqslant 1$, let $\mathrm{Soc}_{n+1}(B)$ be the ideal of B such that

$$\mathrm{Soc}_{n+1}(B)/\mathrm{Soc}_n(B) = \mathrm{Soc}\left(B/\mathrm{Soc}_n(B)\right).$$

Then, following [**3**], B is said to have *finite multipermutational level* m, if m is the smallest natural such that the ascending sequence $\{\mathrm{Soc}_n(B)\}_{n\in\mathbb{N}}$ reaches B.

Recall that multipermutation solutions were first studied for involutive non-degenerate solutions of the YBE which have associated skew left braces of abelian type. The first result which relates multipermutation solutions and the socle series of skew left braces of abelian type is due to Rump.

Recall that if (X,r) and (Y,s) are two set-theoretic solutions of the YBE, a map $f\colon (X,r) \to (Y,s)$ is an isomorphism if f is bijective and $(f \times f) \circ r = s \circ (f \times f)$. In this case, we say that (X,r) and (Y,s) are isomorphic.

Proposition 5.1 [**14**, proposition 7]. *Let B be a skew left brace of abelian type and let (B, r_B) be the involutive non-degenerate solution of the YBE associated with B. Then, $(B/Soc_n(B), r_n)$ is isomorphic to the nth-retraction $\mathrm{Ret}^n(B, r_B)$, for every $n \geqslant 1$.*

In [**8**] multipermutation solutions associated to skew left braces of abelian type are characterized by means of right nilpotency.

Theorem 5.2 [**8**, theorem 4.21]. *Let B be a skew left brace of abelian type and let (B, r_B) be the involutive non-degenerate solution of the YBE associated with B. Then, (B, r_B) is a multipermutation solution of level m if, and only if, B is right nilpotent of nilpotent class $m+1$.*

Our first result in this section generalizes proposition 5.1 to the general universe of all skew left braces.

Proposition 5.3. *Let B be a skew left brace and let (B, r_B) be the solution of the YBE associated with B. Then, $(B/\mathrm{Soc}_n(B), r_n)$ is isomorphic to the nth-retraction $\mathrm{Ret}^n(B, r_B)$, for every $n \geqslant 1$. As a consequence, (B, r_B) is a multipermutation solution of level m if, and only if, B has finite multipermutational level m.*

Proof. We argue by induction on n. Assume that $n = 1$. We prove

$$b \in \mathrm{Soc}(B) \Leftrightarrow \lambda_b = \rho_b = \mathrm{id}_B\,.$$

If $b \in \mathrm{Soc}(B)$ then $b \in \mathrm{Ker}\,\lambda$ and so $\lambda_b = \mathrm{id}_B$. Moreover, $b \in Z(B,+)$ and then, $\rho_b(a) = (a^{-1}+b)^{-1}b = (b+a^{-1})^{-1}b = (ba^{-1})^{-1}b = a$, for every $a \in B$. Thus, $\rho_b = \mathrm{id}_B$.

https://doi.org/10.1017/prm.2023.37 Published online by Cambridge University Press

Conversely, suppose that $\lambda_b = \rho_b = \mathrm{id}_B$, for some $b \in B$. It remains to prove that $b \in Z(B,+)$. For every $a \in B$, it holds

$$\rho_b(a) = a \Leftrightarrow (a^{-1}+b)^{-1}b = a \Leftrightarrow (a^{-1}+b)^{-1} = ab^{-1} \Leftrightarrow a^{-1}+b = ba^{-1}$$
$$\Leftrightarrow a^{-1}+b = b+\lambda_b(a^{-1}) = b+a^{-1},$$

as desired.

Then, for the retraction relation on the solution r_B, it occurs that $a \sim b$ if, and only if, $ab^{-1} \in \mathrm{Soc}(B)$, for every $a, b \in B$. Therefore, the map $\varphi\colon B/{\sim} \to B/\mathrm{Soc}(B)$, defined as $\varphi([b]) = b\,\mathrm{Soc}(B)$, turns out to be an isomorphism between the solutions $\mathrm{Ret}(B, r_B)$ and $(B/\mathrm{Soc}(B), r_{B/\mathrm{Soc}(B)})$, as $([\lambda_a(b)], [\rho_b(a)]) = (\lambda_a(b)\,\mathrm{Soc}(B), \rho_b(a)\,\mathrm{Soc}(B))$, for every $a, b \in B$.

