MEAN-CONTINUOUS INTEGRALS

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Introduction. Descriptive definitions of Cesàro-Denjoy integrals (CD-integrals) equivalent to the Cesàro-Perron integrals (CP-integrals) introduced by J. C. Burkill [1, 2] have been given by Miss Sargent [6] (see §2). The CD-integrals are generalizations of the special Denjoy integral [5, p. 201]. They are somewhat complicated in that modifications of the definitions of continuity, generalized absolute continuity in the restricted sense (ACG*) [5, p. 231], and of derivatives are required for each order. In the present paper a scale of integrals is obtained which is based on the descriptive definition of the general Denjoy integral [5, p. 241]. The approximate derivative and a slightly modified definition of generalized absolute continuity (ACG) are used for all orders so that the only concept generalized for increasing orders is that of continuity. The resulting $r^{th}$ order integral, $r = 0, 1, 2, \ldots$, called the $r^{th}$ generalized mean integral ($GM_r$-integral), contains the corresponding $C_rD$- and $C_rP$-integrals.

In §1 the descriptive definition of the $GM_r$-integral is given and some of the more important properties of the integral, including a theorem on integration by parts, are derived. The relation between the $GM_r$-integral and the $C_rD$, $C_rP$-integrals is considered in §2. In §3 a constructive definition of the $GM_r$-integral is given and shown to be equivalent to the descriptive definition. The paper concludes with a proof that the indefinite $GM_r$-integral takes all values between its upper and lower bounds on any interval over which the integral exists.

1. The descriptive generalized mean integrals. We shall obtain a scale or series of generalized mean integrals, $GM_r$-integrals, $r = 0, 1, \ldots$ of increasing generality in the sense that each integral will be contained in but not equivalent to all those with higher subscripts. We take as our starting point the general Denjoy integral.

Notation. We number theorems by Roman numerals, lemmas by Arabic numerals, and the definitions which change with the order by groups of letters. The order concerned in each case is indicated by a subscript, e.g. Definition ($M_rC$) is the definition of mean continuity of order $r$. With this notation we refer, for example, to “Theorem II,” rather than “Theorem II for the $GM_r$-integral.”

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DEFINITION ($M_r$). We define the $M_r$-mean of $F(x)$ on $(a, b)$ as $F(b)$ for $r = 0$ and as

$$M_r(F, a, b) = \frac{1}{(b-a)^r} \int_a^b (b-t)^{r-1} F(t) dt,$$

for $r$ a positive integer, where the integral in the definition of the $M_1$-mean is in the general Denjoy sense and the sense in which the integral involved in the $M_r$-mean is required to exist will be stated below.

DEFINITION ($M_r$-C). The function $F(x)$ is continuous in the $r$th mean sense ($M_r$-continuous) at $x_0$ if $M_r(F, x_0, x_0 + \varepsilon) \to F(x_0)$ as $\varepsilon \to 0$.

DEFINITION ($M_r$I). The function $f(x)$ is $GM_r$-integrable on $(a, b)$ if there exists an $M_r$-continuous function $F(x)$ that is ACG on $(a, b)$ and is such that the approximate derivative of $F(x)$, $ADF(x)$, [5, p. 220] exists and is equal to $f(x)$ almost everywhere on $(a, b)$. The function $F(x)$ is then called an indefinite $GM_r$-integral of $f(x)$ on $(a, b)$. The definite $GM_r$-integral of $f$ over $(a, b)$ is designated by

$$GM_r(f, a, b) = (GM_r) \int_a^b f(x) dx = F(b) - F(a).$$

For $r = 0$ this definition is seen to be equivalent to the descriptive definition of the general Denjoy integral.

The integral in the definition of the $M_r$-mean, $r \geq 2$, is required to exist in the sense of the $GM_{r-1}$-integral. In order to define the $GM_r$-integral we must therefore assume that the $GM_{r-1}$-integral has been defined. To establish the properties of the $GM_r$-integral we must assume the following properties for the $GM_{r-1}$-integral.

PROPERTY I$_{r-1}$. If $f_1(x), f_2(x)$ are $GM_{r-1}$-integrable on $(a, b)$ and $f_1(x) \geq f_2(x)$ almost everywhere on $(a, b)$ then

$$GM_{r-1}(f_1, a, b) \geq GM_{r-1}(f_2, a, b).$$

PROPERTY II$_{r-1}$. The $GM_{r-1}$-integral contains the $GM_{r-2}$-integral.

