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ON A PROBLEM POSED BY MAHLER

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Abstract

Maillet proved that the set of Liouville numbers is preserved under rational functions with rational coefficients. Based on this result, a problem posed by Mahler is to investigate whether there exist entire transcendental functions with this property or not. For large parametrized classes of Liouville numbers, we construct such functions and moreover we show that they can be constructed such that all their derivatives share this property. We use a completely different approach than in a recent paper, where functions with a different invariant subclass of Liouville numbers were constructed (though with no information on derivatives). More generally, we study the image of Liouville numbers under analytic functions, with particular attention to $f(z) = z^q$, where q is a rational number.

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1. Introduction

1.1. Definitions. As usual, for a real number α we will write $\lfloor\alpha\rfloor$ for the largest integer not greater than α, $\lceil\alpha\rceil$ for the smallest integer not smaller than α and $\{\alpha\} = \alpha - \lfloor\alpha\rfloor$. Moreover, $\|\alpha\|$ will denote the distance from α to the closest integer, and we will write $A \asymp B$ if both $A \ll B$ and $B \ll A$ are satisfied. For a function $f : X \mapsto Y$ and a set $A \subseteq X$, we will write $f(A) := \{f(x) : x \in A\}$.

A transcendental function is defined as an analytic function $f(z)$ which is algebraically independent of its variable z over some field. We will usually assume this field to be $\mathbb{C}$, and when at times we deal with $\overline{\mathbb{Q}}$ or $\mathbb{Q}$ instead this will be explicitly mentioned. On the other hand, an analytic function that satisfies some polynomial identity $P(z, f(z)) = 0$ with $P \in \mathbb{C}[X, Y]$ not equal $P(X, Y) \equiv 0$, is called an algebraic function. It is a widely known fact that the set of algebraic entire functions (over $\mathbb{C}$) coincides with the set of complex polynomials $\mathbb{C}[X]$. The nontrivial inclusion can be inferred from the great Picard theorem; see [8, Theorem 4.2 and Corollary 4.4].

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At the end of the XIXth century, after the proof by Hermite and Lindemann of the transcendence of e^{α} for all nonzero algebraic α, a question arose:

Does a transcendental analytic function usually take transcendental values at algebraic points?

In the example of the exponential function e^z, the word 'usually' stands for avoiding the exception $z = 0$. The set of the exceptions of this 'rule' was named by Weierstrass as the *exceptional set* of a function f, which is defined as

$$S_f := \{\alpha \in \overline{\mathbb{Q}} : f(\alpha) \in \overline{\mathbb{Q}}\}.$$

The study of exceptional sets started in 1886 with a letter of Weierstrass to Strauss. Clearly, for algebraic functions over the field $\overline{\mathbb{Q}}$, one has $S_f = \overline{\mathbb{Q}}$. In 2009, Huang *et al.* [12] proved, in particular, that all subset of $\overline{\mathbb{Q}}$ is the exceptional set of uncountably many transcendental entire functions (including their derivatives); see [19] for a more general result.

1.2. Liouville numbers and Mahler's classification. The *irrationality exponent* of a real number α, denoted by $\mu(\alpha)$, is defined as the (possibly infinite) supremum of all $\eta \geq 0$ such that

$$\left|\alpha - \frac{y}{x}\right| \leq x^{-\eta} \tag{1.1}$$

has infinitely many rational solutions y/x. We point out that (1.1) can be written equivalently using linear forms as $|\alpha x - y| \leq x^{-\eta+1}$. Mostly in this paper, the linear form representation will be more convenient. By Dirichlet's theorem [34, Corollary 2], $\mu(\alpha) \geq 2$ for all $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and the equality holds for nonrational real algebraic numbers α (by Roth's theorem), whereas $\mu(p/q) = 0$.

Real numbers with irrationality exponent equal to infinity are called *Liouville numbers*. We will write ζ for Liouville numbers in contrast to α for arbitrary real numbers and denote the set of Liouville numbers by $\mathscr{L}$. The elements of $\mathscr{L}$ are known to be transcendental by Liouville's theorem, which also led to the first construction of a transcendental number, namely the Liouville constant

$$L = \sum_{n\geq 1} 10^{-n!} = 0.110\,001\,000\,000\,000\,000\,000\,001\,0\ldots. \tag{1.2}$$

By altering the exponents in L slightly and adding fixed rational numbers, it is not hard to construct uncountably many elements of $\mathscr{L}$ within any set $A \subseteq \mathbb{R}$ with nonempty interior; see also Theorem 1.1 in Section 1.3. Furthermore, the set $\mathscr{L}$ is known to be a dense G_δ set, since it can be written $\mathscr{L} = \bigcap_{n\geq 1} U_n$, where

$$U_n := \bigcup_{q\geq 2}\bigcup_{p\in\mathbb{Z}}\left(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n}\right)\Bigg\backslash\left\{\frac{p}{q}\right\}$$

are open dense sets. Thus, $\mathscr{L}$ is a residual set, that is, the complement of a first category set. However, $\mathscr{L}$ is very small in sense of measure theory, as its Hausdorff dimension is 0; see [13].

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Some results of the paper are related to Mahler's U-numbers, so we want to give a short introduction of Mahler's classification of real transcendental numbers into S-, T- and U-numbers regarding their properties concerning approximation by algebraic numbers. In fact, we will introduce Koksma's classes S^*, T^* and U^*; however, the corresponding classes are known to be pairwise identical [3, cf. Theorem 3.6]. For real transcendental ζ and $n \geq 1$ an integer, define $w_n^*(\zeta)$ as the (possibly infinite) supremum of $\nu > 0$ such that

$$0 < |\alpha - \zeta| \leq H(\alpha)^{-\nu-1}$$

has infinitely many solutions in algebraic numbers α of degree at most n for arbitrarily large X, where $H(\alpha)$ is the largest absolute value of the coefficients of the irreducible (over $\mathbb{Z}$) minimal polynomial $P \in \mathbb{Z}[X]$ of α. Obviously, $w_1^*(\zeta) \leq w_2^*(\zeta) \leq \cdots$. The set of S-numbers is defined as the set of real transcendental numbers that satisfy

$$\limsup_{n\to\infty} \frac{w_n^*(\zeta)}{n} < \infty.$$

The T-numbers are defined by the properties

$$\limsup_{n\to\infty} \frac{w_n^*(\zeta)}{n} = \infty, \quad w_n^*(\zeta) < \infty \quad \text{for } n = 1, 2, \ldots.$$

Finally, the U-numbers are defined as numbers that satisfy $w_n^*(\zeta) = \infty$ for some finite index n. If m is the smallest such index, then ζ is a U_m-number. The definitions imply that the set $\mathscr{L}$ coincides with the set of U_1-numbers, and the set of U-numbers is the disjoint union of the sets of U_m-numbers over $m \geq 1$. We quote some important facts. Two algebraically dependent numbers belong to the same class [6, 16]. Almost all ζ, in the sense of Lebesgue measure, are S-numbers (this follows immediately from a result of Sprindžuk [32]), but the sets of T-numbers and U_m-numbers for $m \geq 1$ are nonempty. Schmidt was the first to construct T-numbers [27], and the first construction of U_m-numbers of arbitrary prescribed degree m was due to LeVeque [15]. See also [29, Ch. 3] or [3, Ch. 3].

1.3. The image of $\mathscr{L}$ under analytic functions. In his pioneering book, Maillet [18, Chapitre III] discusses some arithmetic properties of Liouville numbers. In particular, he proved the following result concerning the image of $\mathscr{L}$ under analytic functions.

THEOREM 1.1 (Maillet). *If f is a nonconstant rational function with rational coefficients, then $f(\mathscr{L}) \subseteq \mathscr{L}$.*

We observe that a kind of converse of this result is not valid in general; for example, taking $f(x) = x^2$ and any number of the form $\ell = \sum a_j 10^{-j!}$ with $a_j \in \{2, 4\}$, the number $\zeta = \sqrt{(3+\ell)/4}$ is not a Liouville number [3, Theorem 7.4], but $f(\zeta)$ is. Also, the rational coefficients cannot be taken algebraic (with at least one of them nonrational). For instance, for L in (1.2) and $m \geq 2$, the number $L\sqrt[m]{3/2}$ is not a Liouville number; see [18, Théorème I_3]. In fact, $L\sqrt[m]{3/2}$ is a U_m-number [7].

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A problem posed by Mahler [17] is to study which analytic functions share this property. In particular, he asked whether there exist nonconstant entire transcendental functions for which this is true.

In 1886, Weierstrass made a construction of entire transcendental functions with the property $f(\mathbb{Q}) \subseteq \mathbb{Q}$. Stäckel [33] proved that for any countable set $A \subseteq \mathbb{C}$ and any dense set $B \subseteq \mathbb{C}$, there exists an entire transcendental function f with the property $f(A) \subseteq B$. Gramain [11] showed that this is true for subsets of $\mathbb{R}$ as well. Several other generalizations are known; we refer the reader to [12, 20, 21, 24] for references. However, due to the uncountable cardinality of $\mathscr{L}$, the used classical methods dealing with recursive constructions do not to provide an obvious construction of entire transcendental functions with $f(\mathscr{L}) \subseteq \mathscr{L}$. More generally, Mahler's problem suggests studying the set $f(\mathscr{L} \cap I) \cap \mathscr{L}$ for functions f analytic on some interval $I \subseteq \mathbb{R}$ with real Taylor coefficients. A recent result due to Kumar *et al.* [14] shows that the set is always rather large; we will carry this out in Section 5.

1.4. Continued fractions. We introduce the notation we will use throughout the paper for continued fractions and gather various related results. The proofs can be found in [23] if not stated otherwise.

Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Let $\alpha_0 = \alpha$, $r_0 = \lfloor \alpha \rfloor$ and define the sequences $(r_j)_{j\geq 0}$, $(\alpha_j)_{j\geq 0}$ via the recursive formulas $r_{j+1} = \lfloor 1/\{\alpha_j\} \rfloor$ and $\alpha_{j+1} = \{1/\{\alpha_j\}\}$. Then, if we define

$$[r_0; r_1, r_2, \ldots, r_n] := r_0 + 1/(r_1 + 1/(r_2 + 1/(r_3 + \cdots + 1/r_n))\cdots),$$

the identity $\alpha = \lim_{n\to\infty}[r_0; r_1, r_2, \ldots, r_n]$ holds. This representation is unique and $[r_0; r_1, r_2, \ldots]$ is called the continued fraction expansion of α; r_j are called *partial quotients*. Denote

$$\frac{s_n}{t_n} = [r_0; r_1, \ldots, r_n], \quad n \geq 0,$$

the nth *convergent* of α in lowest terms. If we put $t_{-2} = 1, t_{-1} = 0$,

$$t_n = r_n t_{n-1} + t_{n-2}, \quad n \geq 0. \tag{1.3}$$

The analogous recursive formula for the s_n holds but we do not need it. Moreover, for any $n \geq 0$, we have $|s_n t_{n+1} - s_{n+1} t_n| = 1$, such that $(s_n, s_{n+1}) = 1$ and $(t_n, t_{n+1}) = 1$.

THEOREM 1.2 (Legendre). *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. If $|\alpha q - p| < (1/2)q^{-1}$ holds for integers p, q, then the fraction p/q equals a convergent of the continued fraction expansion of α.*

THEOREM 1.3 (Lagrange). *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and s_n/t_n be the nth convergent of $\alpha = [r_0; r_1, r_2, \ldots]$. Then*

$$\frac{r_{n+2}}{t_{n+2}} < |\alpha t_n - s_n| < \frac{1}{t_{n+1}} = \frac{1}{t_n r_{n+1} + t_{n-1}} < \frac{1}{t_n r_{n+1}}.$$

In particular, it follows from (1.3) that $\lim_{n\to\infty} \log r_{n+1}/\log t_n = \infty$ is equivalent to $\lim_{n\to\infty} \log t_{n+1}/\log t_n = \infty$, and in this case $\alpha \in \mathscr{L}$ follows. More precisely,

$$\limsup_{n\to\infty} \frac{\log t_{n+1}}{\log t_n} = \infty \quad \Longleftrightarrow \quad \alpha \in \mathscr{L}.$$

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1.5. Outline. This paper is organized in the way that Sections 2–4 deal with the main topic of $f(\mathscr{L}) \subseteq \mathscr{L}$ for entire transcendental functions, whereas Sections 5 and 6 discuss related topics indicated in Section 1.1. The assertion of our main result concerning the first category, Theorem 4.3, at first sight appears similar to a recent result [20], which we will state in Section 3. We will show in Section 3, though, that the classes considered in the respective theorems are in fact significantly different, and want to point out also that the proofs differ vastly. Moreover, we point out the advantage of Theorem 4.3, in that it makes assertions on the derivatives too. See Remark 4.4 for another difference. Concerning results on related topics, the main result we will prove in Section 6 is basically the following theorem.

THEOREM 1.4. *For any $q \in \mathbb{Q}\backslash\{0\}$, let $f_q(z) = z^q$. Then there exist uncountably many ζ, some of which can be explicitly constructed, such that $f_q(\zeta) \in \mathscr{L}$ if and only if $q \in \mathbb{Z}$.*

2. An approach connected to $f(\mathbb{Q})$

For a function f analytic in some open interval $I \subseteq \mathbb{R}$, we will establish sufficient conditions for $f(\mathscr{L} \cap I) \subseteq \mathscr{L}$, connected with the image $f(\mathbb{Q})$. More precisely, if we assume that $f(\mathbb{Q}) \subseteq \mathbb{Q}$ as in various constructions, see Section 1.3, and additionally assume certain upper bounds for the complexity of the fractions in the image, we will be able to deduce that $f(\mathscr{L} \cap I) \subseteq \mathscr{L}$. Keep in mind that $I = \mathbb{R}$ leads to entire functions. The method can be applied to confirm Theorem 1.1.

THEOREM 2.1. *Suppose that f is nonconstant analytic in some open interval $I \subseteq \mathbb{R}$ and $f(\mathbb{Q} \cap I) \subseteq \mathbb{Q}$. Moreover, assume that there exists a function $\psi : \mathbb{R}_{>0} \mapsto \mathbb{R}_{>0}$ with the properties:*

- *$\psi(m) = o(m)$ as $m \to \infty$;*
- *for $\zeta \in \mathscr{L} \cap I$ and any $m \geq 1$, we can find coprime $p_m, q_m \geq 2$ such that*

$$\left|\zeta - \frac{p_m}{q_m}\right| \leq q_m^{-m} \tag{2.1}$$

and, writing $f(p_m/q_m) = p'_m/q'_m$ in lowest terms, we have $q'_m \leq q_m^{\psi(m)}$.

Then $f(\mathscr{L} \cap I) \subseteq \mathscr{L}$.

PROOF. Let $\zeta \in \mathscr{L}$ be arbitrary. Let $J \subseteq I$ be nonempty and compact. Then $U := \max_{z \in J} |f'(z)|$ is well defined. Since $\zeta \in \mathscr{L}$, we can write

$$\zeta = \frac{p_m}{q_m} + \epsilon_m, \quad |\epsilon_m| \leq \frac{1}{U} q_m^{-m}$$

for any integer $m \geq 1$ with coprime integers p_m, q_m, where $q_m > 0$. Say $f(p_m/q_m) = p'_m/q'_m$ and, by assumption, $q'_m \leq q_m^{\psi(m)}$. Now for m sufficiently large that $p_m/q_m \in J$ the intermediate value theorem of differentiation gives

$$\left|f(\zeta) - \frac{p'_m}{q'_m}\right| = |f(\zeta) - f(p_m/q_m)| \leq U|\epsilon_m| \leq q_m^{-m} \leq q_m'^{-m/\psi(m)}. \tag{2.2}$$

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Since $\psi(m) = o(m)$, we conclude that $\mu(f(\zeta)) = \infty$ with μ the irrationality exponent unless $f(\zeta) \in \mathbb{Q}$. To exclude $f(\zeta) \in \mathbb{Q}$, assume the opposite and write $f(\zeta) = l_1/l_2$. Since f is not constant in I, by the identity theorem for analytic functions, see [8, Theorem 3.7 and Corollary 3.10], there exists some neighborhood $W \ni \zeta$ of ζ such that $f(z) \neq f(\zeta)$ for $z \in W$. Since p_m/q_m converges to ζ as $m \to \infty$, we infer that $f(p_m/q_m) \neq f(\zeta)$ for large m. Thus,

$$\left|f(\zeta) - f(p_m/q_m)\right| = \left|f(\zeta) - \frac{p'_m}{q'_m}\right| = \left|\frac{l_1}{l_2} - \frac{p'_m}{q'_m}\right| \geq \frac{1}{q'_m l_2},$$

which contradicts (2.2) for large m since $\psi(m) = o(m)$. □

We check that, as indicated above, rational functions with rational coefficients satisfy the conditions of Theorem 2.1. Let f be such a function and p, q integers. Then we can write

$$f(p/q) = \frac{P(p,q)}{Q(p,q)} = \frac{p'}{q'}$$

with fixed polynomials $P, Q \in \mathbb{Z}[X, Y]$ and $p', q' \in \mathbb{Z}$. Consider $\zeta \in \mathscr{L}$ fixed and let $p = p_m, q = q_m$ satisfy (2.1) and put $p' = p'_m, q' = q'_m$. First observe that we may assume that $q' \neq 0$ since ζ is transcendental and there are only finitely many algebraic poles of f, so there is no pole of f (and hence f is analytic) in a neighborhood of ζ. From (2.1), we deduce that $|p_m - \zeta q_m| < 1$ and thus $p_m \asymp q_m$ with implied constants depending on ζ, P, Q but not on m. It follows that $q'_m \ll q_m^k$, where k is the degree of Q and again the implied constant depends on ζ, P, Q only. Hence, the constant function $\psi(z) = k + 1$ (or $\psi(z) = k + \epsilon$ for any $\epsilon > 0$) satisfies the conditions of Theorem 2.1.

Considering constant functions $\psi(z)$, we stem a corollary from Theorem 2.1 whose conditions do not explicitly involve ζ but are solely conditions on the image $f(\mathbb{Q})$.

Corollary 2.2. *Suppose that f is nonconstant analytic in some open interval $I \subseteq \mathbb{R}$ and $f(\mathbb{Q} \cap I) \subseteq \mathbb{Q}$. Moreover, assume that there exists $\eta \in \mathbb{R}$ such that*

$$f(p/q) = p'/q'$$

implies $q' \leq q^\eta$ provided $(p, q) = 1$, $(p', q') = 1$ and $q \geq 2$. Then $f(\mathscr{L} \cap I) \subseteq \mathscr{L}$.

Proof. Since $\zeta \in \mathscr{L}$, for any $m \geq 1$ there exist p_m, q_m with (2.1). Apply for any such choice Theorem 2.1 with the constant function $\psi(m) = \eta$. □

Incorporating the additional condition of Theorem 2.1 or Corollary 2.2 for transcendental functions seems difficult with the common methods, as used for instance in [12] or [20]. In this context, [20, Theorem 1.2] asserts that there exist entire transcendental functions with $q' < q^{8q^2}$ in the notation of Corollary 2.2. See also [21, Theorem 2] for a related result concerning the image of algebraic numbers of bounded height under certain entire transcendental functions.

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3. Special classes of Liouville numbers

We define a few interesting subclasses of $\mathscr{L}$. The first one, which is new and will be considered in the main result Theorem 4.3, is parametrized by real functions.

