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Geometry of the Katok examples

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Abstract. We consider examples of Finsler metrics symmetric or not) on S^n, $P^n\mathbb{C}$, $P^n\mathbb{H}$, and P^2Ca with only finitely many closed geodesics or with only few short closed geodesics. The number of closed geodesics in these examples and properties of the closed geodesics are considered.

0. *Introduction*

It is an old problem in variational calculus to estimate the number of closed extremals for a one-dimensional variational problem on a compact manifold. In order to apply the usual methods of calculus of variations, the variational problem has to be positive and positive regular. This reduces the problem to the existence of closed geodesics for a Finsler metric on a compact manifold. By a Finsler metric we mean a norm on each tangent space, possibly not symmetric, such that the unit sphere in each tangent space is a strictly convex submanifold which depends differentiably on the footpoint. One calls the Finsler metric symmetric (or reversible) if the length of v and $-v$ is the same and non-symmetric (or non-reversible) otherwise.

It was quite surprising when Katok [**12**] in 1973 found some non-symmetric Finsler metrics on S^n with only finitely many closed geodesics. It is therefore of some interest to examine these examples more closely, which will be done in this paper. The examples are simple enough so that one can easily compute all invariants of the closed geodesics and make a number of interesting observations.

It turns out that these examples exist in any neighbourhood of the standard metrics on S^n, P^nC, P^nH, and P^2Ca and that all closed geodesics are non-degenerate and elliptic. The smallest number of closed geodesics that one obtains in these examples is $2n$ on S^{2n} and S^{2n-1}, $n(n+1)$ on P^nC, $2n(n+1)$ on P^nH, and 24 on P^2Ca.

For these spaces we can show that any Finsler metric sufficiently close to the standard metric and with all closed geodesics non-degenerate has at least as many closed geodesics as in the examples. In addition to the Poincaré map, one can also compute the lengths and the Morse indices of all closed geodesics and one obtains some remarkable pictures of what the Morse theory on the free loop space $C^\infty(S^1, M)$ has to look like in order to build up the homology of this space with only finitely many closed geodesics. For $M = S^2$ for example, one can build up the homology below any prescribed dimension with only one closed geodesic. On the other hand these examples are rather exceptional, since we will show that for a generic Finsler metric on a compact manifold, the initial vectors to closed geodesics are dense in the unit tangent bundle.

The same construction also shows that there exist Finsler metrics on S^n, P^nC, P^nH, and P^2Ca with all geodesics closed, but not of the same least period. On the other hand one can show that for any Finsler metric with all geodesics closed there exists a common period for the closed geodesics, and for any symmetric Finsler metric on S^2 with all geodesics closed, the closed geodesics have to have the same least period.

We will also generalize the construction of Katok to find some other interesting examples. The Lusternik–Schnirelmann theory implies that, if one considers all Finsler metrics close to the standard metrics on $M = S^n, P^nC, P^nH$, or P^2Ca (instead of only those which have only non-degenerate closed geodesics), then any such metric has at least $\dim M$ closed geodesics with lengths close to 2π. We will construct examples on these spaces of Finsler metrics with only $\dim M$ closed geodesics of lengths close to 2π and with the length of all other closed geodesics larger than any prescribed number.

If one considers only symmetric Finsler metrics on S^n which are close to the standard metric, then Lusternik–Schnirelmann theory implies the existence of at least $g(n) = 2n - s - 1$ closed geodesics with lengths close to 2π, where

$$n = 2^k + s < 2^{k+1}.$$

Notice that

$$(3n-1)/2 \leq g(n) \leq 2n - 1.$$

We will show that there exist symmetric Finsler metrics on S^n with only $2n-1$ closed geodesics of lengths close to 2π and with the length of all other closed geodesics larger than any prescribed number. Using an unpublished example of J. Milnor of a function on the Grassmannian of unoriented two planes in four space with only four critical points, we will also construct Finsler metrics on S^3 with only $g(3) = 4$ closed geodesics of lengths close to 2π and with the length of all other closed geodesics larger than any prescribed number. These examples on S^3 seem to be a counter-example to one of the main theorems in [**13**], namely theorem 5.1.1, which claims that any metric on S^n has at least $2n - 1$ short closed geodesics.