PROPERTY III$_{r-1}$. Let $f_n(x)$ be $GM_n$-integrable on $(a, b)$, let $F_n(x) = GM_n(f_n, a, x)$, $n = r - 1, r - 2, \ldots 0$, and let

$$g_n(x) = \int_a^x dt \int_a^t dt \ldots \int_a^{t_{n-1}} g(t) dt,$$

where $g(x)$ is of bounded variation on $(a, b)$. Then $f_n(x)g_n(x)$ is $GM_n$-integrable on $(a, b)$ and

$$(GM_n) \int_a^b f_n(x)g_n(x) dx = F_n(b)g_n(b) - (GM_{n-1}) \int_a^b F_n(x)g_{n-1}(x) dx.$$

Saks' definition of ACG [5, p. 222] implies that $F(x)$ is continuous and does not require the sets $E_n$ to be closed. The condition that the sets $E_n$ be closed gives no restriction when $F(x)$ is continuous since the continuity of $F(x)$ is sufficient to ensure that if $F(x)$ is $AC$ on an arbitrary set it is $AC$ on the closure of this set.
(Property III\(_{r-1}\) implies that, if \(f(x)\) is \(GM_{r-1}\)-integrable, then \(f(x)g_{r-1}(x)\) can be integrated by parts \(r\) times.)

That properties I\(_{r-1}\)-III\(_{r-1}\) may be presumed is justified by induction. Since the \(GM_{r}\)-integral is the general Denjoy integral it is clear that Property I\(_{0}\) is true and that Property III\(_{0}\) is true provided that the \(GM_{r-1}\)-integral is interpreted as a Stieltjes integral and \(g(x)dx\) is replaced by \(dg(x)\) [5, p. 246]. If \(f(x)\) is \(GM_{r}\)-integrable on \((a, b)\), \(F(x) = GM_{r}(f, a, x)\) is \(ACG\) on \((a, b)\) and \(ADF(x) = f(x)\) almost everywhere on \((a, b)\). To establish II\(_{1}\) we need only prove that \(F(x)\) is \(M_{1}\)-continuous and, since \(F(x)\) is continuous, this is easily done. Our inductive process will be complete if, when we define the \(GM_{r}\)-integral and assume Properties I\(_{r-1}\)-III\(_{r-1}\), we can establish Properties I\(_{r}\)-III\(_{r}\).

**Lemma 1.** If there exists an interval \((x_0, x_0 + h)\), \(h > 0\) and a positive number \(d\) such that \(F(t) - F(x_0) < d\) for all \(t\) except at most a set of measure zero, then \(F(x)\) cannot be \(M_{r}\)-continuous at \(x_0\).

Let \(x\) be any point in \((x_0, x_0 + h)\). Then, using Property I\(_{r-1}\),

\[
M_r(F, x_0, x) = r(x - x_0)^{-r}(GM_{r-1}) \int_{x_0}^{x} (x - t)^{-r-1}F(t)dt \\
\geq r(x - x_0)^{-r} \int_{x_0}^{x} (x - t)^{-r-1}[F(x_0) + d] dt \\
= F(x_0) + d.
\]

It follows that \(F(x)\) cannot be \(M_r\)-continuous at \(x_0\). Similar results hold for \(h < 0\) and also if \(F(t) - F(x_0) > d\) is replaced by \(F(t) - F(x_0) < -d\).

**Theorem I.** If \(F(x)\) is monotone and \(M_r\)-continuous for \(a < x < b\) then \(F(x)\) is monotone and continuous for \(a \leq x \leq b\).

If we suppose that \(F(x)\) is not continuous at some point \(x, a \leq x \leq b,\) Lemma 1, gives a contradiction.

**Theorem II.** If \(F(x)\) is \(M_r\)-continuous and \(ACG\) on \((a, b)\) and if \(ADF(x) \geq 0\) almost everywhere on \((a, b)\), then \(F(x)\) is non-decreasing on \((a, b)\).

**Definition** [4, p. 130]. A function \(F(x)\) is lower semi-absolutely continuous (\(AC\)) over a set \(E\) if to a given positive number \(\epsilon\) there corresponds a positive number \(\delta\) such that for any non-overlapping set of intervals \((a_j, a'_j)\) with \(a_j, a'_j\) points of \(E\), \(\Sigma_j \{F(a'_j) - F(a_j)\} > -\epsilon\) when \(\Sigma_j (a'_j - a_j) < \delta\).

**Definition.** A function \(F(x)\) is generalized lower semi-absolutely continuous (\(ACG\)) on \(E\) if \(E\) can be covered by a denumerable sequence of closed sets \(E_1, E_2, \ldots\) such that \(F(x)\) is \(AC\) on each set \(E_j\).

If \(\geq -\epsilon\) is replaced by \(\leq \epsilon\) in the above definitions, the corresponding definitions of upper and generalized upper semi-absolutely continuous (\(AC\)) and \((ACG)\) functions are obtained. If \(F(x)\) is \(ACG\) on \((a, b)\) it is both \(ACG\) and \(ACG\) on \((a, b)\).
Lemma 2. If $F(x)$ is ACG on $(a, b)$ and if $ADF(x)^{\geq 0}$ almost everywhere on $(a, b)$, then there is an interval $(l, m)$ on $(a, b)$ over which $F(x)$ is non-decreasing.

Let $E_1, E_2, \ldots$ be the sets over which $F(x)$ is ACG. Since these sets are closed there exists, by Baire’s theorem [5, p. 54], an interval $(l, m)$ on $(a, b)$ such that $E_n(l, m) = (l, m)$ for some $n$.

