Fibrations and Grothendieck topologies

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Given a site \( T \), that is, a category equipped with a fixed Grothendieck topology, we provide a definition of fibration for morphisms of the presheaves on \( T \). We verify that the notion is well-behaved with respect to composition, base change, and exponentiation, and is trivial on the topos of sheaves. We compare our definition to that of Kan fibration in the semi-simplicial setting. Also we show how we can obtain a notion of fibration on our ground site \( T \) and investigate the resulting notion in certain ring-theoretic situations.

1. Introduction

Let \( T \) be a site; that is, a category equipped with a fixed Grothendieck topology. We have the adjoint pair

\[
\begin{align*}
S & \xrightarrow{sh} [T^0, \text{Sets}] \\
& \text{sh}
\end{align*}
\]

where \( sh \) is the associated sheaf functor and \( S \) is the full topos of sheaves with respect to the topology. We define a notion of fibration for morphisms of presheaves that is well behaved with respect to composition, base change and exponentiation, and trivializes on the topos \( S \). We investigate how our notion compares with that of Kan fibrations, when \( T = \text{Ord} \), the category of finite ordered sets equipped with an appropriate topology. We then observe we can pull our notion of fibration back to the ground site \( T \) and we investigate it in certain ring-theoretic situations.

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2. Basic notions

Let \( p : E \to B \) be a map (that is, natural transformation) of presheaves. We define

**DEFINITION.** The map \( p \) is a (weak) fibration if the following diagram in \( \text{Sets} \) is (weak) cartesian:

\[
\begin{array}{ccc}
E(U) & \to & B(U) \\
\downarrow & & \downarrow \\
\ker \left( \prod_i E(U_i) \to \prod_{i,j} E(U_i \times_U U_j) \to \prod_i B(U_i) \right)
\end{array}
\]

for every covering \( \{U_i \to U\} \) in \( T \).

As usual we also define

**DEFINITION.** \( X \) is a (weak) fibrant object in \( [T^0, \text{Sets}] \) if \( X \to e \) is a (weak) fibration. (\( e \) is the final object of \( [T^0, \text{Sets}] \); 
\( e(U) = \{\ast\} \) for all \( U \) in \( \text{ob}(T) \).)

We have three immediate trivialities.

**FACTS.** 1 Every isomorphism is a fibration.

2 A morphism of sheaves is a fibration.

3 \( X \) is (weak) fibrant iff \( X \) is a (weak) sheaf.

Weak sheaf is the "dual" notion to separated presheaf; that is, it means the canonical map of sets

\[
X(U) \to \ker \left( \prod_i X(U_i) \to \prod_{i,j} X(U_i \times_U U_j) \right) = H^0(\{U_i \to U\}, X)
\]

is epic for all coverings \( \{U_i \to U\} \) in \( T \). (We freely use the above cohomological abbreviation in the following.)

We now check the desired stability properties.

**PROPOSITION 1.** If \( p : E \to B \) is a (weak) fibration, and \( f : B' \to B \) is arbitrary then \( p' : E \times_B B' \to B' \) is a (weak) fibration where
is a cartesian square in \([T^0, \text{Sets}]\).

Proof. Let \( \{ U_i \xrightarrow{\psi_i} U \} \) be a covering; we check

\[(E \times_{B'} B')(U) \to B'(U)\]

\[H^0(\{ U_i + U \}, E \times_{B'} B') \to \prod_i B'(U_i)\]

is (weak) cartesian in \( \text{Sets} \).