In § 1 we explain the Katok examples, compute the smallest number of closed geodesics that one can obtain, and explain the topological significance of these numbers. In § 2 we derive the geometric properties of the Katok examples and in § 3 we give examples of symmetric and non-symmetric Finsler metrics with few short closed geodesics.

1. *Examples with finitely many closed geodesics*

Let M be a differentiable manifold. A Finsler metric on M is a norm on each tangent space, possibly not symmetric, such that the unit sphere in each tangent space T_qM is a strictly convex submanifold which depends differentiably on q. Alternatively a Finsler metric is a function N: $TM \to R$, differentiable off the zero-section, such that $D_F^2(N^2)$ is positive definite and such that

$$N(\lambda x) = \lambda N(x) \quad \text{for all } \lambda > 0 \text{ and } x \in T_qM.$$

Here D_F^2 denotes the second derivative in the fibre direction (see [**22**], chapter 1, for details). N is called symmetric if $N(-x)=N(x)$. If N is not symmetric and $c(t)$ is a closed geodesic, then $c(-t)$ need not be a closed geodesic any more. If $c(-t)$, or a reparametrization of it, happens to be a closed geodesic, as will be the case in the Katok examples, it will therefore be counted as a second closed geodesic. If N is symmetric they will be counted as only one.

Finsler metrics are sometimes conveniently described on the cotangent bundle. We consider T^*M as a symplectic manifold with canonical symplectic form ω. Any function $H: T^*M \to R$ gives rise to a Hamiltonian vector field X_H defined by

$$dH(y)=\omega(X_H, y) \quad \text{for all } y \in T(T^*M).$$

If $D_F^2 H$ is positive definite, the Legendre transform

$$L_H = D_F H : T^*M \to TM$$

is a local diffeomorphism. If H is homogeneous of degree two,

$$H(\lambda x)=\lambda^2 H(x) \quad \text{for } \lambda>0,$$

then L_H is a global diffeomorphism and

$$N^2 = H \circ L_H^{-1}$$

is a Finsler metric on M. X_H describes the geodesics of the Finsler metric since the projection of the integral curves of X_H under $\pi: T^*M \to M$ are the geodesics of N. If $2N^2$ is a Riemannian metric, $2H$ is the dual metric and L_H is the canonical identification between TM and T^*M.

To obtain the examples of Katok one starts with a Riemannian metric g on M with all geodesics closed and which admits a one-parameter group of isometries. By a theorem of Wadsley [**5**, p. 182], the closed geodesics of g have a common period r and we normalize the metric such that $r=2\pi$. After changing the one-parameter subgroup ϕ_t if necessary we can also assume $\phi_{2\pi}=\text{id}$. (Since the isometry group of g is a compact Lie group, it contains a closed one-parameter subgroup once it contains a non-trivial one-parameter subgroup.) Let V be the vector field generated by ϕ_t. Define $H_0, H_1: T^*M \to R$ by

$$H_0(x)=\|x\|_* \quad \text{and} \quad H_1(x)=x(V)$$

where $\| \ \|_*$ is the dual norm of g. Let

$$H_\alpha = H_0 + \alpha H_1.$$

H_α is differentiable off the zero-section and homogeneous of degree one. For α small $D_F^2(H_\alpha^2)$ is positive definite since this is true for $\alpha=0$ and hence

$$N_\alpha = H_\alpha \circ L_{\frac{1}{2}H_\alpha^2}^{-1}$$

defines a Finsler metric on M. Since $H_\alpha(-x)\neq H_\alpha(x)$, N_α is not symmetric.