Let $G$ be the set of points of $(l, m)$ at which $ADF \geq 0$. For $x$ a point of $G$ we then have

$$
\frac{F(x + h_i) - F(x)}{h_i} > -\epsilon,
$$

with $x + h_i$ on a set of density unity at $x$, $h_i \to 0$. We can then use the Vitali covering theorem [5, p. 109] to get a finite non-overlapping set of intervals $(x_k, x'_k)$ satisfying this relation and such that $\Sigma_k (x'_k - x_k) > mG - \delta = m - l - \delta$, where $\delta$ is sufficiently small to ensure that for $(a_j, a'_j)$ a set of non-overlapping intervals with $\Sigma_j (a'_j - a_j) < \delta$ then

$$
\Sigma_j \{ F(a'_j) - F(a_j) \} > -\epsilon.
$$

Let $(a_j, a'_j)$ be the intervals complementary to the intervals $(x_k, x'_k)$. Then

$$
F(m) - F(l) = \Sigma_k \{ F(x'_k) - F(x_k) \} + \Sigma_j \{ F(a'_j) - F(a_j) \} > -\epsilon (m - l) - \epsilon.
$$

Since $\epsilon$ is arbitrary, $F(m) - F(l) \geq 0$.

In a similar manner it may be shown that $F(m') - F(l') \geq 0$ for $(l', m')$ any interval on $(l, m)$. Hence $F(x)$ is non-decreasing on $(l, m)$.

Lemma 3r. Let $F(x)$ be $M_r$-continuous and ACG on $(a, b)$ and let $ADF(x) \geq 0$ almost everywhere on $(a, b)$. If $P$ is a perfect set on $(a, b)$ with $F(x)$ non-decreasing on the intervals complementary to $P$, then there is an interval $(l, m)$ containing points of $P$ with $F(x)$ non-decreasing on $(l, m)$.

Since $P$ is perfect, by Baire’s theorem there exists an interval $(l, m)$ containing points of $P$ and such that $P(l, m)$ is identical with $E_k(l, m)$ for some $k$, where $E_k$ is one of the sets over which $F(x)$ is ACG.

Let $F_1(x) = F(x)$ on $P(l, m)$ and let $F_1(x)$ be linear in the intervals complementary to $P(l, m)$ in such a way that $F_1(x)$ is continuous on $(l, m)$. Then $F_1(x)$ is ACG on $(l, m)$. Since $F(x)$ is $M_r$-continuous and non-decreasing on an interval $a_j < x < b_j$ complementary to $P$ it follows from Theorem I, that $F(x)$ is continuous and non-decreasing on $a_j \leq x \leq b_j$. Hence $ADF_1(x) \geq 0$ almost everywhere on $(l, m)$. We can therefore use the argument of the preceding lemma to show that $F_1(x)$ is non-decreasing on $(l, m)$ and $F(x)$ therefore non-decreasing on $P(l, m)$. We conclude that $F(x)$ is non-decreasing on $(l, m)$.

Theorem II'. If $F(x)$ is $M_r$-continuous and ACG on $(a, b)$ and if $ADF(x) \geq 0$ almost everywhere on $(a, b)$ then $F(x)$ is non-decreasing on $(a, b)$.

If we assume that Theorem II' is false we can use Lemmas 2 and 3, as in
the proof of Theorem I, [4, p. 133], to obtain a contradiction. Theorem II, then follows as a corollary.

**Theorem III.** If \( f_1(x) \) and \( f_2(x) \) are \( GM_r \)-integrable on \((a, b)\) and \( f_1(x) \geq f_2(x) \) almost everywhere on \((a, b)\) then \( GM_r(f_1, a, b) \geq GM_r(f_2, a, b) \).

If we set \( F_1(x) = GM_r(f_1, a, x) \) and \( F_2(x) = GM_r(f_2, a, x) \), then \( F_1(x) - F_2(x) \) is \( M_r \)-continuous and \( ACG \) on \((a, b)\) and \( AD[F_1(x) - F_2(x)] = ADF_1 - ADF_2(x) = f_1(x) - f_2(x) \geq 0 \) almost everywhere on \((a, b)\). Hence, by Theorem II, \( F_1(x) - F_2(x) \) is non-decreasing and, since \( F_1(a) = F_2(a) = 0 \), \( F_1(b) \geq F_2(b) \). We have therefore established Property I.

**Theorem IV.** If \( f(x) \) is \( GM_{r-1} \)-integrable on \((a, b)\) it is necessarily \( GM_r \)-integrable.

We need only show that \( F(x) = GM_{r-1}(f, a, x) \) is \( M_r \)-continuous. By hypothesis

\[
(GM_{r-1}) \int_x^{x+h} (x + h - t)^{r-2} \{ F(t) - F(x) \} \, dt = o(h^{r-1})
\]
as \( h \to 0 \). The equality

\[
\frac{r}{h^r} (GM_{r-2}) \int_x^{x+h} (x + h - t)^{r-1} \{ F(t) - F(x) \} \, dt = \frac{r(r-1)}{h^r} (GM_{r-2}) \int_x^{x+h} dt (GM_{r-3}) \int_x^t (t - \eta)^{r-2} \{ F(\eta) - F(x) \} \, d\eta
\]
may be established by integrating the integral on the left side by parts \( r - 1 \) times and the inner integral on the right \( r - 2 \) times [2, p. 543], operations which are justified by Property III_{r-1}. By Property II_{r-1} the integral on the left side exists in the \( GM_{r-1} \)-sense and has the same value. The right hand side is equal to

\[
r(r-1)h^{-r} \int_x^{x+h} o [(t - x)^{r-1}] \, dt = o(1)
\]
as \( h \to 0 \). We have therefore established Property II.