First observe that pullbacks in \([T^0, \text{Sets}]\) are computed pointwise; so \( (E \times_{B'} B')(V) = E(V) \times_{B(V)} B'(V) \), for \( V \) in \( \text{ob}(T) \) and induced maps are the obvious projections. Let \( s \) be in \( B'(U) \) and \( \{ u_i, \omega_i \} \) be in \( H^0(\{ U_i + U \}, E \times_{B'} B') \) where \( \omega_i = p'[u_i] \{ u_i, \omega_i \} = B'(u_i)(s) \). Consider \( f(U)(s) \) in \( B(U) \) and \( \{ v_i \} \) in \( \prod_i E(U_i) \). We first observe that

\[B[u_i]f(U)(s) = f(u_i)B'[u_i](s) = f(u_i)p'[u_i] \{ v_i, \omega_i \} = p(u_i)f'[u_i] \{ v_i, \omega_i \} = p(u_i) \{ v_i \},\]

since \( f' \) is a projection onto the first factor at each "point". Since \( p \) is a (weak) fibration, there exists a (unique) \( t \) in \( E(U) \) such that

1. \( p(U)(t) = f(U)(s) \), and
2. \( E[u_i](t) = v_i \).

Consider \( (t, s) \) in \( (E \times_{B'} B')(U) \), by dint of (1) above. Certainly \( p'(U)(t, s) = s \) and

\[(E \times_{B'} B')[u_i](t, s) = (E[u_i](t), B'[u_i](s)) = (v_i, \omega_i)\]

by (2) above. This completes the proof.
PROPOSITION 2. Let \( q : X \to E \), \( p : E \to B \) be (weak) fibrations; then \( pq : X \to B \) is a (weak) fibration.

Proof. Let \( \{ U_i \xrightarrow{u_i} U \} \) be a covering in \( T \) and consider

\[
\begin{array}{ccc}
X(U) & \xrightarrow{pq(U)} & B(U) \\
\downarrow & & \downarrow \\
H^0(\{ U_i \to U \}, X) & \longrightarrow & \bigoplus_i B(U_i).
\end{array}
\]

Let \( s \) be in \( B(U) \) and \( \{ \omega_i \} \) be in \( H^0(\{ U_i \to U \}, X) \) such that

\[ B(u_i)(s) = (pq)(u_i)(\omega_i). \]

Certainly \( q(u_i)(\omega_i) \) is in \( H^0(\{ U_i \to U \}, X) \) and is compatible with \( s \) in the obvious sense. So since \( p : E \to B \) is a (weak) fibration, there exists a (unique) \( t \) in \( E(U) \) such that

1. \( p(U)(t) = s \), and
2. \( E[u_i](t) = q(u_i)(\omega_i) \).

Since \( q : X \to E \) is a (weak) fibration and the second equality gives us "compatibility", there exists a (unique) \( z \) in \( X(U) \) such that \( X(u_i)(z) = \omega_i \) and \( q(U)(z) = t \). So then

\[ (pq)(U)(z) = p(U)q(U)(z) = p(U)(t) = s. \]

This completes the proof.

(Note that Fact 1 and Propositions 1 and 2 verify the (isolated) properties of a fibration in the sense of Quillen's model categories [4].)

We recall now the notion of exponentiation in our functor category \([T^0, \text{Sets}]\). Categorically one defines \((-)^Y\) as the right adjoint to the functor \((-) \times Y\). Along with the Yoneda Lemma, this forces the definition in the category of presheaves

\[ X^Y(U) \cong \text{nat}(\text{hom}_T(-, U), X^Y) \cong \text{nat}(\text{hom}_T(-, U) \times Y, X). \]

We then have
PROPOSITION 3. If $p : E \to B$ is a fibration and $K$ is a presheaf, then $p^K : E^K \to B^K$ is a fibration.

Proof. See Appendix.

COROLLARY. If $E$ is a sheaf and $K$ a presheaf then $E^K$ is a sheaf.
(This is well-known; see [6], p. 258.)

3. Semi-simplicial application

We now consider a particular situation. Let $T = \text{Ord}$, the category whose objects are finite ordered sets and the morphisms are weakly monotone maps. It is customary to consider the obvious countable skeletal subcategory whose objects are denoted $n = \{0 < 1 < 2 < \ldots < n\}$. As usual, the simplicial sets are the set-valued presheaves on this category. We describe a Grothendieck topology on $\text{Ord}$ and investigate the resulting notions of fibration and fibrant object. First we define a modified notion of topology.

DEFINITION. A weak Grothendieck topology is a category with a notion of covering which satisfies all but the composition axiom for Grothendieck topologies.

