10

Symmetry breaking in model theories

In Chapter 9, 'effective' weak interaction Lagrangian densities were constructed. When used in low orders of perturbation theory, these account well for the observed phenomena at low energies. Difficulties arise in higher order perturbation theory, as they do in quantum electrodynamics. There is, however, an important difference: it has been proved that these effective Lagrangian theories cannot be renormalised and they are therefore unsatisfactory. Furthermore, at higher energies new phenomena appear, and it is now well established experimentally that the weak interaction is mediated by the W^+, W^- and Z bosons. How are these particles to be incorporated in a theory of the weak interaction that can be renormalised, and which has the same seeming inevitability as QED? The answer lies in the Weinberg–Salam unified theory of the electromagnetic and weak interactions. As an introduction to the Weinberg–Salam theory we shall in this chapter consider 'model' theories, the mathematics of which is fairly simple, but which contain the basic ideas we shall need.

10.1 Global symmetry breaking and Goldstone bosons

A possible Lagrangian density for a complex scalar field $\Phi = (\phi_1 + \mathrm{i}\phi_2)/\sqrt{2}$ is

$$\mathcal{L} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi \tag{10.1}$$

(cf. equation (3.32)).

In this expression $(\partial\Phi^\dagger/\partial t)(\partial\Phi/\partial t)$ can be regarded as the kinetic energy density and $\nabla\Phi^\dagger \cdot \nabla\Phi + m^2\Phi^\dagger\Phi$ as the potential energy density (see Section 3.3). If Φ is constant, independent of space and time, the only contribution to the energy is $m^2\Phi^\dagger\Phi$. Since m^2 is positive this will be a minimum when $\phi_1 = \phi_2 = 0$. Thus $\Phi = 0$ corresponds to the 'vacuum' state. Consider now the Lagrangian density obtained by changing the sign in front of m^2. This would be unstable: the potential

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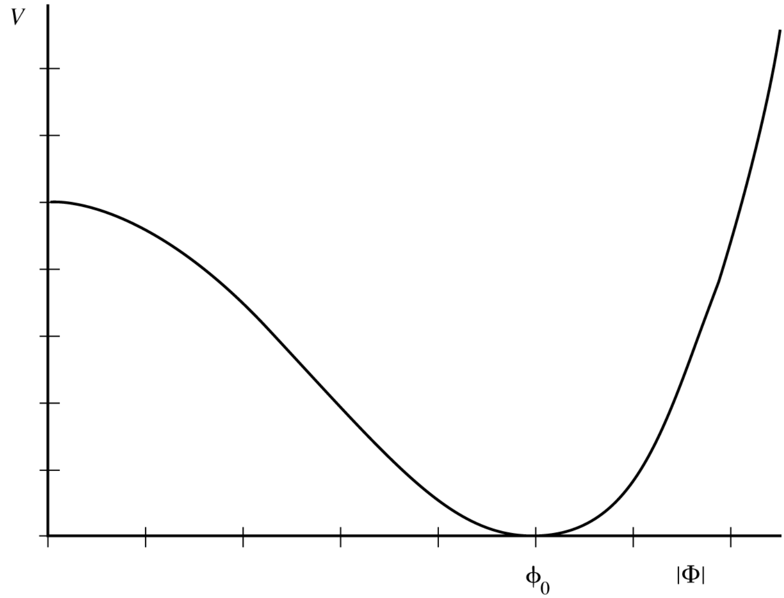

Figure 10.1 Plot of $V = (m^2/2\phi_0^2)[\Phi^*\Phi - \phi_0^2]^2$ as a function of $|\Phi|$; Φ is here a classical field.

energy density is then unbounded below. Stability can be restored by introducing a term $(m^2/2\phi_0^2)(\Phi^\dagger\Phi)^2$ where ϕ_0^2 is another (real) parameter. For convenience we add a constant term $m^2\phi_0^2/2$, and then

$$\mathcal{L} = \partial_\mu\Phi^\dagger\partial^\mu\Phi - V(\Phi^\dagger\Phi)$$

where

$$V(\Phi^\dagger\Phi) = \frac{m^2}{2\phi_0^2}\left[\Phi^\dagger\Phi - \phi_0^2\right]^2. \tag{10.2}$$

The form of V is shown in Fig. (10.1). The minimum field energy is now obtained with Φ constant independent of space and time, but such that $\Phi^\dagger\Phi = |\Phi|^2 = \phi_0^2$. Such a field is not unique but is defined by a point on the circle $|\Phi| = \phi_0$ in the state space (ϕ_1, ϕ_2), so that the number of possible vacuum states is infinite.