**Theorem V.** Let \( F(x) \) be \( M_r \)-continuous at \( x \) and let

\[
g_r(t) = \int_a^t \int_a^{t_1} \ldots \int_a^{t_{r-1}} g(t) \, dt,
\]
where \( g(t) \) is of bounded variation on \((a, x)\). Then \( F(x)g_r(x) \) is \( M_r \)-continuous at \( x \).

Using the Cesàro-Perron analogue of Property III_{r-1} Burkill [2, p. 549] establishes Theorem V, for the \( C_P \)-integral. The method of proof applies as well to the \( GM_r \)-integral.

**Theorem VI.** (Integration by parts). Let \( f(x) \) be \( GM_r \)-integrable on \((a, b)\) and let \( g_r(x) \) be defined as in Theorem V. Then \( f(x)g_r(x) \) is \( GM_r \)-integrable on \((a, b)\) and

\[
(GM_r) \int_a^b f(x)g_r(x) = F(b)g_r(b) - (GM_{r-1}) \int_a^b F(x)g_{r-1}(x) \, dx.
\]
By Property III, \( F(x)g_{r-1}(x) \) is \( GM_{r-1} \)-integrable. We set
\[
H(x) = F(x)g_r(x) - (GM_{r-1}) \int_a^x F(t)g_{r-1}(t) \, dt.
\]

By Theorem V, \( F(x)g_r(x) \) is \( M_r \)-continuous on \((a, b)\). By Theorem IV, \( F(t)g_{r-1}(t) \) is also \( GM_r \)-integrable and to \( GM_{r-1}(Fg_{r-1}, a, x) \). Hence \( GM_{r-1}(Fg_{r-1}, a, x) \) is \( M_r \)-continuous on \((a, b)\). Since the sum of two \( M_r \)-continuous functions is \( M_r \)-continuous, \( H(x) \) is \( M_r \)-continuous.

Now \( F(x) \) is \( ACG \) and \( g_r(x) \) is \( AC \) on \((a, b)\). Since the product of two \( AC \) functions is \( AC \), \( F(x)g_r(x) \) is \( AC \) and therefore \( F(x)g_r(x) \) is \( ACG \) on \((a, b)\). Since \( GM_{r-1}(Fg_{r-1}, a, x) \) is also \( AC \) on \((a, b)\) it follows that \( H(x) \) is \( ACG \) on \((a, b)\).

As \( GM \)-integrals the functions \( F(x) \) and \( GM_{r-1}(Fg_{r-1}, a, x) \) are approximately derivable almost everywhere on \((a, b)\). Since \( ADF(x) = f(x) \) almost everywhere on \((a, b)\), and \( AD [GM_{r-1}(Fg_{r-1}, a, x)] \) is equal to \( F(x)g_{r-1}(x) \) almost everywhere on \((a, b)\), \( ADH(x) = ADF(x)g_r(x) = f(x)g_r(x) \) almost everywhere on \((a, b)\).

It follows that \( H(x) \) is an indefinite \( GM_r \)-integral of \( f(x)g_r(x) \) and
\[
(GM_r) \int_a^b f(x)g_r(x) \, dx = H(b) = F(b)g_r(b) - (GM_{r-1}) \int_a^b F(x)g_{r-1}(x) \, dx.
\]

We have therefore proved Theorem VI, and established Properties I-III by induction. Property II shows that the \( GM \)-scale of integrals is consistent, i.e. that each integral contains those with lower subscripts. Each integral of the scale is more general than the preceding integral. This is shown by the example \( f(x) = 0, x = 0, f(x) = (d/dx)^p x^q \sin(1/x), p, q \) integers \( x \neq 0 \) for, given \( r \), \( f(x) \) will be a \( GM_r \)-integrable but not \( GM_{r-1} \)-integrable function if \( p \) and \( q \) are properly chosen.

A further property of the \( GM_r \)-integral that follows easily from Theorem II, is the following.

The definite \( GM_r \)-integral is determined uniquely, and the indefinite \( GM_r \)-integral is determined uniquely apart from an additive constant.

We list other properties for which the proofs are essentially the same for higher orders as for the \( GM \)-or general Denjoy integral.

A function which is \( GM_r \)-integrable is necessarily measurable and almost everywhere finite [5, p. 243].

The function \( F(x) = GM_r(f, a, x), a \leq x \leq b \), satisfies Lusin's condition (N) [5, p. 225].