A sheaf with respect to a weak Grothendieck topology has the obvious meaning. Certainly it also makes sense to speak of the (weak) Grothendieck topology generated by a partial collection of "coverings". Hence consider the set $C$:

$$C = \{n \xrightarrow{i_0} n\} \cup \left\{ \begin{array}{c}
\begin{array}{c}
d_{i_0} \\
\downarrow \\
n \\
\end{array} \\
\begin{array}{c}
n+1 \\
\downarrow \\
i_1 \\
\vdots \\
\downarrow \\
q \\
\end{array} \\
\end{array} : 0 \leq i_0 \leq i_1 \leq \ldots \leq i_r \leq q+1, r \leq q \right\}.
$$

We thus obtain a (weak) Grothendieck topology generated by $C$. We call $\text{Ord}$ with this topology the (weak) combinatorial site.

PROPOSITION 4. $X$ is a Kan fibration iff $X$ is a weak fibration on the weak combinatorial site.

First we have an easy lemma.

LEMMA. The following square is cartesian in $\text{Ord}$ if $i < j$;
Proof. The diagram commutes by the usual "simplicial" identities in \( \text{Ord} \). Suppose we want to fill in the dotted arrow in the following commutative diagram:

\[
\begin{array}{c}
\text{m} \\
\downarrow d_i \\
n-1 \quad \text{n-1} \quad i \\
\downarrow d_j \\
n \quad n \quad d_i \\
\downarrow d_j \\
n \quad \text{n+1} \\
\end{array}
\]

Let \( l \) be in \( m = \{0 < 1 < 2 < \ldots < m\} \). First we claim \( e(l) \neq j - 1 \). Otherwise \( d_j[e'(l)] = d_j[l] = d_j[j-1] = j \). But \( j \) is never in the image of \( d_j \). Similarly \( e'(l) \neq i \). Hence there exist \( x, y < n \) such that \( d_{j-1}(x) = e(l) \) and \( d_{i}(y) = e'(l) \). We must show \( x = y \). We have two cases.

Case 1. Suppose \( e'(l) < i \). Then \( e'(l) < j \); so

\[
d_i[e(l)] = d_j[e'(l)] = e'(l) < i.
\]

Hence \( e(l) = e'(l) < i < j \); thus \( e(l) < j - 1 \) and (in the notation above) \( y = e'(l) \), \( x = e(l) \); so \( x = y \).

Case 2. Suppose \( e'(l) > i \). This splits up into two subcases.

(a) Suppose \( j \leq e'(l) \). Then

\[
d_i[e(l)] = d_j[e'(l)] = e'(l) + i > i.
\]

Hence \( e(l) = (e'(l)+1) - 1 = e'(l) \). So \( y = e'(l) - 1 \), and \( x = y \).

(b) Suppose \( e'(l) < j \). Then

\[
d_i[e(l)] = d_j[e'(l)] = e'(l) > i.
\]
Hence $e(l) = e'(l) - 1$. Since $e'(l) > i$, $y = e'(l) - 1$.
Also $e(l) = e'(l) - 1 < j - 1$. So $x = e(l)$ and $x = y$.

This completes Case 2 and the proof.

Proof of Proposition 4. (ONLY IF) Let $s_j$ be in $\bigcap_{0 \leq j \leq n} E(n)$ and $t$ in $B(n+1)$ such that $\partial_i(s_j) = \partial_j(s_{i-1})$, $i < j$, $i, j \neq k$, and $\partial_i(t) = p(n)(s_l)$. We consider the covering hypothesis gives us

$$
\begin{array}{c}
\begin{array}{c}
E(n+1) \\
\downarrow \\
\ker \left( \bigcap_i E(n) \rightarrow \bigcap_{i,j} E(n \times_{n+1} n) \rightarrow \bigcap_i B(n) \right)
\end{array}
\end{array}
$$

is weak cartesian.

The lemma identifies $n \times_{n+1} n$ and the maps; hence our assumption implies $\{s_0, \ldots, \hat{s}_k, \ldots, s_{n+1}\}$ is in $\ker$. We thus obtain the desired $(n+1)$-simplex in $E$ from the diagram.