Definition 3.1. Let Φ be the set of all functions $\varphi : \mathbb{R}_{\geq 2} \mapsto \mathbb{R}_{\geq 2}$ which are nondecreasing and satisfy $\lim_{x\to\infty} \varphi(x) = \infty$. For $\varphi \in \Phi$, define $\mathscr{L}_\varphi$, the (possibly empty) subclass of $\zeta \in \mathscr{L}$, for which for any given positive integer N, the estimate

$$-\frac{\log \|\zeta q\|}{\log q} \geq N \tag{3.1}$$

has an integer solution $q = q(N)$ with $2 \leq q \leq \varphi(N)$. Similarly, let $\mathscr{L}_\varphi^* \supset \mathscr{L}_\varphi$ be the set of $\zeta \in \mathscr{L}$ for which the condition holds for all $N \geq N_0(\zeta)$.

Remark 3.2. Observe that by Theorem 1.2, for $N \geq 2$ the smallest q for which (3.1) holds equals some denominator t_n of a convergent of ζ.

Remark 3.3. Only evaluations of $\varphi \in \Phi$ at integers will be of importance, so we could alternatively work with sequences. For $\varphi \in \Phi$ of low growth, the sets $\mathscr{L}_\varphi, \mathscr{L}_\varphi^*$ are indeed empty. However, we will see soon that the sets are large for φ of sufficiently fast growth.

Define orderings on Φ by $\psi \leq \varphi$ (respectively $\psi \leq_* \varphi$) if $\psi(x) \leq \varphi(x)$ for all $x \geq 2$ (respectively $x \geq x_0 = x_0(\varphi, \psi)$). These relations are clearly reflexive and transitive. The relation $\leq$ is also antisymmetric and hence $(\Phi, \leq)$ is a partially ordered set. Furthermore, the pointwise maximum of two functions lies above both functions in these partial orders, such that $(\Phi, \leq)$ and $(\Phi, \leq_*)$ can be viewed as directed sets. Obviously, $\psi \leq \varphi$ implies $\mathscr{L}_\psi \subseteq \mathscr{L}_\varphi$ and $\psi \leq_* \varphi$ implies $\mathscr{L}_\psi^* \subseteq \mathscr{L}_\varphi^*$, such that the set of all $\{\mathscr{L}_\varphi\}$ (respectively $\{\mathscr{L}_\varphi^*\}$), partially ordered by inclusion, is a directed set as well. For any $\zeta \in \mathscr{L}$, say $\mathscr{A}(\zeta) \subseteq \Phi$ is the set of $\varphi \in \Phi$ such that $\zeta \in \mathscr{L}_\varphi$. There is a unique $\varphi \in \mathscr{A}(\zeta)$ with the property that $\varphi \leq \psi$ for any $\psi \in \mathscr{A}(\zeta)$ (in particular, $\mathscr{A}(\zeta) \neq \emptyset$). This function is locally constant, right-continuous, has image in $\mathbb{Z}_{\geq 2}$ and increases in a discontinuous way at integer values q, where an estimate $\|\zeta q\| \leq q^{-N}$ for some integer $N > 0$ is satisfied for 'the first time' (for q but no smaller integer). We call it the *minimum function* for $\zeta \in \mathscr{L}$.

Example 3.4. For L as in (1.2) and for any integer $n \geq 1$,

$$\|10^{n!}L\| \leq 10^{n!-(n+1)!} + 2 \cdot 10^{n!-(n+2)!} = 10^{-n\cdot n!} + 2 \cdot 10^{n!-(n+2)!}$$

and hence

$$-\frac{\log \|10^{n!}L\|}{\log 10^{n!}} = \frac{n \cdot n! \log 10}{n! \log 10}(1 + o(1/n)) = n + o(1).$$

The remainder term tends to 0 fast, such that certainly $\varphi(x) = 10^{(x+1)!}$ is a proper choice for which $L \in \mathscr{L}_\varphi$, where we extend the definition of the factorials to real numbers by $x! := x(x-1)(x-2)\cdots(1 + \{x\})$.

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Example 3.5. For either $\varphi(x) = 2^{(x!)!}$ or $\varphi(x) = 2^{2^{x!}}$, it is easy to check that all numbers of the form $L_M := \sum_{j\geq 1} M^{-j!}$, for $M \geq 2$ an integer, belong to $\mathscr{L}_\varphi^*$ simultaneously.

Proposition 3.6. *Let $\varphi \in \Phi$ for which $\mathscr{L}_\varphi \neq \emptyset$, for example the minimum function of arbitrary $\zeta \in \mathscr{L}$. Then the set $\mathscr{L}_\varphi$ is uncountable. Moreover, for any nonempty open interval J, the set $\mathscr{L}_\varphi^* \cap J$ is uncountable.*

Proof. Say $\zeta = [r_0; r_1, r_2, \ldots]$ belongs to $\mathscr{L}_\varphi$. By the properties we have established, we may assume that φ is the minimum function of ζ.

By Remark 3.2, any rise of the locally constant minimum function of ζ is induced by some convergent (in general, not every convergent induces a rise). It is also obvious that there are infinitely many rises since $\zeta \in \mathscr{L}$. Define the subsequence $j(n)$ of $\{0, 1, 2, \ldots\}$ such that the nth rise is induced by $s_{j(n)}/t_{j(n)} = [r_0, r_1, \ldots, r_{j(n)}]$, that is, $q = t_{j(n)}$ but no smaller integer satisfies (3.1) for some integer N. Then $r_{j(n)+1}$ is large. For any subset $T \subseteq \{j(1), j(2), \ldots\}$ with infinite complement, define ζ_T, the number that arises from ζ by deleting precisely those partial quotients r_i for which $i - 1 \in T$. By virtue of (1.3) and Theorem 1.3 and since T^c is infinite, one checks that $\zeta_T \in \mathscr{L}$. On the other hand, the recurrence (1.3) implies that $\psi \leq \varphi$ for ψ the minimum function of ζ_T. Hence, $\zeta_T \in \mathscr{L}_\psi \subseteq \mathscr{L}_\varphi$. Since there are uncountably many choices for T and the continued fraction expansion is uniquely determined, this yields uncountably many elements in $\mathscr{L}_\varphi$. The assertion on $\mathscr{L}_\varphi^* \cap J$ can be inferred from the above by altering initial partial quotients, which only yields a rational transformation of ζ. □

Unfortunately, for any given $\varphi \in \Phi$ it is not hard to construct continued fraction expansions of elements in $\mathscr{L} \setminus \mathscr{L}_\varphi$ either, such that $\mathscr{L}_\varphi \subsetneq \mathscr{L}$. It suffices to choose many successive small partial quotients between rather large ones, such that the maximum of the left-hand side in (3.1) for bounded q tends to infinity more slowly than φ. More generally, a diagonal method argument shows that there is no representation of $\mathscr{L}$ as a countable union of classes $\mathscr{L}_\varphi$. However, obviously $\mathscr{L}$ can be written as the uncountable union $\bigcup_{\zeta\in\mathscr{L}} \mathscr{L}_{\varphi(\zeta)}$, where $\varphi(\zeta)$ is the minimum function of $\zeta \in \mathscr{L}$.

We compare the classes $\mathscr{L}_\varphi$ with certain other subclasses of $\mathscr{L}$ that have been studied. LeVeque [15] introduced strong Liouville numbers. This concept was refined by Alniacik [1], who defined semi-strong Liouville numbers. The following definition comprises these concepts and some additional ones that fit our purposes.

Definition 3.7. For $\zeta \in \mathscr{L}$, denote s_n/t_n $(n \geq 0)$, the sequence of its convergents. Then ζ is called *semi-strong* if one can find a subsequence $(v_i)_{i\geq 0}$ of $\{0, 1, 2 \ldots\}$ with the properties

$$|t_{v_i}\zeta - s_{v_i}| = t_{v_i}^{-\omega(v_i)}, \quad \lim_{i\to\infty} \omega(v_i) = \infty, \tag{3.2}$$

$$\limsup_{i\to\infty} \frac{\log t_{v_{i+1}}}{\log t_{v_i+1}} < \infty. \tag{3.3}$$

It is called *strong* if (3.2) is true for $v_i = i$ (note that (3.3) is trivial then). Denote the sets of semi-strong (respectively strong) Liouville numbers by $\mathscr{L}^{ss}$ (respectively $\mathscr{L}^s$). Further, for any nondecreasing function $\Lambda : \mathbb{R}_{\geq 1} \mapsto \mathbb{R}_{\geq 1}$ with $\lim_{x\to\infty} \Lambda(x) = \infty$,

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let $\mathscr{L}^{s,\Lambda} \subseteq \mathscr{L}^{s}$ (respectively $\mathscr{L}^{ss,\Lambda} \subseteq \mathscr{L}^{ss}$) be the sets for which $\omega(v_i) \geq \Lambda(i)$ for some sequence $(v_i)_{i\geq 1}$ as above.

Conversely to the sets $\mathscr{L}_\varphi$, the sets $\mathscr{L}^{s,\Lambda}$ and $\mathscr{L}^{ss,\Lambda}$ get smaller the faster Λ tends to infinity. For any Λ as in Definition 3.7, choosing the partial quotients sufficiently large, it is easy to check that all defined sets are nonempty (in fact, uncountable).

It is not hard to see that $\mathscr{L}^{s} \subsetneq \mathscr{L}^{ss} \subsetneq \mathscr{L}$. Unfortunately (in view of Section 4.2), for any given $\varphi \in \Phi$, there exist (semi-)strong Liouville numbers not contained in $\mathscr{L}^*_\varphi$, that is, $\mathscr{L}^{s} \nsubseteq \mathscr{L}^*_\varphi$. To ensure inclusion, we need some (arbitrarily weak) additional minimum growth condition on the sequence $\omega(v_i)$ in (3.2).

Proposition 3.8. *Fix any function Λ as in Definition 3.7. Then there exists $\varphi = \varphi(\Lambda) \in \Phi$ such that $\mathscr{L}^{s,\Lambda} \subseteq \mathscr{L}_\varphi$. Furthermore, there exists $\psi = \psi(\Lambda) \in \Phi$ for which $\mathscr{L}^{s,\Lambda} \subseteq \mathscr{L}^{ss,\Lambda} \subseteq \mathscr{L}^*_\psi$.*

Proof. First we construct $\varphi \in \Phi$ such that $\mathscr{L}^{s,\Lambda} \subseteq \mathscr{L}_\varphi$ and prove this rigorously; subsequently, we sketch how to derive the other inclusion in a similar way.