A function which is \( GM_r \)-integrable and almost everywhere non-negative on an interval \((a, b)\) is necessarily Lebesgue integrable on \((a, b)\) [5, p. 242].

Given a non-decreasing sequence of functions \( f_n(x) \) which are \( GM_r \)-integrable on an interval \((a, b)\) and whose \( GM_r \)-integrals over \((a, b)\) constitute a sequence bounded above, then the function \( f(x) = \lim f_n(x) \) is itself necessarily \( GM_r \)-integrable on \((a, b)\) and
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$$\left(GM_r\right) \int_a^b f(x) \, dx = \lim_{n \to \infty} \left(GM_r\right) \int_a^b f_n(x) \, dx,$$

[5, p. 243].

2. The relation between the $GM_r$- and the $C_r$-$T$, $C_r$-$P$-integrals. The $C_0$-$P$-integral is the Perron integral and the $C_r$-$P$-integral is defined in an analogous manner [1, 2] using major and minor functions but with ordinary continuity and derivatives replaced by mean (Cesàro) continuity and derivatives.

**Definition (AC$^*$,$_r$)** [6, p. 221]. The function $F(x)$ is said to be $AC^*$ ($C_r$-sense) over a set $E$ if it is $C_r$-integrable in an interval which contains $E$ and if to each positive number $\epsilon$ corresponds a number $\delta$ such that

$$\sum_{i=1}^{n} \text{bound}_{a_i < x < b_i} |C_r(F, a_i, x) - F(a_i)| < \epsilon,$$

$$\sum_{i=1}^{n} \text{bound}_{a_i < x < b_i} |C_r(F, b_i, x) - F(b_i)| < \epsilon,$$

for all finite sets of non-overlapping intervals $(a_i, b_i)$ with end points on $E$ and such that $\sum (b_i - a_i) < \delta$.

**Definition (ACG*$_r$)** [6, p. 221]. The function $F(x)$ will be said to be $ACG^*_r$ ($C_r$-sense) over a set $E$ if $F(x)$ is $C_r$-continuous at points of $E$ and if $E$ is the sum of a denumerable number of sets over each of which $F(x)$ is $AC^*$($C_r$-sense).

**Definition (DI$_r$)** [6, p. 232]. The function $f(x)$ is said to be $C_r$-$D$-integrable in $(a, b)$ if there is a function $F(x)$ that is $ACG^*(C_r$-sense) over the closed interval $(a, b)$ and such that $C_rDF(x) = f(x)$ almost everywhere in $(a, b)$. Then $F(x)$ is an indefinite $C_r$-$D$-integral of $f(x)$, $F(b) - F(a)$ the definite $C_r$-$D$-integral in $(a, b)$.

By basing our generalizations on the general Denjoy integral we were able to obtain a considerably simpler descriptive definition of an $r$th order integral. The condition that $F(x)$ be $C_r$-continuous could be separated from Definition (ACG*$_r$) and included separately in Definition (DI$_r$). The concept of (ACG) is then seen to be simpler than the modified Definition (ACG*$^r$). Furthermore, in Definition (DI$_r$) the concepts of both $ACG^*_r(C_r$-sense) and $C_r$-derivatives must be modified for each $r$. We prove that the simpler $GM_r$-integral is actually more general than the $C_r$-$D$- and equivalent $C_r$-$P$-integrals.

**Theorem VII.$_r$.** The $GM_r$-integral contains the $C_r$-$D$- and $C_r$-$P$-integrals.

Since Miss Sargent [6] proved the equivalence of the $C_r$-$D$- and $C_r$-$P$-integrals we need only show that the $GM_r$-integral contains the $C_r$-$D$-integral. We proceed by induction. Since it is well known that the $GM_r$-integral contains the $C_0$-$D$- (or special Denjoy integral we may suppose the theorem true for orders less than $r$ and prove that it is then true for order $r$.

*The $r$th Cesàro mean of $F$ on $(a, b)$ is denoted by $C_r(F, a, b)$ and differs from the $M_r$-mean only in that the integral is required to exist in the $C_r$-$D$- sense rather than the $GM_r$-sense.
We suppose that $f$ is $C_r$-$D$-integrable and set $F(x) = C_rD(f, a, x)$. Then, since $F(x)$ is $C_r$-continuous,

$$
\frac{1}{h^r} (C_{r-1}D) \int_a^{x+h} (x + h - t)^{-r-1} F(t) dt \to F(x)
$$

as $h \to 0$. By hypothesis the integral exists in the $GM_{r-1}$-sense and has the same value. We conclude that $F(x)$ is $M_r$-continuous on $(a, b)$.

By the descriptive definition of the $C_r$-$D$-integral, $(a, b)$ can be covered by a sequence of closed sets $(E_n)$ over each of which $F(x)$ is $AC^*(C_r$-sense). By Theorem II [6, p. 227] a necessary condition for $F(x)$ to be $AC^*(C_r$-sense) over a set $E_n$ is that $F(x)$ be $AC$ on $E_n$. It follows that $F(x)$ is $ACG$ on $(a, b)$.