An analogy with magnetism is helpful. The Hamiltonian describing a Heisenberg ferromagnet has rotational symmetry: all directions in space are equivalent. However, in its ground state a ferromagnet is magnetised in some particular direction, which is not determined within the theory, and the rotational symmetry is lost. This is an example of *spontaneous symmetry breaking*.

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The Lagrangian density (10.2) has a 'global' $U(1)$ symmetry: $\Phi \to \Phi' = e^{-i\alpha}\Phi$, $\mathcal{L} \to \mathcal{L}' = \mathcal{L}$, for any real α. Equivalently,

$$\phi_1' = \phi_1 \cos\alpha + \phi_2 \sin\alpha,$$
$$\phi_2' = -\phi_1 \sin\alpha + \phi_2 \cos\alpha.$$

The transformation rotates the state round a circle $|\Phi|^2 =$ constant in the state space (ϕ_1, ϕ_2). If we pick out the particular direction in (ϕ_1, ϕ_2) space for which Φ is real, and take the vacuum state to be $(\phi_0, 0)$, we break the $U(1)$ symmetry.

Expanding about this ground state $(\phi_0, 0)$, we put $\Phi = \phi_0 + (1/\sqrt{2})(\chi + i\psi)$. The Lagrangian density becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu \chi \partial^\mu \chi + \frac{1}{2}\partial_\mu \psi \partial^\mu \psi - \frac{m^2}{2\phi_0^2}\left[\sqrt{2}\phi_0 \chi + \frac{\chi^2}{2} + \frac{\psi^2}{2}\right]^2. \quad (10.3)$$

After breaking the $U(1)$ symmetry we must interpret the new fields. (In much the same way, the excited states of a ferromagnet cannot be discussed until the spatial symmetry has been broken.) In place of the complex field Φ, we have two coupled scalar real fields χ and ψ. We write

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}$$

where

$$\mathcal{L}_{\text{free}} = \frac{1}{2}\partial_\mu \chi \partial^\mu \chi - m^2\chi^2 + \frac{1}{2}\partial^\mu \psi \partial^\mu \psi. \quad (10.4)$$

$\mathcal{L}_{\text{free}}$ represents free particle fields, and contains all the terms in $\mathcal{L}$ that are quadratic in the fields. For classical fields and small oscillations, these terms dominate. The rest of the Lagrangian density, $\mathcal{L}_{\text{int}}$, corresponds to interactions between the free particles and higher order corrections to their motion.

There is a quadratic term $-m^2\chi^2$ in (10.4), so that the χ field corresponds to a scalar spin-zero particle of mass $\sqrt{2}m$ (by comparison with (3.18)). In the case of the ψ field there is no such quadratic term: the corresponding scalar spin-zero particle is therefore massless. The massless particles that always arise as a result of global symmetry breaking are called *Goldstone bosons*.

10.2 Local symmetry breaking and the Higgs boson

We now generalise further, and construct a Lagrangian density that is invariant under a *local* $U(1)$ gauge transformation,

$$\Phi \to \Phi' = e^{-iq\theta}\Phi,$$

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where $\theta = \theta(x)$ may be space and time dependent. This requires the introduction of a (massless) gauge field A_μ, as in Section 7.5, and we take

$$\mathcal{L} = [(\partial_\mu - \mathrm{i}qA_\mu)\Phi^\dagger][(\partial^\mu + \mathrm{i}qA^\mu)\Phi] - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(\Phi^\dagger\Phi), \tag{10.5}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and again

$$V(\Phi^\dagger\Phi) = \frac{m^2}{2\phi_0^2}\left[\Phi^\dagger\Phi - \phi_0^2\right]^2.$$