Consider an arbitrary but fixed integer $N \geq 1$. We will construct suitable $\varphi(N)$. Let $\iota_N := \lceil \Lambda^{-1}(N) \rceil$, that is, the smallest index i such that $\Lambda(i) \geq N$. Consider integers $T_1, \ldots, T_{\iota_N}$ given by the recurrence relation $T_0 = 1, T_1 = N + 1$ and $T_{j+1} = T_j^{N+1}$ for $1 \leq j \leq \iota_N - 1$ and put $D_N := T_{\iota_N}$. We show that $\varphi(N) := D_N$ is a suitable choice. We use the notation of Section 1.4 for the continued fraction expansion of ζ. First assume that all partial denominators $t_1, \ldots, t_N$ of the convergents of some ζ are bounded by $t_j \leq T_j$. It follows from (1.3) that $t_{\iota_N} \leq T_{\iota_N} = D_N$, but, on the other hand, the inequality $|t_j\zeta - s_j| < t_j^{-N}$ is satisfied for the index $j = \iota_N$ by definition of ι_N. Thus, if we put $q = T_{\iota_N}$ in Definition 3.1, we see that $\varphi(N) := D_N$ is indeed a proper choice. On the other hand, if for some $1 \leq j \leq \iota_N - 1$ we have $t_j > T_j$, then again by (1.3) and Theorem 1.3 we infer that $|t_{j-1}\zeta - s_{j-1}| < t_{j-1}^{-N}$, and if j is the smallest such index then moreover $t_{j-1} \leq D_N$. Again this shows that we may put $q = q(N) = t_{j-1}$ in Definition 3.1 and $\varphi(N) := D_N$ is a proper choice.

For the inclusion $\mathscr{L}^{ss,\Lambda} \subseteq \mathscr{L}^*_\psi$, construct $\psi(N) = D_N$ as above, with the replacement $T_{j+1} := T_j^{j(j+1)}$ in each inductive step. Observe that for any $\zeta \in \mathscr{L}^{ss}$, condition (3.3) guarantees that we will have $t_{\iota_N} < T_{\iota_N} =: D_N$ for sufficiently large $N = N(\zeta)$. □

Conversely, it can be shown that for any fixed $\varphi \in \Phi$ we have $\mathscr{L}_\varphi \nsubseteq \mathscr{L}^{ss}$. We will not need this, though. Another subclass of $\mathscr{L}$ was recently defined in [20].

Definition 3.9. Recursively define $\exp^{[0]}(x) := x$ and $\exp^{[k+1]}(x) = \exp(\exp^{[k]})(x)$. Then $\zeta \in \mathscr{L}$ is called *ultra-Liouville* if for any $k \geq 0$ there exists a rational number p/q such that

$$\left|\zeta - \frac{p}{q}\right| \leq \frac{1}{\exp^{[k]}(q)}. \tag{3.4}$$

We denote the set of ultra-Liouville numbers by $\mathscr{L}_{ultra}$.

Theorem 1.1 in [20], which relies on [20, Theorem 1.2] mentioned at the end of Section 2, asserts the following result.

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THEOREM 3.10 (Marques and Moreira [20]). *There exist uncountably many entire transcendental functions f such that $f(\mathscr{L}_{ultra}) \subseteq \mathscr{L}_{ultra}$. In particular, $f(\mathscr{L}_{ultra}) \subseteq \mathscr{L}$.*

It is important to notice that the previous result is strong in the sense that it ensures the existence of an uncountable subset of Liouville numbers which is invariant for uncountably many transcendental analytic functions.

It is not hard to check that there exist functions $\varphi \in \Phi$ for which $\mathscr{L}_\varphi \nsubseteq \mathscr{L}_{ultra}$. It suffices to take φ, the minimum function of any $\zeta \in \mathscr{L}$, for which we cannot find a rational for which (3.4) holds for $k = 1$ (or any larger k), which clearly exists. Conversely, one checks that $\mathscr{L}_{ultra} \nsubseteq \mathscr{L}^*_\varphi$ for any fixed function $\varphi \in \Phi$, as the frequency of values q inducing very good approximations p/q in (3.4) can be arbitrarily low. For similar reasons, also a combination of the concepts (semi-)strong and ultra is not sufficient to provide φ with inclusion; in other words, $\mathscr{L}_{ultra} \cap \mathscr{L}^s \nsubseteq \mathscr{L}_\varphi$ for any $\varphi \in \Phi$. Moreover, there is no inclusion within $\mathscr{L}^s$ (respectively $\mathscr{L}^{ss}$) and $\mathscr{L}_{ultra}$. Finally, we also refer to [21] for a result similar to Theorem 3.10 concerning the image of more general sets (in general, no longer subsets of $\mathscr{L}$). There is again no immediate correlation to Theorem 4.3.

4. Entire transcendental functions with large invariant set

4.1. Preparatory results. We put our focus on entire functions f now. We gather some results that we will utilize in the proof of Theorem 4.3. The following Lemma 4.1 on its own leads to another proof of Theorem 1.1 in the special case of polynomials. In the proof we will use the following elementary fact. For a real number α and a positive integer k, the estimate

$$|q\alpha - p| \le q^{-\nu} \tag{4.1}$$

implies that

$$|q^k\alpha^k - p^k| = |q\alpha - p| \cdot |q^{k-1}\alpha^{k-1} + \cdots + p^{k-1}| \le D(k,\alpha)q^{-\nu+k-1} \tag{4.2}$$

with a constant $D(k,\alpha)$ depending only on k and α. This argument was actually used in a slightly more general way in the proof of Lemma 1 in [4] and will be frequently applied in Section 6 as well.

LEMMA 4.1. *Let $\alpha \in \mathbb{R}$ and $P \in \mathbb{Q}[X]$ be given as*

$$P(z) = \frac{a_0}{b_0} + \frac{a_1}{b_1}z + \cdots + \frac{a_m}{b_m}z^m$$

with a_j/b_j in lowest terms. Put $A := \max_{0\le j\le m} |a_j|$, $B := \mathrm{lcm}(|b_0|, \ldots, |b_m|)$. Assume for a positive integer q and (large) $\nu > 0$ that

$$\|q\alpha\| \le q^{-\nu}. \tag{4.3}$$

Then $Bq^m \in \mathbb{Z}$ and

$$\|Bq^m \cdot P(\alpha)\| \le m^2(1 + |\alpha|)^{m-1} \cdot ABq^{-\nu+m-1}.$$

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Proof. By definition, $d_k := |B/b_k|$ is an integer with $1 \le d_k \le B$ for $0 \le k \le m$. Recall that for any integer M and $\alpha \in \mathbb{R}$, we have $\|M\alpha\| \le |M| \cdot \|\alpha\|$. For $0 \le k \le m$, we estimate the monomial

$$\left\| Bq^m \frac{a_k}{b_k} \alpha^k \right\| = \|d_k a_k q^m \alpha^k\| \le |a_k| d_k q^{m-k} \|q^k \alpha^k\| \le ABq^{m-k} \|q^k \alpha^k\|. \tag{4.4}$$

Moreover, for $k = 0$, the left-hand side of (4.4) is 0, which will improve the result slightly. As ν is large and thus p/q is very close to α for some $p \in \mathbb{Z}$, we may apply (4.2) to estimate $\|q^k \alpha^k\|$ with the bound $D(k, \alpha) \le k(1 + |\alpha|)^{k-1} \le m(1 + |\alpha|)^{m-1}$ for any $1 \le k \le m$. Since $\|\mu_0 + \cdots + \mu_m\| \le \|\mu_1\| + \cdots + \|\mu_m\|$ for all real $\mu_0, \mu_1, \ldots, \mu_m$ with $\mu_0 \in \mathbb{Z}$, we infer the lemma if we put μ_k in the left-hand side of (4.4) for $0 \le k \le m$. □

We will need an additional technical coprimeness result for special choices of coefficients c_j in Lemma 4.1 in the proof of Theorem 4.3.

Proposition 4.2. *Let $\alpha \in \mathbb{R}$ and $P \in \mathbb{Q}[X]$ be as in Lemma 4.1, where $c_j = 1/b_j$ and $b_j | b_{j+1}$ for $0 \le j \le m - 1$. Define A, B as in the lemma, such that $A = 1, B = b_m$.*

There exists $\nu_0 = \nu_0(P)$ which depends on P but not on q, such that if $q \ge 2$ satisfies (4.3) for $\nu \ge \nu_0$, and if for p the closest integer to $q\alpha$ we have $(p, q) = 1$, and if R denotes the closest integer to $Bq^m \cdot P(\alpha)$, we have $(q, R) = 1$.