(IF) The converse follows from a standard fact about Kan fibrations (see [3], p. 26) and the fact that $C$ is closed under fibre products; hence is itself the weak Grothendieck topology. To prove the latter claim we first recall the unique factorization of morphisms in $\text{Ord}$ as strings of $d_i$'s and $s_j$'s (see [3], p. 4). Since juxtapositions of cartesian squares are cartesian, it suffices to check closure under fibre products induced by the $d_i$'s and $s_j$'s individually. This is tedious and left to the reader.

**COROLLARY.** $X$ is a Kan complex iff $X$ is a weak sheaf on the weak combinatorial site.
4. Fibrations on $T$: examples

It is also possible to obtain a notion of fibration on our ground site $T$. We have the fully faithful Yoneda embedding

$$T \xrightarrow{h} \mathcal{T}^0, \text{Sets}$$

along which we can in some sense "pull back". Suppose $p^* : \text{hom}_T(-, E) \rightarrow \text{hom}_T(-, B)$ is a morphism of representable presheaves induced by $p : E \rightarrow B$. By definition, $p^*$ is a fibration if the following square is cartesian:

$$
\text{hom}_T(U, E) \xrightarrow{h^0} \text{hom}_T(U, B)
$$

for every covering $\{U_i \rightarrow U\}$ in $T$.

In other words we have the following lifting property;

where "compatible" means the diagram

commutes for all $i$ and $j$. Similarly $E$ in $\text{ob}(T)$ is fibrant if the following diagram can always be completed:
Fibrations

Numerous sites appear in algebro-geometric contexts. To consider a particularly simple example let $T$ be the category of affine schemes over $\text{spec}(R)$; that is, the opposite of the category of commutative $R$-algebras and declare a covering to be a single faithfully flat morphism $\text{spec}(B) \to \text{spec}(A)$. (These "affine" sites appear in Dobbs [1] under the name $R$-based topologies.) What are the fibrant $R$-algebras? We have the following observation.

**PROPOSITION 5.** $E$ is a fibrant $R$-algebra iff for any faithfully flat morphisms $S' \to S$, and homomorphism $f : E \to S$, for every $e$ in $E$, $f(e) \otimes 1 = 1 \otimes f(e)$ in $S \otimes_S, S$.

Proof. By faithfully flat descent the lifting below exists iff the bottom oblique arrows are equal:

Also the following is true.

**PROPOSITION 6.** If $E$ is fibrant then $B \to E$ is always a fibration.

Proof. Suppose we have the diagram

Since $E$ is fibrant there exists a map $E \to S'$ making the resulting upper triangle commute. But since $S' + S$ is a monomorphism the lower triangle also commutes.

For simplicity let us suppose $R = \mathbb{Z}$, so we are considering the
category of commutative rings. We have three properties of fibrations.

**PROPOSITION 7.** If $p_i : B_i \rightarrow E_i$, $i = 1, 2$, are fibrations then so is $p_1 \otimes p_2 : B_1 \otimes B_2 \rightarrow E_1 \otimes E_2$.

**COROLLARY.** Fibrant rings are closed under tensor product (equals coproduct).

**PROPOSITION 8.** Epimorphisms of rings are fibrations.

**COROLLARY.** Fibrant rings are closed under homomorphic images.

**PROPOSITION 9.** If $A$ is a ring, $S$ a multiplicatively closed subset of $A$, then the localization map $A \rightarrow S^{-1}A$ is a fibration.

**COROLLARY.** Fibrant rings are closed under taking rings of fractions.

**PROPOSITION 10.** Fibrant rings are rigid (that is, have no non-trivial automorphisms).

We now can produce many examples and non-examples of fibrant rings. $\mathbb{Z}$ is trivially fibrant, and all subrings of the rationals are fibrant by Corollary 3. The finite cyclic rings are fibrant by Corollary 2. The rings $R \times R$, for $R$ arbitrary, and the complex numbers are non-examples by Proposition 1. If $R$ is noetherian, $R[t]$ is never fibrant by considering the faithfully flat morphism $R \rightarrow R[[t]]$. We provide a representative proof.