$\mathcal{L}$ is invariant under the local gauge transformation

$$\Phi(x) \to \Phi'(x) = \mathrm{e}^{-\mathrm{i}q\theta}\Phi(x), \quad A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu\theta(x).$$

A minimum field energy is obtained when the fields A_μ vanish, and Φ is constant, defined by a point on the circle $|\Phi| = \phi_0$. Any gauge transformation on this field configuration is also a minimum. Again we have an infinity of vacuum states.

Given $\Phi(x)$, we can always choose $\theta(x)$ so that the field $\Phi'(x) = \mathrm{e}^{-\mathrm{i}q\theta}\Phi(x)$ is real. This breaks the symmetry, since we are no longer free to make further gauge transformations.

Putting $\Phi'(x) = \phi_0 + h(x)/\sqrt{2}$, where $h(x)$ is real, gives

$$\begin{aligned}\mathcal{L} = {} & [(\partial_\mu - \mathrm{i}qA'_\mu)(\phi_0 + h/\sqrt{2})][(\partial^\mu + \mathrm{i}qA'^\mu)(\phi_0 + h/\sqrt{2})] \\ & - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} - \frac{m^2}{2\phi_0^2}\left[\sqrt{2}\phi_0 h + \frac{1}{2}h^2\right]^2.\end{aligned} \tag{10.6}$$

For clarity, we again separate this into

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}$$

where, dropping the primes on the gauge field,

$$\begin{aligned}\mathcal{L}_{\text{free}} &= \frac{1}{2}\partial_\mu h\partial^\mu h - m^2h^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + q^2\phi_0^2 A_\mu A^\mu, \\ \mathcal{L}_{\text{int}} &= q^2A_\mu A^\mu\left(\sqrt{2}\phi_0 h + \frac{1}{2}h^2\right) - \frac{m^2h^2}{2\phi_0^2}\left(\sqrt{2}\phi_0 h + \frac{1}{4}h^2\right).\end{aligned} \tag{10.7}$$

Before symmetry breaking, we had a complex scalar field $\Phi = (\phi_1 + \mathrm{i}\phi_2)/\sqrt{2}$, and a massless vector field with two polarisation states (Section 4.4). In $\mathcal{L}_{\text{free}}$ we have a single scalar field *h(x)* corresponding to a spinless boson of mass $\sqrt{2}m$, and a vector field A_μ, corresponding to a vector boson of mass $\sqrt{2}q\phi_0$, with three independent components (Section 4.9).

This mechanism for introducing mass into a theory was invented by Higgs (1964) and others (for example Anderson, 1963), and the particle corresponding to the field *h(x)* is called a *Higgs boson*. As a consequence of local symmetry breaking the gauge

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field acquires a mass, and the massless spin-zero Goldstone boson that appeared in our example of global symmetry breaking in Section 10.1 is replaced by the longitudinal polarised state of this massive spin one boson.

In the Weinberg and Salam ‘electroweak’ theory, the masses of the $W^{\pm}$ and Z particles arise as a result of symmetry breaking. The resulting theory can be renormalised, whereas the phenomenological theory of Chapter 9 cannot be renormalised. The form of $V(\Phi^{\dagger}\Phi)$ that has been introduced in this chapter appears also in the electroweak theory. It may seem a somewhat arbitrary feature. However, it can be shown to be the most general form that can be renormalised.

Problems

10.1 What interaction term in the model Lagrangian density (10.3) allows the massive boson to decay into two Goldstone bosons? Show that the decay rate in lowest order perturbation theory is

$$\frac{1}{\tau(\chi \to \psi\psi)} = \frac{m_\chi}{128\pi}\left(\frac{m_\chi}{\phi_0}\right)^2.$$

10.2 Show that with the model Lagrangian density (10.7), the vector boson would be stable, but if the coupling constant $q < m/(2\phi_0)$ the scalar boson would decay into two vector bosons.

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