Proof. There exists some large $\nu_1 = \nu_1(P)$ independent of q such that for $\nu \ge \nu_1$, all left-hand sides in (4.4) in the proof of Lemma 4.1 are sufficiently small to add up to a number smaller than $1/2$. Then R equals the sum of the $m + 1$ closest integers to the monomials $Bq^m a_k / b_k \alpha^k$; call them Z_k. In view of (4.2),

$$q^m \alpha^k = q^{m-k} (q\alpha)^k = q^{m-k} p^k + q^{m-k} \|q\alpha\|^k$$

is very close to $q^{m-k} p^k$ uniformly in $0 \le k \le m$, provided that $\|q\alpha\|$ is sufficiently small. More precisely, it is not hard to check that if ν in (4.3) satisfies $\nu \ge \nu_2$ with large $\nu_2 = \nu_2(P)$ independent of q, again writing $d_k = B/b_k \in \mathbb{Z}$ for $0 \le k \le m$,

$$Z_k = q^{m-k} p^k a_k d_k = q^{m-k} p^k d_k, \quad 0 \le k \le m.$$

Note that $d_m = 1$ since $b_m = B$ follows from the divisibility conditions on the b_j. Combining these results, if we let $\nu \ge \nu_0$ in (4.3) with $\nu_0 := \max\{\nu_1, \nu_2\}$, we infer that

$$R = Z_0 + \cdots + Z_m = q^m d_0 + q^{m-1} p d_1 + q^{m-2} p^2 d_2 + \cdots + q p^{m-1} d_{m-1} + p^m.$$

Clearly, any prime divisor of q divides any other expression in the sum but certainly not p^m since $(p, q) = 1$ by assumption. The assertion follows. □

4.2. The main result. Now we state the main theorem, which provides nonconstant entire transcendental functions f that map large prescribed subclasses of $\mathscr{L}$ to $\mathscr{L}$. It will turn out that all derivatives have the same property. The idea is to look at entire functions whose Taylor coefficients decrease fast by absolute value, in order to apply Lemma 4.1 with gain. To exclude the case that an element of the image is rational is slightly technical. We agree that $f(\emptyset) = \emptyset$ in the trivial case $\mathscr{L}_\varphi^* = \emptyset$.

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THEOREM 4.3. *Let $\varphi \in \Phi$ be arbitrary but fixed. Then there exist uncountably many entire transcendental functions $f(z) = c_0 + c_1 z + \cdots$ with $c_j \in \mathbb{Q} \setminus \{0\}$ and the property that for any $s \in \{0, 1, 2, \ldots\}$:*

- $f^{(s)}(0) \in \mathbb{Q}$;
- $f^{(s)}(\mathbb{Q} \setminus \{0\}) \subseteq \mathscr{L}$;
- $f^{(s)}(\mathscr{L}_\varphi^*) \subseteq \mathscr{L}$.

Suitable functions f can be explicitly constructed.

PROOF. First we prove the assertion for $s = 0$ and subsequently describe how the proof extends to $s > 0$.

Let $(T_m)_{m\geq 1}$ be any sequence of positive real numbers that tends to infinity, for instance $T_m = m$. We recursively construct the rational Taylor coefficients c_j of suitable functions f. Note that the first assertion of the theorem will follow immediately from $c_j \in \mathbb{Q}$. Let $c_0 = 1$. Assume that the Taylor polynomial $P_m(z) = c_0 + c_1 z + \cdots + c_m z^m$ of f of degree $m \geq 0$ is already constructed and has rational coefficients $c_j = 1/b_j$ and $b_j | b_{j+1}$ for $0 \leq j \leq m-1$, as in Proposition 4.2. We construct c_{m+1}. Let $P := P_m$ in Lemma 4.1 and similarly define $A := A_m, B := B_m$ with A_m, B_m arising from the present a_j, b_j as in the lemma. In fact, the conditions show that $A_m = 1, B_m = b_m$. Let the positive integer k_m be large enough such that

$$q^{k_m} > m^2 (T_m + 1)^{m-1} A_m B_m q^{m-1} \cdot 2(B_m q^m)^m =: q^{m^2+m-1} D_m \tag{4.5}$$

for any integer $q \geq 2$, which is possible since D_m and the exponent $m^2 + m - 1$ are constants. Since we can make k_m larger if necessary, we may assume that $k_m \geq \nu_0(P_m)$, where $\nu_0(P_m)$ is as in Proposition 4.2 for $P = P_m$. By definition of the set $\mathscr{L}_\varphi$, for any $\zeta \in \mathscr{L}_\varphi$, the inequality

$$\|q\zeta\| \leq q^{-k_m} \tag{4.6}$$

has a solution $q =: \widetilde{q}_m$ that may depend on ζ but with $2 \leq \widetilde{q}_m \leq \varphi(k_m)$ uniformly. First consider only the class $\mathscr{L}_\varphi$ instead of $\mathscr{L}_\varphi^*$. Restricting to $\zeta \in \mathscr{L}_\varphi \cap [-T_m, T_m]$, application of Lemma 4.1 with $\nu := k_m$ in view of (4.5) yields

$$\|(B_m \widetilde{q}_m^m) \cdot P_m(\zeta)\| \leq m^2 (1 + |\zeta|)^{m-1} \cdot A_m B_m \widetilde{q}_m^{-k_m+m-1} \leq \tfrac{1}{2} |B_m \widetilde{q}_m^m|^{-m}. \tag{4.7}$$

Put $\widetilde{Q}_m := B_m \widetilde{q}_m^m$; then (4.7) turns into

$$\|\widetilde{Q}_m P_m(\zeta)\| \leq \tfrac{1}{2} \widetilde{Q}_m^{-m}. \tag{4.8}$$

Moreover, if we write $\tau_m := B_m \varphi(k_m)^m$, then

$$|\widetilde{Q}_m| \leq \tau_m. \tag{4.9}$$

Now we determine $c_{m+1} \in \mathbb{Q} \setminus \{0\}$ of very small modulus. Assume that the coefficients $c_{m+2}, c_{m+3}, \ldots$ do not vanish but are of very small and fast decreasing modulus too. More precisely, for now we assume that all the coefficients $c_{m+1}, c_{m+2}, \ldots$ satisfy

$$|c_{m+h}| < \min\{(1/4)(1 + T_m)^{-m-2h} \tau_m^{-m-1}, 1/(m+h)!\}, \quad h \geq 1, \tag{4.10}$$

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where the purpose of $1/(m+h)!$ is solely to guarantee convergence. Pick any suitable $c_{m+1} = 1/b_{m+1} \in \mathbb{Q} \setminus \{0\}$ for b_{m+1} a sufficiently large integral multiple of b_m such that (4.10) is satisfied for $h = 1$. Then

$$|f(z) - P_m(z)| = \left|\sum_{h=1}^{\infty} c_{m+h} z^{m+h}\right| \leq \sum_{h=1}^{\infty} |c_{m+h}| T_m^{m+h} < \frac{1}{2}\tau_m^{-m-1}$$

uniformly for $z \in [-T_m, T_m]$. Thus, in particular for $\zeta \in \mathscr{L}_\varphi \cap [-T_m, T_m]$, condition (4.9) implies that

$$|\widetilde{Q}_m \cdot (f(\zeta) - P_m(\zeta))| \leq |\widetilde{Q}_m| \cdot \tfrac{1}{2}\tau_m^{-m-1} \leq \tfrac{1}{2}|\widetilde{Q}_m|^{-m}. \tag{4.11}$$

Combination of (4.8), (4.11) and the triangular inequality yields

$$\|\widetilde{Q}_m \cdot f(\zeta)\| \leq |\widetilde{Q}_m|^{-m}. \tag{4.12}$$

Now we repeat the procedure with the polynomial $P_{m+1}(z) = c_0 + \cdots + c_{m+1}z^{m+1}$, where we have to satisfy the condition (4.10) for m and $m+1$, which however we may easily do by choosing any sufficiently small rational $c_{m+2} = 1/b_{m+2}$ with $b_{m+1}|b_{m+2}$. Proceeding in this manner, we obtain integer solutions to the estimate (4.12) for any $m \geq 1$ and any $\zeta \in \mathscr{L}_\varphi \cap [-T_m, T_m]$. Any ζ belongs to $[-T_m, T_m]$ for all large $m \geq m_0(\zeta)$; hence, indeed $\mu(f(\zeta)) = \infty$ or $f(\zeta) \in \mathbb{Q}$ for any $\zeta \in \mathscr{L}_\varphi$, where μ denotes the irrationality exponent. We have to exclude the case $f(\zeta) \in \mathbb{Q}$ to infer that $f(\zeta) \in \mathscr{L}$, simultaneously for all $\zeta \in \mathscr{L}_\varphi$.

Assume that $f(\zeta) \in \mathbb{Q}$ for some $\zeta \in \mathscr{L}_\varphi$, say $f(\zeta) = l_1/l_2$ with coprime integers l_1, l_2. For $\widetilde{q}_m$ as constructed in the proof, let $\widetilde{p}_m/\widetilde{q}_m$ be the good approximation to ζ with denominator $\widetilde{q}_m$, that is, $\widetilde{p}_m$ is the closest integer to $\zeta\widetilde{q}_m$. Recalling the definition of $\widetilde{q}_m$ in (4.6), we may assume that $(\widetilde{p}_m, \widetilde{q}_m) = 1$, otherwise we could divide both $\widetilde{p}_m, \widetilde{q}_m$ by their greatest common divisor and (4.6) still holds (in fact, the left-hand side is even smaller and the right-hand side larger) and all above works analogously. Further, say that $\widetilde{R}_m$ is the closest integer to $\widetilde{Q}_m f(\zeta)$ for $m \geq 1$. The estimate (4.12) can be written

$$|\widetilde{Q}_m f(\zeta) - \widetilde{R}_m| \leq |\widetilde{Q}_m|^{-m}, \quad m \geq 1. \tag{4.13}$$

On the other hand, if for some m we have $\widetilde{R}_m/\widetilde{Q}_m \neq l_1/l_2$, then

$$|\widetilde{Q}_m f(\zeta) - \widetilde{R}_m| = \left|\widetilde{Q}_m \frac{l_1}{l_2} - \widetilde{R}_m\right| \geq \frac{1}{l_2}, \quad m \geq 1. \tag{4.14}$$

Since both (4.13) and (4.14) cannot hold for large m, we must have

$$\frac{\widetilde{R}_m}{\widetilde{Q}_m} = f(\zeta) = \frac{l_1}{l_2}, \quad m \geq m_0. \tag{4.15}$$

Since $\widetilde{Q}_m = B_m \widetilde{q}_m^m$ and $\lim_{m\to\infty} \widetilde{q}_m = \infty$, it suffices to show that $\widetilde{R}_m$ and $\widetilde{q}_m$ are coprime for any fixed m to contradict (4.15). Due to (4.11), $\widetilde{R}_m$ equals the closest integer to $\widetilde{Q}_m P_m(\zeta)$ as well. Hence, recalling (4.6) and $k_m \geq \nu_0(P_m)$, Proposition 4.2 indeed

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implies that $(\widetilde{R}_m, \widetilde{q}_m) = 1$. This contradicts the hypothesis $f(\zeta) \in \mathbb{Q}$, which finishes the proof of $f(\mathscr{L}_\varphi) \subseteq \mathscr{L}$.