**Proof of Proposition 9.** Suppose we have a commutative square

$$
\begin{array}{ccc}
A & \xrightarrow{f} & R \\
\downarrow & & \downarrow i \\
S^{-1}A & \xrightarrow{g} & R'
\end{array}
$$

with $i$ faithfully flat. We must check $f(S)$ is contained in $R'$, the invertible elements of $R$. Let $s$ be in $S$. Consider the $R$-module $R/f(s)R$. We claim that $R' \otimes_R (R/f(s)R) = 0$. We compute
Fibrations

\[ s \otimes (r + f(s)R) = g\begin{pmatrix} 0 \\ 1 \end{pmatrix} g\begin{pmatrix} 1 \\ 0 \end{pmatrix} s \otimes (r + f(s)R) \]
\[ = if(s)g\begin{pmatrix} 1 \\ 0 \end{pmatrix} s \otimes (r + f(s)R) \]
\[ = g\begin{pmatrix} 1 \\ 0 \end{pmatrix} s \otimes f(s)(r + f(s)R) = 0. \]

Hence by faithful flatness, \( R = f(s)R \); so \( 1 = f(s)r \) for some \( r \) in \( R \). The desired map \( S^{-1}A + R \) can now be constructed.

There exist two other examples where we can identify the fibrations.

EXAMPLE 1. Let \( T \) be an arbitrary category with topology defined by the universally effective epimorphisms; that is, \( \{U \to U\} \) is a covering in \( T \) iff for all objects \( X \) of \( T \),
\[ \text{hom}_T(U, X) \to H^0(\{U \to U\}, \text{hom}_T(-, X)) \]
is an isomorphism. Then since the definition forces every representable functor to be a sheaf, by Fact 2 above, every morphism is a fibration.

EXAMPLE 2. Let \( R \) be a commutative ring. If \( S \) is an \( R \)-algebra and \( M \) an \( S \)-module, Quillen [5] defines a cohomology theory \( D^*(S/R, M) \) based on a Grothendieck topology on the category of \( S \)-algebras where a covering is a single \( S \)-algebra epimorphism with nilpotent kernel. Since all our notions are dualized, \( p : B \to E \) is a fibration iff for every commutative square the dotted arrow exists;

In the terminology of Grothendieck [7] we conclude the fibrations are precisely the formally unramified morphisms.

Appendix

We provide here a detailed proof of Proposition 4 ("Exponentiation").

Proof. We must check the following square is cartesian;
for an arbitrary covering \( \{ U_i \xrightarrow{u_i} U \} \). Letting \((-,-)\) denote \( \text{hom}_X(-,-) \), this square becomes

\[
\begin{array}{ccc}
\text{nat}(K(-) \times (-, U) \to E(-)) & \longrightarrow & \text{nat}(K(-) \times (-, U) \to B(-)) \\
\downarrow & & \downarrow \\
H^0[U \to U, K(-) \times (-, x) \to E(-)] & \longrightarrow & \prod_i \left( \text{nat}(K(-) \times (-, U_i) \to B(-)) \right)
\end{array}
\]

So consider some \( h : K(-) \times (-, U) \to B(-) \) and a compatible collection \( \{ t_i : K(-) \times (-, U_i) \to E(-) \} \) of natural transformations such that

\[
(*) \quad \text{pot}_{\xi} = h \circ (1 \times (u_i)_{\xi})
\]

We want a natural transformation \( t : K(-) \times (-, U) \to E(-) \) such that

\[(1) \quad \text{pot} = h \]

and

\[(2) \quad t \circ (1 \times (u_i)_{\xi}) = t_i\,.
\]

Let \( X \) be an object of \( T \) and consider \((s, f)\) in \( K(X) \times (X, U) \). We have a cartesian square in \( T \),

\[
\begin{array}{ccc}
X \times_U U_i & \xrightarrow{g_{\xi}} & U_i \\
\downarrow e_{\xi} & & \downarrow u_i \\
X & \xrightarrow{f} & U 
\end{array}
\]