In [6, p. 228] it is shown that if $f(x)$ is $AC^*(C_r$-sense) on a set $E_n$, then the $C_r$-derivative [2, p. 542] $C_rDf(x)$ exists, is finite and equal to $ADF(x)$ at almost all points $x$ of $E_n$. Since $(a, b)$ is covered by at most a denumerable sequence of such sets it follows that $C_rDf(x) = ADF(x)$ almost everywhere on $(a, b)$. Since, by the definition of a $C_r$-$D$-integral, $C_rDf(x) = f(x)$ almost everywhere it follows that $ADF(x) = f(x)$ almost everywhere on $(a, b)$. It then follows that $f(x)$ is $GM_r$-integrable to $F(x)$.

On the other hand the $C_r$-$P$, $C_r$-$D$-integrals do not contain the $GM_r$-integral. This is well known for $r = 0$ since the special Denjoy integral is contained in but not equivalent to the general Denjoy integral. A similar relation holds for other values of $r$. We therefore have two distinct scales: (1) the $CD$, $CP$-scale of integrals similar to and generalizing the Denjoy-Perron integral; and (2) the $GM$-scale of integrals similar to and generalizing the general Denjoy integral and such that the $GM_r$-integral contains the $C_r$-$D$, $C_r$-$P$-integrals.

3. The constructive $GM_r$-integral. To obtain a constructive definition of the $GM_r$-integral we modify the definitions and conditions for integrability in the general Denjoy sense by using limits involving $M_r$-means.

**Definition (a).** If the function $f(x)$ is summable over a measurable set $E$ then $GM_r(f, E)$ is $L(f, E)$ the Lebesgue integral of $f(x)$ over $E$.

**Definition (b).** Let $(a_i, \beta_i)$ be any interval and suppose that $GM_r(f, a, \beta)$ has been determined for every interval $(a, \beta)$ interior to $(a_i, \beta_i)$. Let $\xi$ be a point with $a_i < \xi < \beta_i$ and let $F(t) = GM_r(f, t, \xi)$. Let $K_r(a_i, \xi)$ and $K_r(\xi, \beta_i)$ be the respective limits as $h \to 0^+$ of

$$
\frac{1}{h^r} (GM_{r-1}) \int_{a_i}^{a_i+h} (a_i + h - t)^{-r-1} F(t) dt,
$$

$$
\frac{1}{h^r} (GM_{r-1}) \int_{\beta_i-h}^{\beta_i} (t - \beta_i + h)^{-r-1} F(t) dt,
$$

where the integrals are supposed to tend to limits which are finite. If $f(x)$ is such that $K_r(a_i, \xi) + K_r(\xi, \beta_i)$ is independent of $\xi$ then

$$
GM_r(f, a_i, \beta_i) = K_r(a_i, \xi) + K_r(\xi, \beta_i).
$$
DEFINITION (c). Let $E$ be a closed set over which $f(x)$ is summable, $(a_i, \beta_i)$ the intervals complementary to $E$ on $(a, b)$; suppose that $GM_r(f, a_i, \beta_i)$ has been determined for all the intervals $(a_i, \beta_i)$, and that $(l, m)$ is an interval for which

$$\sum_{(l, m)} |GM_r(f, a_i, \beta_i)|$$

converges. Then

$$GM_r(f, l, m) = \int_{E(l, m)} f(x)dx + \sum_{(l, m)} GM_r(f, a_i, \beta_i),$$

where it is understood that if $l$ is interior to an interval $(a_k, \beta_k)$ then the term in the sum arising from this interval is $GM_r(f, l, \beta_k)$. A similar understanding holds if $m$ is not a point of $E$.

The function $f(x)$ is then $GM_r$-integrable on $(a, b)$ if it satisfies the following conditions:

1. If $E$ is any closed set on $(a, b)$ there exists an interval $(l, m)$ containing points of $E$ and such that $f(x)$ is summable over $E(l, m)$.
2. The function $f(x)$ is such that the limits in (b) exist.
3. If $GM_r(f, l, x)$ exists for all $x$ in an interval $(l, m)$ then it is $M_r$-continuous as a function of $x$ in $(l, m)$.
4. The function $f(x)$ is such that if $E$ is any closed set for which $GM_r(f, a_i, \beta_i)$ has been determined for all intervals $(a_i, \beta_i)$ complementary to $E$, there exists an interval $(l, m)$ containing points of $E$ and such that

$$\sum_{(l, m)} |GM_r(f, a_i, \beta_i)|$$

converges.

Definitions (a), (b) and (c) together with conditions (1), (2), and (4) permit the determination of $GM_r(f, a, b)$ in a finite or denumerable number of steps as in [3, p. 20 ff.]. Further conditions are needed to ensure that $F(x) = GM_r(f, a, x)$ is $M_r$-continuous. These conditions are discussed for an integral equivalent to the $GM_r$-integral in [3]. We have postulated mean continuity by adding condition (3).