We now describe how the above generalizes to the larger class $\mathscr{L}_\varphi^*$. We may assume that the sequence $(k_m)_{m\geq 1}$ tends to infinity, otherwise we can choose larger values in any step. Thus, by definition of $\mathscr{L}_\varphi^*$, for any $\zeta \in \mathscr{L}_\varphi^*$ the estimate (4.6) has a solution $2 \leq \widetilde{q} \leq \varphi(k_m)$ for all large $m \geq m_0(\zeta)$. Hence, we deduce solutions to (4.12) for $m \geq m_0(\zeta)$ which guarantee $f(\zeta) \in \mathscr{L} \cup \mathbb{Q}$. The exclusion of $f(\zeta) \in \mathbb{Q}$ obviously works as for $\zeta \in \mathscr{L}_\varphi$.

Next we show that $f(\mathbb{Q} \setminus \{0\}) \subseteq \mathscr{L}$. Let $l_1/l_2 \in \mathbb{Q}$ be arbitrary and write $B_m/b_j = d_{m,j} \in \mathbb{Z}$ for $m \geq 1$ and $0 \leq j \leq m$. Then, on the one hand,

$$B_m l_2^m P_m(l_1/l_2) = B_m l_2^m \sum_{j=0}^{m} c_j \left(\frac{l_1}{l_2}\right)^j = \sum_{j=0}^{m} d_{m,j} l_1^j l_2^{m-j} =: \mathscr{A}_m \in \mathbb{Z}$$

by construction; on the other hand,

$$|B_m l_2^m (f(l_1/l_2) - P_m(l_1/l_2))| \leq \left| B_m l_2^m \sum_{j=m+1}^{\infty} c_j \left(\frac{l_1}{l_2}\right)^j \right| \leq (B_2 l_2^m)^{-m}$$

for large m by the fast decay of $c_j = 1/b_j = 1/B_j$. The triangular inequality shows that $\mu(f(l_1/l_2)) = \infty$ unless $f(l_1/l_2) \in \mathbb{Q}$, and that $\mathscr{A}_m$ is the closest integer to $B_m l_2^m f(l_1/l_2)$. By virtue of the same principle as in (4.14), it suffices to check that $\mathscr{A}_m/(B_m l_2^m) = P_m(l_1/l_2)$ is not constant for all $m \geq m_0$ to exclude the case $f(l_1/l_2) \in \mathbb{Q}$ and thus $f(l_1/l_2) \in \mathscr{L}$. However, since $P_{m+1}(z) = P_m(z) + c_{m+1} z^{m+1}$, the equality $P_m(l_1/l_2) = P_{m+1}(l_1/l_2)$ for some m implies that $c_{m+1} = 0$, which is false, unless $l_1/l_2 = 0$. This yields the assertion.

We check that f has the remaining desired properties. The expression $1/(m+h)!$ in (4.10) guarantees that f is an entire function, which by construction has rational coefficients and is not a polynomial. Hence, it is transcendental as described in Section 1.1. Clearly, this method is flexible enough to provide uncountably many suitable f.

It remains to extend the assertion to the derivatives. We may assume that in every recursive step the condition $b_m | b_{m+1}$ is strengthened to $m! b_m | b_{m+1}$. All derivatives of f are then again of the form $f^{(s)}(z) = \sum_{j\geq 0} (1/b_j^{(s)}) z^j$ for integers $b_j^{(s)}$ with the property $b_j^{(s)} | b_{j+1}^{(s)}$ for all pairs $j \geq 0, s \geq 0$. Let $s \geq 0$ be fixed now. If we define $A_m^{(s)}, B_m^{(s)}$ for $P_m^{(s)}$ the mth Taylor polynomial of $f^{(s)}$ as in Lemma 4.1, then by the above $A_m^{(s)} = 1, B_m^{(s)} = b_m^{(s)}$ for all $m \geq 0$, as in the case $s = 0$. By construction, also $B_m^{(t+1)} = (m+1)^{-1} B_{m+1}^{(t)} < B_{m+1}^{(t)}$ for all $m \geq 0, t \geq 0$ and thus $B_m^{(s)} < B_{m+s}$. Thus, if we put $k_m^{(s)} := k_{m+s}$, then similarly to (4.5) the estimate

$$q^{k_m^{(s)}} > m^2 (T_m + 1)^{m-1} A_m^{(s)} B_m^{(s)} q^{m-1} \cdot 2 (B_m^{(s)} q^m)^m =: q^{m^2+m-1} D_m^{(s)}$$

will be satisfied for all $q \geq 2$ with $D_m^{(s)} := D_{m+s}$. Similarly to (4.6), we infer that

$$\|q\zeta\| \leq q^{-k_m^{(s)}}$$

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has a solution $q =: \widetilde{q}_m^{(s)}$ that may depend on ζ but with $2 \le \widetilde{q}_m \le \varphi(k_m^{(s)})$ uniformly. Proceeding further as in the case $s = 0$, the analogue of (4.7) holds again and with $\widetilde{Q}_m^{(s)} := B_m^{(s)} \widetilde{q}_m^{(s)m}$ we further obtain

$$\|\widetilde{Q}_m^{(s)} P_m^{(s)}(\zeta)\| \le \tfrac{1}{2} \widetilde{Q}_m^{(s)-m}. \tag{4.16}$$

Moreover, with $\tau_m^{(s)} := \tau_{m+s}$, we also obtain

$$|\widetilde{Q}_m^{(s)}| \le \tau_m^{(s)}. \tag{4.17}$$

For the estimate of the remainder term, first note that the coefficients $c_j^{(s)}$ of $f^{(s)}$ satisfy

$$c_j^{(s)} = \frac{1}{b_j^{(s)}} = \frac{1}{b_{j+s}} j(j+1)\cdots(j+s-1) \le (j+s)^s \frac{1}{b_{m+s}} = (j+s)^s c_{m+s}.$$

Hence,

$$|f(z)^{(s)} - P_m^{(s)}(z)| = \left|\sum_{h=1}^{\infty} c_{m+h}^{(s)} z^{m+h}\right| \le \sum_{h=1}^{\infty} |c_{m+h}^{(s)}| T_m^{m+h} \le \sum_{h=1}^{\infty} (m+h+s)^s |c_{m+h+s}| T_m^{m+h}$$

uniformly for $z \in [-T_m, T_m]$. If we strengthen the condition (4.10) in any inductive step by replacing τ^{-m} by τ^{-m^2} if necessary, from the fast decay of $(c_m)_{m\ge1}$ and since s is fixed, it clearly follows that at least for large m the above can be bounded by

$$|f(z)^{(s)} - P_m^{(s)}(z)| \le \sum_{h=1}^{\infty} (m+h+s)^s |c_{m+h+s}| T_m^{m+h} \le \frac{1}{2} \tau_m^{(s)-m-1}.$$

In combination with (4.17) for large m again,

$$|\widetilde{Q}_m^{(s)} \cdot (f^{(s)}(\zeta) - P_m^{(s)}(\zeta))| \le |\widetilde{Q}_m^{(s)}| \cdot \tfrac{1}{2} \tau_m^{(s)-m-1} \le \tfrac{1}{2} |\widetilde{Q}_m^{(s)}|^{-m}$$

and, together with (4.16) and the triangular inequality, eventually

$$\|\widetilde{Q}_m^{(s)} \cdot f^{(s)}(\zeta)\| \le |\widetilde{Q}_m^{(s)}|^{-m}.$$

As this holds for all $\zeta \in \mathscr{L}_\varphi$ and large m, indeed $f^{(s)}(\mathscr{L}_\varphi) \subseteq \mathscr{L}$. The generalization to $\mathscr{L}_\varphi^*$ such as the proof of $f^{(s)}(\zeta) \notin \mathbb{Q}$ and $f^{(s)}(\mathbb{Q} \setminus \{0\}) \subseteq \mathscr{L}$ works very similarly to the case $s = 0$. □

We give several remarks.

Remark 4.4. The assertion $f(\mathbb{Q} \setminus \{0\}) \subseteq \mathscr{L}$ implies that $f(\mathbb{Q} \setminus \{0\})$ is a purely transcendental set; see Section 1.2. Observe the contrast to Theorems 1.1, 2.1, Corollary 2.2 and Theorem 3.10, where we had $f(\mathbb{Q}) \subseteq \mathbb{Q}$. Moreover, since a function f algebraic over $\overline{\mathbb{Q}}$ satisfies $S_f = \overline{\mathbb{Q}}$, this leads to a proof that all constructed functions are transcendental over the base field $\overline{\mathbb{Q}}$ instead of $\mathbb{C}$. This is weaker but avoids the rather deep great Picard theorem; see Section 1.1.

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Remark 4.5. We only needed $\zeta \in \mathscr{L}_\varphi$ to obtain a uniform bound of $\widetilde{q}_m$ in (4.6). If we weaken this to $\zeta \in \mathscr{L}$, we further have no uniform bound in (4.9), which is needed to bound the left-hand side in (4.11), even restricting to ζ in a given compact interval.

Remark 4.6. For any finite set $\{\zeta_1, \zeta_2, \ldots, \zeta_u\} \subseteq \mathscr{L}^u$, the proof of Theorem 4.3 provides a method of constructing entire transcendental functions f that map all ζ_j simultaneously to elements of $\mathscr{L}$. It suffices to define the involved function φ as the pointwise maximum of the individual minimum functions for ζ_j, as carried out subsequent to Definition 3.1. However, such functions f can alternatively be constructed with the Weierstrass factorization theorem; see [8, Ch. 7, paragraph 5].