By the fibre-product axiom for Grothendieck topologies we have
\( \{ X \times_U U_i + X \} \) is a covering of \( X \). Since \( p : E \to B \) is a fibration we have the following cartesian square;
Consider $h(x)(s, f)$ in $B(X)$ and $t_i(x \times_U u_i)(ke_i(s), g_i)$ in $E(x \times_U u_i)$. We claim

\[(B e_i)(h(x)(s, f)) = p(x \times_U u_i) t_i(x \times_U u_i)(ke_i(s), g_i)\]

and

\[t_i(x \times_U u_i)(ke_i(s), g_i) \text{ is in } H^0([x \times_U u_i, E]).\]

Proof of (4). By naturality of $h$ and (\*),

\[(B e_i)(h(x)(s, f)) = h(x \times_U u_i)(ke_i \times e_i^j)(s, f)\]

\[= h(x \times_U u_i)(1 \times [u_i]) (ke_i(s), g_i)\]

\[= p(x \times_U u_i) t_i(x \times_U u_i)(ke_i(s), g_i).\]

Proof of (5). This requires verifying

\[E(1x p_1)[t_i(x \times_U u_i)(ke_i(s), g_i)] = E(1x p_2)[t_j(x \times_U u_j)(ke_j(s), g_j)]\]

where we have maps

Using the compatibility of the $t_i$'s we know the following diagram commutes;
By naturality of $t_i$, \[ E(1 \times p_1) (t_i(X \times_U U_z) [K e_i(s), g_i]) = t_i(Z) [K(1 \times p_1) \times (1 \times p_1^*)] [K e_i(s), g_i] \]
\[ = t_i(Z) [K(e_i \circ (1 \times p_1)) (s), g_i \circ (1 \times p_1)] \]
\[ = t_i(Z) [K(e_i \circ (1 \times p_1)) (s), p \circ q] . \]

By considering $K(e_i \circ (1 \times p_1)) (s, q)$ in $K(Z) \times (Z, U_z \times_U U_j)$ appearing in diagram (6) we can continue our computation;
\[ = t_j(Z) [K(e_j \circ (1 \times p_2)) (s), g_j \circ (1 \times p_2)] \]
\[ = t_j(Z) [K(1 \times p_2) \times (1 \times p_2^*)] [K e_j(s), g_j] \]
\[ = E(1 \times p_2) (t_j(X \times_U U_j) [K e_j(s), g_j]) , \]
by the naturality of $t_j$. This completes the proof of (5).

Now by our cartesian square (3), there exists a unique $z$ in $E(X)$ such that
\[ p(X)(z) = h(X)(s, f) \]
and
\[ [E e_i](z) = t_i(X \times_U U_z) [K e_i(s), g_i] . \]
We then define $t(X)(s, f) = z$. First we claim that
\[ t : K(-) \times (-, U) \rightarrow E(-) \]
is a natural transformation.

Proof of (9). Let $F : Y \rightarrow X$ be a morphism in $T$ and consider the diagram
\[ \begin{array}{c}
K(Z) \times [Z, U_i] \\
\downarrow t_i(Z) \\
K(Z) \times [Z, U_j] \\
\downarrow t_j(Z) \\
E(Z)
\end{array} \]
\[ 1 \times (p_1) \]
\[ 1 \times (p_2) \]
\[ \text{diagram (6)} \]
\[ By\text{ naturality of } t_i, \]
\[ E(1 \times p_1) (t_i(X \times_U U_i) [K e_i(s), g_i]) = t_i(Z) [K(1 \times p_1) \times (1 \times p_1^*)] [K e_i(s), g_i] \]
\[ = t_i(Z) [K(e_i \circ (1 \times p_1)) (s), g_i \circ (1 \times p_1)] \]
\[ = t_i(Z) [K(e_i \circ (1 \times p_1)) (s), p \circ q] . \]
\[ By\text{ considering } K(e_i \circ (1 \times p_1)) (s, q) \text{ in } K(Z) \times (Z, U_i \times U_j) \text{ appearing in} \]
\[ \text{diagram (6)} \text{ we can continue our computation;} \]
\[ = t_j(Z) [K(e_j \circ (1 \times p_2)) (s), g_j \circ (1 \times p_2)] \]
\[ = t_j(Z) [K(1 \times p_2) \times (1 \times p_2^*)] [K e_j(s), g_j] \]
\[ = E(1 \times p_2) (t_j(X \times_U U_j) [K e_j(s), g_j]) , \]
by the naturality of $t_j$. This completes the proof of (5).