Now, suppose that $\mathrm{Ret}^n(B, r_B)$ is isomorphic to the solution associated with $B/\mathrm{Soc}_n(B)$, for some $n \geqslant 1$. Recall that

$$\mathrm{Ret}^{n+1}(B, r_B) = \mathrm{Ret}(\mathrm{Ret}^n(B, r_B)),$$
$$B/\mathrm{Soc}_{n+1}(B) \cong (B/\mathrm{Soc}_n(B))/\mathrm{Soc}(B/\mathrm{Soc}_n(B)).$$

A similar argument to the case $n = 1$ shows that the solutions $\mathrm{Ret}^{n+1}(B, r_B)$ and $B/\mathrm{Soc}_{n+1}(B)$ are isomorphic. □

In [3], finite multipermutational level of a skew left brace is characterized in terms of nilpotency.

THEOREM 5.4 [3, theorem 2.20]. *Let B be a skew left brace. Then, B has finite multipermutation level if, and only if, B is of nilpotent type and right nilpotent.*

The study of the decomposition into chief factors of a skew left brace allows us to complete this theorem. The following result shows that the multipermutational level is bounded by the chief length of a skew left brace having chief series.

THEOREM 5.5. *Let B be a skew left brace with chief series. Then, B has finite multipermutational level m if, and only if, every chief factor of B is an s-factor. In such case, $m \leqslant \mathrm{fcl}(B)$.*

Proof. Assume that B has finite multipermutational level m. We show that every chief factor of B is an s-chief factor by induction on m. If $m = 1$, then $B = \mathrm{Soc}(B)$ and the result follows. Suppose that $m \geqslant 1$ and the statement holds for every skew left brace with chief series of multipermutational level $m - 1$. f

Note that $\mathrm{Soc}(B)$ is a non-trivial ideal of B. By lemma 3.7, $B/\mathrm{Soc}(B)$ has chief series. Furthermore, $B/\mathrm{Soc}(B)$ has finite multipermutational level $m - 1$. Thus, by induction hypothesis, every chief factor of $B/\mathrm{Soc}(B)$ is an s-factor and $m - 1$ is less or equal than the chief length of $B/\mathrm{Soc}(B)$. According to lemma 3.7, there exists a chief series of B passing through $\mathrm{Soc}(B)$. Moreover every chief factor of this series is an s-chief factor of B. Applying theorem 3.10, it follows that every chief factor of B is an s-factor. Furthermore, m is less or equal than the chief length of B.

https://doi.org/10.1017/prm.2023.37 Published online by Cambridge University Press

Conversely, suppose that every chief factor of B is an s-factor. If B has not finite multipermutational level, then there exists $n_0 \in \mathbb{N}$ such that $\operatorname{Soc}_n(B) = \operatorname{Soc}_{n_0}(B) < B$, for every $n \geqslant n_0$ since B is noetherian by theorem 3.12. Now, we consider a chief series of B passing through $\operatorname{Soc}_{n_0}(B)$,

$$1 = I_1 < \ldots < I_i = \operatorname{Soc}_{n_0}(B) < I_{i+1} < \ldots < I_m = B.$$

Since every chief factor is an s-factor, it follows that $1 \neq I_{i+1}/I_i \subseteq \operatorname{Soc}(B/I_i) = \operatorname{Soc}(B/\operatorname{Soc}_{n_0}) = \operatorname{Soc}_{n_0+1}(B)/\operatorname{Soc}_{n_0}(B)$. Therefore, $\operatorname{Soc}_{n_0}(B) \lneq \operatorname{Soc}_{n_0+1}(B)$, which is a contradiction. $\square$

Corollary 5.6. *Let B be a skew left brace with chief series. The following statements are pairwise equivalent:*

1. *B has finite multipermutational level m.*
2. *(B, r_B) is a multipermutational solution of level m.*
3. *B is of nilpotent type and is right nilpotent.*
4. *Every chief factor of B is an s-factor.*

In this case,

- $m \leqslant \operatorname{fcl}(B)$*,*
- $\operatorname{nil\,class}_r(B) \leqslant \operatorname{fcl}(B)$ *and*
- $\operatorname{nil\,class}_r(B) \leqslant m + 1$*.*

In particular, if $\operatorname{nil\,class}_r(B) = \operatorname{fcl}(B)$*, then* $m = \operatorname{nil\,class}_r(B) = \operatorname{fcl}(B)$*.*

Proof. It remains to prove that if B has finite multipermutational level m, then $\operatorname{nil\,class}_r(B) \leqslant m + 1$. We show that $B^{(k)} \subseteq \operatorname{Soc}_{m-k+1}(B)$, for every $1 \leqslant k \leqslant m$, by induction on k. Clearly $B^{(1)} = B = \operatorname{Soc}_m(B)$. Suppose that $B^{(k)} \subseteq \operatorname{Soc}_{m-k+1}(B)$, for some $1 \leqslant k < m$. Since

$$\operatorname{Soc}_{m-k+1}(B)/\operatorname{Soc}_{m-k}(B) = \operatorname{Soc}\left(B/\operatorname{Soc}_{m-k}(B)\right).$$

The induction hypothesis and lemma 3.3 yield

$$B^{(k+1)} = B^{(k)} * B \subseteq \operatorname{Soc}_{m-k+1}(B) * B \subseteq \operatorname{Soc}_{m-k}(B),$$

as desired.