It is evident that Theorem 4.3 becomes more interesting the faster the function φ tends to infinity. See Section 3 for examples of φ inducing large sets $\mathscr{L}_\varphi$. From Proposition 3.8 and Theorem 4.3, we further infer a last corollary.

Corollary 4.7. *Let Λ be any function as in Definition 3.7. Then there exist uncountably many entire transcendental functions f with $f(\mathscr{L}^{s,\Lambda}) \subseteq f(\mathscr{L}^{ss,\Lambda}) \subseteq \mathscr{L}$.*

Proof. Given Λ, by Proposition 3.8 we can choose φ such that $\mathscr{L}^{s,\Lambda} \subseteq \mathscr{L}^{ss,\Lambda} \subseteq \mathscr{L}^*_\varphi$. By virtue of Theorem 4.3, on the other hand, we can find suitable f such that $f(\mathscr{L}^*_\varphi) \subseteq \mathscr{L}$. Thus, $f(\mathscr{L}^{s,\Lambda}) \subseteq f(\mathscr{L}^{ss,\Lambda}) \subseteq f(\mathscr{L}^*_\varphi) \subseteq \mathscr{L}$. □

5. The converse problem: $f(\mathscr{L}) \cap \mathscr{L} = \emptyset$

Up to this point, we have tried to find examples of analytic functions with a large set $f(\mathscr{L}) \cap \mathscr{L}$. This suggests the following converse problem.

Problem 5.1. *Are there nonconstant analytic functions f with real coefficients such that $\mathscr{L} \cap f(\mathscr{L}) = \emptyset$?*

A negative answer can be readily inferred from a recent result on Liouville numbers [14], which is based solely on the topological property of $\mathscr{L}$ being a G_δ dense set.

Theorem 5.2 (Kumar *et al.* [14]). *Let I be a nonempty open interval of $\mathbb{R}$ and let $(f_n)_{n\geq 0}$ be a sequence of real continuous functions on I which are nowhere locally constant. Then there exists an uncountable G_δ-set $E \subseteq \mathscr{L} \cap I$ such that $f_n(E) \subseteq \mathscr{L}$ for all $n \geq 0$.*

See also [2], [10], [25], [30] and [5] (however, as pointed out in the *MathSciNet* review, the proof in [5] has a small gap and it does not work in general. See Silva [31] for a recent slightly weaker result). As a corollary, we indeed obtain the following result.

Theorem 5.3. *Let $I \subseteq \mathbb{R}$ be a nonempty open interval and $f : I \mapsto \mathbb{R}$ be a nonconstant analytic function. Then there exists an uncountable set $E \subseteq \mathscr{L} \cap I$ such that $f^{(s)}(E) \subseteq \mathscr{L}$ for all $s \geq 0$.*

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Proof. Apply Theorem 5.2 with $f_n := f^{(n)}$ for $n \geq 0$, and note that the condition clearly holds due to the identity theorem. It claims that an analytic function on a connected open set I is determined by its values on a set with limit point in I. In particular, the constant 0 function is the only analytic function that takes the value 0 on a set with limit point. Hence, any entire function which is constant on some real interval must already be constant on $\mathbb{C}$ or, equivalently, a nonconstant entire function is not constant on any real interval. □

6. The set $f(\mathscr{L}) \cap \mathscr{L}$ for the functions $f(z) = z^{a/b}$

Theorem 1.1 implies that $f(z) = z^k$ for an integer $k \neq 0$ satisfies $f(\mathscr{L}) \subseteq \mathscr{L}$. A more general class of functions one may consider is $f(z) = z^{a/b}$ for rational numbers a/b. In certain subsets of $\mathbb{C}$, there might be several representatives of f; however, we are only interested in the real representative $f : (0, \infty) \mapsto (0, \infty)$. Any such function f is algebraic even over the base field $\mathbb{Q}$ as $f(z)^b - z^a = 0$; in particular, $S_f = \overline{\mathbb{Q}}$. Further, at any $s \in (0, \infty)$ the function f admits a local power series expansion $f(z) = c_0 + c_1(z - s) + c_2(z - s)^2 + \cdots$ with radius of convergence s. Moreover, one checks that the power series expansion at a point $s \in \overline{\mathbb{Q}} \cap \mathbb{R}$ has coefficients $c_j \in \overline{\mathbb{Q}} \cap \mathbb{R}$.

The first result is an easy observation and more for the sake of completeness. It provides explicit constructions of ζ fixed under given f, as in Theorem 5.3.

Theorem 6.1. *For integer parameters $a \neq 0, b \neq 0$, let $f_{a,b}(z) = z^{a/b}$. Further, let $I \subseteq (0, \infty)$ with nonempty interior. Then there exist uncountably many $\zeta \in \mathscr{L}$ such that $f_{a,b}(\zeta) \in \mathscr{L}$ simultaneously for all a, b. Moreover, for fixed a, b, uncountably many such $\zeta \in \mathscr{L} \cap I$ can be explicitly constructed.*

Proof. The first assertion follows from Theorem 5.2 with $(f_n)_{n\geq 1}$ any enumeration of the set of functions $f_{a,b}$. Now consider a, b fixed and let $f := f_{a,b}$. Due to Theorem 1.1, we may assume that $a > 0, b > 0$. Clearly, if we take arbitrary $\zeta' \in \mathscr{L} \cap (0, \infty)$ and put $\zeta = \zeta'^b$, then Theorem 1.1 implies that $\zeta \in \mathscr{L}$ and $f(\zeta) = \zeta^{a/b} = \zeta'^a$ is in $\mathscr{L}$. Moreover, since $x \mapsto x^b$ induces a homeomorphism on $(0, \infty)$, the suitable set $\mathscr{L}^b := \{\zeta^b : \zeta \in \mathscr{L}\}$ inherits the property of being uncountable in any positive interval from the analogous property of $\mathscr{L}$. □

Now we state the main result of Section 6, which was already indicated in Section 1.5.

Theorem 6.2. *Let $f_{a,b}$ and I be as in Theorem 6.1. Then there exist uncountably many $\zeta \in \mathscr{L} \cap I$ such that $f_{a,b}(\zeta) \in \mathscr{L}$ if and only if a/b is an integer. Suitable ζ can be explicitly constructed. In particular, for any fixed coprime a, b with $|b| \geq 2$, we have $f_{a,b}(\mathscr{L}) \cap \mathscr{L}$ is uncountable but $f_{a,b}(\mathscr{L}) \not\subseteq \mathscr{L}$.*

We compare Theorem 6.2 with a result connected to U-numbers in Mahler's classification introduced in Section 1.2. Theorem 7.4 and its proof in [3] provide an explicit example of a number ζ_0 whose mth root is a U_m-number for any integer $m \geq 1$. This implies Theorem 6.2 for $a = 1$, and is in fact stronger for $b > 2$ since

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the latter only yields that $\zeta^{a/b}$ is a U_l-number for some $2 \le l \le b$. In contrast to [3, Theorem 7.4], Theorem 6.2 provides no information on approximation by algebraic irrational numbers. However, it seems that the general assertion of Theorem 6.2 cannot be deduced entirely from [3, Theorem 7.4] or related results.

Theorems 6.1 and 6.2, in view of Theorem 1.1, suggest the following conjecture.

Conjecture 6.3. Let I and $f_{a,b}$ be as in Theorem 6.1. Further, let $A \subseteq \mathbb{Q} \setminus \{0\}$ be arbitrary with the properties $1 \in A$ and for any element of A any nonzero integral multiple belongs to A as well. Then there exist uncountably many $\zeta \in \mathscr{L} \cap I$ with the property that $f_{a,b}(\zeta) \in \mathscr{L}$ if $a/b \in A$ and $f_{a,b}(\zeta) \notin \mathscr{L}$ if $a/b \notin A$.

Obviously, Theorems 6.1 and 6.2 provide the extremal cases $A = \mathbb{Q} \setminus \{0\}$ and $A = \mathbb{Z} \setminus \{0\}$. If we drop the condition $1 \in A$, then the conjecture might be true for some $\zeta \in I$ not necessarily in $\mathscr{L}$. We collect some ingredients for the proof of Theorem 6.2 in the next section.

6.1. Preparatory results. It was shown by Maillet [18] that the bth root of $\zeta \in \mathscr{L}$ is a Liouville number if and only if among the convergents of ζ there are infinitely many bth powers of rationals. We carry out his main argument for the necessity of this condition in the following more general Lemma 6.4, which in particular will allow us to establish effective bounds in Corollary 6.8.

Lemma 6.4. *Let a/b be a rational number in lowest terms. Suppose that $\zeta \in \mathscr{L}$ and $\zeta^{a/b} \in \mathscr{L}$. Then for any $\eta > 0$ the inequality*

$$|q^b \zeta^a - p^b| \le q^{-\eta} \tag{6.1}$$

has a solution in coprime integers p, q. Moreover, if $\eta > b$ is fixed and q is large, then p^b/q^b is a convergent of ζ^a.

Proof. Assume for a real number α and a positive integer k that the estimate (4.1) is satisfied. This implies (4.2) with a constant $D(k, \alpha)$ depending only on k and α. Further, observe that if

$$q^{-\nu+k-1} < \frac{1}{2D(k, \alpha)q}, \tag{6.2}$$

then Theorem 1.2 and (4.2) imply for large q that p^k/q^k is a convergent of α^k. Obviously, for fixed k, α the estimate (6.2) is satisfied for any $\nu > k$ and all large $q \ge q_0(\nu)$.

Suppose that ζ and $\zeta^{a/b}$ both belong to $\mathscr{L}$ for some suitable a, b. The above argument with $k = b$, $\alpha = \zeta^{a/b}$ shows that for arbitrarily large η the estimate (6.1) has a solution $(p, q) \in \mathbb{N}^2$ with p^b/q^b a convergent of ζ^a. □

In the proof of the more technical case $a > 1$ of Theorem 6.2, we will need the following basic result (Lemma 6.5). It can be derived by the combination of Theorem 1.2 and Proposition 4.6 in [26] (or, if one prefers, directly from Minkowksi's second lattice point theorem [22]; see also [28, Section 1]).