Now by our cartesian square (3), there exists a unique $z$ in $E(X)$ such that
\[ p(X)(z) = h(X)(s, f) \]
and
\[ [E e_i](z) = t_i(X \times_U U_i) [K e_i(s), g_i] . \]
We then define $t(X)(s, f) = z$. First we claim that
\[ t : K(-) \times (-, U) \rightarrow E(-) \]
is a natural transformation.

Proof of (9). Let $F : Y \rightarrow X$ be a morphism in $T$ and consider the diagram
We must show that \((EF)(t(X)(s,f)) = t(Y)((KF)(s), fF)\).

By our definition of \(t\) this requires showing

\[
(10) \quad p(Y)((EF)(t(X)(s,f))) = h(Y)(KF(s), fF)
\]

and

\[
(11) \quad \left( E_{i}^{Y} \right)((EF)(t(X)(s,f))) = t_{i}(Y \times U_{i}) \left[ Ke_{i}^{X}(KF(s)), g_{i}^{Y} \right]
\]

where the maps mentioned appear in the following cube:

---

**Proof of (10).**

\[
p(Y)(EF)(t(X)(s,f)) = (BF)p(X)(t(X)(s,f)) \quad \text{by naturality of } p
\]

\[
= (BF)(h(X)(s,f)) \quad \text{by definition of } t
\]

\[
= h(Y)(Ke_{i}^{X})(s,f) \quad \text{by naturality on } h
\]

\[
= h(Y)(Ke(s), fF) .
\]

**Proof of (11).**

\[
\left( E_{i}^{Y} \right)((EF)(t(X)(s,f))) = E \left( E_{i}^{Y} \right)(t(X)(s,f))
\]

\[
= E \left( e_{i}^{X} \left( P_{X}U_{i} \right) \right)(t(X)(s,f))
\]

by cube. Now using our definition of \(t\) and the naturality of \(t_{i}\),
This completes the proof of (11) and, hence, (9). We now assert that $t$ satisfies our original two requirements, (1) and (2).

Proof of (1). This follows immediately from (7).

Proof of (2). This statement translates into
\[ t_i(X)(s, f) = t(X)(s, u_i \circ f), \]
where $s$ is in $K(X)$ and $f$ is in $(X, U_i)$. By definition of $t$ this requires showing that
\[ p(X)(t_i(X)(s, f)) = h(X)(s, u_i \circ f) \]
and
\[ (E_2_i)(t_i(X)(s, f)) = t_i(P)(K^e_i(s), \bar{g}_i) \cdot \]

Proof of (12). This follows immediately from (*).

Proof of (13). The maps mentioned in (13) come from the following cartesian square,

\[ \begin{array}{ccc}
  P & \xrightarrow{g_i} & U_i \\
  e_i & \downarrow & u_i \\
  X & \xrightarrow{f} & U_i & \xrightarrow{u_i} & U
\end{array} \]

By the naturality of $t_i$,
\[ (Ee_i)(t_i(X)(s, f)) = t_i(P)(K^e_i(s) \times e_i)(s, f) = t_i(P)(K^e_i(s), f^e_i). \]

Consider the following dotted arrow $h$,
In diagram (6), let $i = j$, $Z = P$ and consider $(\overline{K\theta}_i(s), h)$ in $K(P) \times (P, U_i \times U_j)$. This gives the equality

$$t_i(P)(\overline{K\theta}_i(s), f_0\overline{e}_i) = t_i(P)(\overline{K\theta}_i(s), \overline{g}_i).$$

Together with computation (14), this completes the proof of (13) and thus, Proposition 4.

References


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