If we set $F(a) = 0$, $F(x) = GM_r(f, a, x)$ for $a < x \leq b$, we can prove Lemmas 4, and 5, as in [3, Theorems I and II].

**Lemma 4.** The function $F(x) = GM_r(f, a, x)$ is ACG on $(a, b)$.

**Lemma 5.** At almost all points of $(a, b)$ $ADF(x)$ exists and is equal to $f(x)$.

**Theorem VIII.** The constructive and descriptive definitions of the $GM_r$-integral are equivalent.

Condition (3) and Lemmas 4, and 5, show that the descriptive integral contains the constructive integral. We must therefore show that if the $M_r$-continuous function $F(x)$ is ACG on $(a, b)$ and such that $ADF(x)$ is finite almost everywhere and equal to $f(x)$, then the constructive definition gives $GM_r(f, a, x) = F(x) - F(a)$. We first prove a lemma.

**Lemma 6.** Let $F(x)$ be ACG on $(a, b)$ and let $ADF(x)$ be finite and equal to $f(x)$ almost everywhere on $(a, b)$. If $E$ is any closed set on $(a, b)$ there then exists an interval $(l, m)$ such that $f(x)$ is summable over $E(l, m)$, 

$$\sum_{(l, m)} |F(\beta_i) - F(a_i)|$$

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converges, where \((a_i, \beta_i)\) are the intervals complementary to \(E(l, m)\), and for any such interval
\[
F(m) - F(l) = \int_{E(l, m)} f(x)dx = \sum_{(l, m)} \{ F(\beta_i) - F(a_i) \}.
\]

Let \(A_1, A_2, \ldots\) be the closed sets over which \(F(x)\) is AC. There then exists, by Baire’s Theorem, an interval \((l, m)\) and an integer \(k\) such that \(A_k\) and \(E\) are identical on \((l, m)\). Since \(F\) is AC on \(A_k\), \(D_{A_k}F(x)\) exists for almost all points of \(A_k\). Then, since for almost all points of \(A_k\) we have \(D_{A_k}F = ADF = f\), it follows that \(f\) is summable over \(E(l, m)\). The convergence of \(\sum_{(l, m)} \left| F(\beta_i) - F(a_i) \right|\) follows from the absolute continuity of \(F\) on \(E(l, m)\).

Let \(G(x) = F(x)\) on \(E(l, m)\) and be linear in the intervals \((a_i, \beta_i)\) in such a way as to be continuous on \((l, m)\). Then \(G(x)\) is AC on \((l, m)\) and, at almost all points of \(E(l, m)\), \(G' = D_{E(l,m)} F = ADF = f\). Hence
\[
F(m) - F(l) = \int f(x)dx + \sum_{(l, m)} \{ F(\beta_i) - F(a_i) \}.
\]

We return to the proof of the theorem and, as in the existence proof for the constructive \(GM_r\)-integral [3, pp. 21-23], we let \(E_1\) be the points of non-summability of \(F\) on \((a, b)\). Then \(E_1\) is closed. If we denote by \((a_1^l, \beta_1^l)\) the intervals complementary to \(E_1\), by \((a, \beta)\) an interval with \(a_1^l < a < \beta < \beta_1^l\), then
\[
F(\beta) - F(a) = L(f, a, \beta) = GM_r(f, a, \beta).
\]

Since \(F(x)\) is \(M_r\)-continuous, \(F(\beta) - F(a)\) tends in the \(M_r\)-sense to the limit \(F(\beta_1^l) - F(a_1^l)\) which is finite and independent of any \(\xi\), \(a_1^\xi < \xi < \beta_1^l\). It follows that Definition (b) applies to \(GM_r(f, a, \beta)\) and
\[
F(\beta_1^l) - F(a_1^l) = \lim_{\beta \to \beta_1^l} GM_r(f, a, \beta) = GM_r(f, a_1^l, \beta_1^l).
\]

By Lemma 6, there exists at least one interval \((l, m)\) containing points of \(E_1\) and such that \(f\) is summable on \(E_1(l, m)\) and \(\sum_{(l, m)} \left| F(\beta_i^l) - F(a_i^l) \right|\) converges. Let \(E_2\) be the points of \(E_1\) that are points of non-summability of \(f\) over \(E_1\) and/or points \(x\) such that \(\sum_{(l, m)} \left| F(\beta_i^l) - F(a_i^l) \right|\) diverges for every interval \((l, m)\) containing \(x\). If \((a_2^\xi, \beta_2^\xi)\) are the intervals complementary to \(E_2\) and \((a, \beta)\) is an interval with \(a_2^\xi < a < \beta < \beta_2^\xi\) then, by Lemma 6,
\[
F(\beta) - F(a) = \int_{E_2(a, \beta)} f(x)dx + \sum_{(a, \beta)} \{ F(\beta_2^l) - F(a_1^l) \},
\]
and the right side is now equal to \(GM_r(f, a, \beta)\) by Definition (c). As before
we can use the \( M_r \)-continuity of \( F \) to determine \( GM_r(f, a^2, \beta^2) = F(\beta^2) - F(\beta^2) \). Continuing this process we can determine \( GM_r(f, l, m) = F(m) - F(l) \) in a finite or denumerable number of steps.