Then, $B^{(m)} \subseteq \operatorname{Soc}(B)$ and, thus, $B^{(m+1)} = 0$. Hence, $\operatorname{nil\,class}_r(B) \leqslant m + 1$. $\square$

The right nilpotency class of a skew left brace of finite multipermutational level m is not $m + 1$ in general.

Example 5.7. Let G be a finite group with nilpotency class $m \geqslant 3$, then the trivial skew left brace $(B, +, \cdot)$, with $(B, +) = (B, \cdot) = G$, satisfies $\operatorname{nil\,class}_r(B) = 2$, as $B^{(2)} = 1$, but B has multipermutational level m, as the socle series of B coincides with the upper central series of G.

https://doi.org/10.1017/prm.2023.37 Published online by Cambridge University Press

Acknowledgements

These results are a part of the R+D+i project supported by the Grant PGC2018-095140-B-I00, funded by MCIN/AEI/10.13039/501100011033 and by 'ERDF A way of making Europe.'

References

1 A. Ballester-Bolinches and R. Esteban-Romero. Triply factorised groups and the structure of skew left braces. *Commun. Math. Stat.* **10** (2022), 353–370.

2 R. Baxter. Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. I. Some fundamental eigenvectors. *Ann. Phys.* **76** (1973), 1–24.

3 F. Cedó, A. Smoktunowicz and L. Vendramin. Skew left braces of nilpotent type. *Proc. London Math. Soc.* **118** (2019), 1367–1392.

4 L. N. Childs. Skew braces and the Galois correspondence for Hopf Galois structures. *J. Algebra* **511** (2018), 270–291.

5 F. Chouraqui. Garside groups and Yang–Baxter equations. *Commun. Algebra* **38** (2010), 4441–4460.

6 V. G. Drinfeld. On some unsolved problems in quantum group theory. In *Quantum Groups. Proceedings of Workshops held in the Euler International Mathematical Institute, Leningrad, Fall 1990* (ed. P. P. Kulish). Lecture Notes in Mathematics, vol. 1510, pp. 1–8 (Berlin: Springer-Verlag, 1992).

7 P. Etingof, T. Schedler and A. Soloviev. Set theoretical solutions to the quantum Yang–Baxter equation. *Duke Math. J.* **100** (1999), 169–209.

8 T. Gateva-Ivanova. Set-theoretic solutions of the Yang–Baxter equation, braces and symmetric groups. *Adv. Math.* **338** (2018), 649–701.

9 T. Gateva-Ivanova and M. Van den Bergh. Semigroups of I-type. *J. Algebra* **206** (1998), 97–112.

10 L. Guarnieri and L. Vendramin. Skew-braces and the Yang–Baxter equation. *Math. Comput.* **86** (2017), 2519–2534.

11 E. Jespers, Ł. Kubat, A. Van Antwerpen and L. Vendramin. Radical and weight of skew braces and their applications to structure groups of solutions of the Yang–Baxter equation. *Adv. Math.* **385** (2021), 107767.

12 V. Lebed and L. Vendramin. On structure groups of set-theoretic solutions to the Yang–Baxter equation. *Proc. Edinb. Math. Soc.* **62** (2019), 683–717.

13 J.-H. Lu, M. Yan and Y.-C. Zhu. On the set-theoretical Yang–Baxter equation. *Duke Math. J.* **104** (2000), 1–18.

14 W. Rump. Braces, radical rings, and the quantum Yang–Baxter equation. *J. Algebra* **307** (2007), 153–170.

15 A. Smoktunowicz and L. Vendramin. On skew braces (with an appendix by N. Byott and L. Vendramin). *J. Comb. Algebra* **2** (2018), 47–86.

16 Y. Sysak. Products of groups and local nearrings. *Note Mat.* **28** (2008), 177–211.

17 C. N. Yang. Some exact results for many-body problem in one dimension with repulsive delta-function interaction. *Phys. Rev. Lett.* **19** (1967), 1312–1315.