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Lemma 6.5. *Let $\alpha \in \mathbb{R}$. For any parameter $Q > 1$, there cannot be two linearly independent integral solution pairs (x, y) to the system*

$$|x| \le Q, \quad |\alpha x - y| < \frac{1}{2Q}.$$

Moreover, if (x, y) is a solution for some Q, then y/x must be a convergent of α.

It will be convenient to apply Dirichlet's theorem on primes in arithmetic progressions [9] to shorten the proof of Theorem 6.2, although more elementary methods would work as well. See also Remark 6.7.

Theorem 6.6 (Dirichlet [9]). *Let A, B be coprime positive integers. Then the arithmetic progression $a_n = An + B$ contains infinitely many prime numbers.*

Now we are ready for the proof of Theorem 6.2.

6.2. Proof of Theorem 6.2. As in Theorem 6.1, we may assume that $a > 0, b > 0$. If a/b is an integer and $\zeta \in \mathscr{L}$, then $f_{a,b}(\zeta) \in \mathscr{L}$ by Theorem 1.1. Thus, it suffices to construct $\zeta \in \mathscr{L}$ with $f_{a,b}(\zeta) \notin \mathscr{L}$ simultaneously for all coprime pairs a, b with $b \ge 2$. At first we drop the restriction $\zeta \in I$. Due to Lemma 6.4, it suffices to find $\zeta \in \mathscr{L}$ such that for each pair a, b we can find $\eta = \eta(a, b) > b$ such that ζ^a has no convergent of the form p^b/q^b for which (6.1) has a solution for $\eta = \eta(a, b)$, to infer $\zeta^{a/b} \notin \mathscr{L}$.

We construct such ζ. We want that the partial quotients of ζ are rapidly increasing and all denominators of convergents of ζ are prime numbers. With the notation as above, suppose that the partial denominators $r_0, r_1, \ldots, r_g$ are constructed with the property that the denominators of all convergents $s_1/t_1, \ldots, s_g/t_g$ are primes. Subsequent to (1.3), we remarked that t_{g-1}, t_g are coprime. By Theorem 6.6 and (1.3), we may choose arbitrarily large r_{g+1} such that t_{g+1} is prime. We require $r_{g+1} \ge t_g^g$ and, for technical reasons, the sequence r_n should moreover grow fast enough that if ν_n is defined by $|\zeta t_n - s_n| = t_n^{-\nu_n}$, then $\nu_{n+1} > \nu_n$ in any step. By Theorem 1.3, obviously $\lim_{n\to\infty} \nu_n = \infty$, such that this procedure indeed leads to $\zeta \in \mathscr{L}$. We have to show that ζ has the requested property. Throughout the remainder of the proof let $\delta > 0$ be arbitrarily small but fixed.

First let $a = 1$. In this case it suffices to put $\eta(1, b) = b + \delta$ and observe that by construction all convergents of $\zeta^a = \zeta$ have prime denominators and hence no convergent is of the form p^b/q^b for $b \ge 2$.

Now let $a \ge 2$. We show that the inequality

$$|x\zeta^a - y| \le x^{-a-\delta} \tag{6.3}$$

can hold for $(x, y) \in \mathbb{N}^2$ with large x only in case of (x, y) an integral multiple of some (q'^a, p'^a), where p'^a/q'^a is a convergent of ζ^a in lowest terms. More precisely, $(p', q') = (s_n, t_n)$ for some n, with s_n, t_n as above. Assume that this is true. Let $\eta = \eta(a, b) = \max\{a + \delta, b + \delta\}$. Assume for this choice of η that there exist solutions of (6.1) that must be convergents of ζ^a of the form p^b/q^b by Lemma 6.4. On the other hand, by the above observation and the choice of η, these solutions must at the

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same time have a representation as a quotient of ath powers of integers p'^a/q'^a. Since a, b are coprime and $q' = t_n$ is a prime number, this is clearly impossible, giving a contradiction. This yields again an indirect proof of $\zeta^{a/b} \notin \mathscr{L}$.

It remains to check the assertion above. We have to check that for $(x, y) \in \mathbb{N}^2$ with large x and linearly independent of any (s_n^a, t_n^a), we cannot have (6.3). Consider large x fixed and let N be the index such that $t_N \le x < t_{N+1}$. Recall that all s_n/t_n are very good approximations to ζ. By construction of ζ and definition of ν_n, in particular we have $|\zeta t_{n+1} - s_{n+1}| < t_{n+1}^{-\nu_n}$. Then, similar to (4.2), we can write

$$|t_N^a \zeta^a - s_N^a| = |t_N \zeta - s_N| \cdot |t_N^{a-1}\zeta^{a-1} + \cdots + s_N^{a-1}| \le D(a, \zeta) t_N^{-\nu_N + a - 1}, \tag{6.4}$$
$$|t_{N+1}^a \zeta^a - s_{N+1}^a| = |t_{N+1} \zeta - s_{N+1}| \cdot |t_{N+1}^{a-1}\zeta^{a-1} + \cdots + s_{N+1}^{a-1}| \le D(a, \zeta) t_{N+1}^{-\nu_N + a - 1}.$$

Moreover, $t_{N+1} \asymp t_N^{\nu_N}$ in view of (1.3) and Theorem 1.3. We distinguish two cases.

Case 1: $t_N \le x < t_N^a$. We apply Lemma 6.5, with $Q := t_N^a$. Since (s_N^a, t_N^a) leads to a good approximation for ζ^a by (6.4), there cannot be another vector $(u, v) \in \mathbb{N}^2$ linearly independent of (s_N^a, t_N^a) with $u < t_N^a$ that leads to a good approximation. As the condition $x < t_N^a$ is satisfied by assumption, Lemma 6.5 more precisely yields that $|\zeta^a x - y| > (1/2) t_N^{-a}$. Since $t_N \le x$, for large x (or N) we conclude that

$$|\zeta^a x - y| > (1/2) t_N^{-a} \ge (1/2) x^{-a} > x^{-a-\delta},$$

which is indeed a contradiction to (6.3).

Case 2: $t_N^a \le x < t_{N+1}$. First assume that x is close to t_{N+1}; more precisely, $t_{N+1}^{1-\epsilon} \le x < t_{N+1}$ for $\epsilon \in (0, \delta/(a+\delta))$. Then we may use the same argument as in Case 1 with $Q = t_{N+1}^a$ instead of $Q = t_N^a$, since $|\zeta^a x - y| > (1/2) t_{N+1}^{-a} > x^{-a-\delta}$ is still valid. So, we may assume that $t_N^a \le x < t_{N+1}^{1-\epsilon}$. In this case we apply Lemma 6.5 with $Q := x$. Assume that (6.3) holds. Then (x, y) is a pair with $|\zeta^a x - y| < (1/2) Q^{-1}$, so by Lemma 6.5 there cannot be another such pair linearly independent of (x, y). However, we show that (s_N^a, t_N^a) satisfies the inequality as well. Recall that $t_{N+1} \asymp t_N^{\nu_N}$ such that $Q = x < t_{N+1}^{1-\epsilon}$ yields $Q^{1/[(1-\epsilon)\nu_N]} \ll t_N$. By (6.4), we infer that

$$|t_N^a \zeta^a - s_N^a| \ll t_N^{-\nu_N + a - 1} \ll Q^{(-\nu_N + a - 1)/((1-\epsilon)\nu_N)} \ll Q^{-(1/(1-\epsilon))}$$

for large N as ν_N is then large too. Since $1/(1-\epsilon) > 1$, the right-hand side is indeed smaller than $(1/2) Q^{-1}$ for large $x = Q$ and the contradiction again shows that (6.3) is false.

Finally, we may allow the continued fraction expansion of ζ to start with arbitrary $[r_0; r_1, r_2, \ldots, r_l]$ and then start the above procedure. Hence, the method is flexible enough to guarantee uncountably many suitable ζ in any subinterval of $(0, \infty)$. This completes the proof. □

Remark 6.7. The constructed $\zeta \in \mathscr{L}$ in the proof of Theorem 6.2 are strong Liouville numbers; see Definition 3.7. Indeed, the method of the proof for $a \ge 2$ with Lemma 6.5 requires that there are no large gaps between denominators of convergents with very

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good approximation to ζ. Conversely, the proof basically does work for any semi-strong Liouville number for which no convergent is of the form p^b/q^b for some $b \geq 2$. Recall that for $a = 1$, it was already shown by Maillet that a sufficient condition is that no convergent p/q of ζ is of the form p^b/q^b for $b \geq 2$, which is rather easy to construct.

The proof of Theorem 6.2 provides explicit upper bounds for the irrationality exponent of $\zeta^{a/b}$ for the involved $\zeta \in \mathscr{L}$.

Corollary 6.8. *Let $f_{a,b}(z)$ be as in Theorem 6.1 and $\zeta \in \mathscr{L}$ be constructed as in the proof of Theorem 6.2. Then $f_{a,b}(\zeta) \in \mathscr{L}$ for a/b an integer but $\mu(f_{a,b}(\zeta)) \leq \max\{|a|, |b|\} + |b|$ simultaneously for all a, b for which a/b is not an integer.*

Proof. If a/b is an integer, then the assertion follows from Theorem 1.1, as already observed in Theorem 6.2. Thus, and since $\mu(\alpha^{-1}) = \mu(\alpha)$, we can restrict to $a > 0, b > 0$ and a/b not an integer. Let $a = 1$. Indeed, the fact that (6.1) has no (large) solution for $\eta = b + \delta$ implies that (4.1) has no (large) solution for $\nu = (b + \delta) + (b - 1) = 2b - 1 + \delta$. With $\delta \to 0$ and adding 1, taking into account the transition from linear forms to fractions, we obtain the bound. The same argument can be applied for $a \geq 2$ with $\eta(a, b) = \max\{a + \delta, b + \delta\}$. □

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