4. A continuity property of \( GM_r \)-integrals. We conclude with a proof that an important property of continuous functions extends to mean continuous indefinite integrals. This result has been stated for the \( C_1P \)-integral [7, p. 238].

**Theorem IX.** If \( F(x) \) is an indefinite \( GM_r \)-integral of \( f(x) \) defined on \((a, b)\) and if \((l, m)\) is any closed interval on \((a, b)\), then \( F(x) \) takes all values between its upper and lower bounds on \((l, m)\) for \( l \leq x \leq m \).

Let \( E_1 \) be the points of non-summability of \( f \) on \((l, m)\). If \((a^1, \beta^1)\) are the intervals complementary to \( E_1 \) on \((l, m)\), \( a^1, \beta^1 \) points of \( E_1 \) and \((a, \beta)\) is an interval with \( a^1 < a < \beta < \beta^1 \), then \( f \) is Lebesgue integrable on \((a, \beta)\). It follows that \( F(x) \) takes all values between its upper and lower bounds on \((a, \beta)\) for \( a \leq x \leq \beta \).

Let \( \beta \) be fixed, let \( a \) tend to \( a^1 \) and consider the intervals \((a^1, a)\). There are three possibilities: (i) Every interval \((a^1, a)\) contains points \( x \) with \( F(x) > F(a^1) \) and points \( x' \) with \( F(x') < F(a^1) \); (ii) There exists an interval \((a^1, a)\) with no point \( x \) such that \( F(x) > F(a^1) \); or (iii) There exists an interval \((a^1, a)\) containing no point \( x \) such that \( F(x) < F(a^1) \).

In the first case it is clear that \( F(x) \) takes all values between its upper and lower bounds on \((a^1, \beta)\) for \( a^1 \leq x \leq \beta \). In the second case, given an arbitrary \( \epsilon > 0 \) there exists \( \delta \) such that \( F(a^1) - F(x_1) < \epsilon \) for some \( x_1, a^1 < x_1 < a^1 + \delta \). If not, Lemma 1, would contradict the \( M_r \)-continuity of \( F(x) \) at \( a^1 \). Since \( F(x) \) takes all values between its upper and lower bounds on \((x_1, \beta)\) and \( \epsilon \) is arbitrary, it follows that it takes all values between its upper and lower bounds on \((a^1, \beta)\). Since a similar argument holds for \( \beta \) tending to \( \beta^1 \), \( F(x) \) takes all values between its upper and lower bounds on \((a^1, \beta^1)\) for \( a^1 \leq x \leq \beta^1 \). A similar argument holds if (iii) applies.

As in the transfinite process by which the \( GM_r \)-integral was built up from the constructive definition [3, p. 20], let \( E_2 \) be the points \( x \) of \( E_1 \) such that one or both of the following conditions hold: (i) For every interval \((c, d)\) containing \( x \) the function \( f \) is not summable over \( E_1(c, d) \); (ii) The sum \( \sum_{(c, d)} [GM_r(f, a^1, \beta^1)] \) diverges for every interval \((c, d)\) containing \( x \). Let \((a_2^2, \beta_2^2)\) be an interval of the set complementary to \( E_2 \) with \( a_2^2, \beta_2^2 \) points of \( E_2 \); \((a, \beta)\) an interval with \( a_2^2 < a < \beta < \beta_2^2 \).

Let \( G(x) = F(x) \) for \( x = a, \beta \) and at points of \( E_1(a, \beta) \) and let \( G(x) \) be linear in the intervals \((a^1, \beta^1)\) on \((a, \beta)\) and in the intervals \((a, a^1), (\beta, \beta^1, \beta)\) where \( a_2^1, \beta_2^1 \), are the upper and lower bounds of \( \beta - \beta_2^2 \) points of \( E_1 \) on \((a, \beta)\).

Since \( a, \beta \) are arbitrary it is sufficient to prove that \( F(x) \) takes all values between \( F(a) \) and \( F(\beta) \) on \((a, \beta)\). Let \( c \) be any value between \( F(a), F(\beta) \). Then \( G(x') = c \) for some \( x' \), \( a < x' < \beta \). If \( x' \in E_1 \), then \( F(x') = c \). If \( G(x) \neq c \) in \( E_1 \), there exists some interval \((a^j, \beta^j)\) with \( a^j < x' < \beta^j \) and \( F(a^j) = G(a^j) < c \).
\( c < G(\beta_j) = F(\beta_j) \) or the reverse inequality. By the first part of the theorem \( F(x) \) takes all values between \( F(a_j) \) and \( F(\beta_j) \) for \( a_j < x < \beta_j \) and therefore takes the value \( c \).

As before we can pass from the intervals \((a, \beta)\) to the intervals \((a^*_j, \beta^*_j)\) complementary to \( E_2 \). Continuing this process we can establish the theorem in a finite or denumerable number of steps.

**References**


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