We will now examine the geodesic flow of N_α. The Hamiltonian vector field X_{H_0} is not quite the geodesic flow of g (under the canonical identification of TM and T^*M via g) since H_0 is homogeneous of degree one. But X_{H_0} and $X_{\frac{1}{2}H_0^2}$ are proportional:

$$X_{\frac{1}{2}H_0^2} = H_0^{-1} X_{H_0}$$

and thus their integral curves are reparametrizations of each other. If $c(t)$ is a geodesic and $(c(t), Y(c(t)))$ an integral curve of $X_{\frac{1}{2}H_0^2}$ in T^*M and if $c_0(s)$ is the geodesic c parametrized by arc length, then $(c_0(s), Y(c_0(s)))$ is an integral curve of X_{H_0}. Thus if $\psi_t^{H_0}$ is the flow of X_{H_0} we have

$$\psi_{2\pi}^{H_0} = \mathrm{id}$$

on all of T^*M. It is well known that the flow of X_{H_1} is

$$\psi_t^{H_1} = D^*\phi_t.$$

To see this one can verify directly that

$$DH_1(y) = \omega\left(y, \left.\frac{d}{dt}\right|_{t=0} (D^*\phi_t)\right)$$

for a basis $y = \partial/\partial q_i$, $\partial/\partial p_i$ of $T(T^*M)$. The flow of X_{H_0} and X_{H_1} commute since ϕ_t are isometries of g. Thus the flow of X_{H_α} is

$$\psi_t^{H_\alpha} = \psi_t^{H_0} \circ \psi_{\alpha t}^{H_1}.$$

The geodesic flow of N_α, i.e. the flow of $X_{\frac{1}{2}H_\alpha^2}$, again differs from $\psi_t^{H_\alpha}$ only by reparametrization: if $(c(t), Y(c(t)))$ is an integral curve of $X_{\frac{1}{2}H_\alpha^2}$ then $(c_0(s), Y(c_0(s)))$ is an integral curve of X_{H_α} where c_0 is a reparametrization of c such that

$$N_\alpha(\dot{c}_0(s)) = 1.$$

To examine the closed geodesics of N_α let $x \in T^*M$ be such that

$$\psi_T^{H_\alpha} x = x$$

where T is the length of the closed geodesic in the N_α metric. Since then

$$\psi_{-T}^{H_0} x = \psi_{\alpha T}^{H_1} x$$

and since the flow of H_0 and H_1 commute, this implies that $\psi_{\alpha T}^{H_1}$ leaves the orbit

$$c(t) = \psi_{-t}^{H_0} x$$

invariant. Hence $\psi_{\alpha n T}^{H_1}$ leaves c invariant for $n = 1, 2, \ldots$. If $\alpha T/2\pi$ is irrational, $\alpha n T/2\pi$ is dense in R/Z and since

$$\psi_{2\pi}^{H_1} = \mathrm{id},$$

c is then invariant under the whole one-parameter group $\psi_t^{H_1}$. If $\alpha T = 2\pi m/n$, then

$$x = \psi_{2\pi m}^{H_1} x = \psi_{\alpha n T}^{H_1} x = \psi_{-nT}^{H_0} x$$

which implies that $T/2\pi$ and hence α must be a rational number. Since the base point curves of $\psi_t^{H_0}$ and $\psi_t^{H_\alpha}$ are, up to parametrization, the geodesics of g and N_α respectively, we obtain:

If α is irrational, then the closed geodesics of N_α are, up to parametrization, the closed geodesics of g which are invariant under the one-parameter group ϕ_t.

The simplest example is $M = S^2$ with its standard metric of constant curvature 1 and ϕ_t the one-parameter group of rotations leaving the north and south poles fixed. The equator traversed in each direction is the only great circle invariant under ϕ_t and hence N_α is a Finsler metric on S^2, which for α small and irrational, has only two closed geodesics. To visualize the geodesic flow of N_α we can identify T^*M and TM via g in which case the flow $\psi_t^{H_\alpha}$ carries over into the flow $G_t \circ D\phi_{\alpha t}$ where G_t is the geodesic flow of g. The base point curves of $G_t \circ D\phi_{\alpha t}$ are then the geodesics of N_α parametrized according to arc length. One can view this as observing the geodesic flow on S^2 with constant curvature from a coordinate system which is rotating at constant irrational speed.

The constant curvature 1 metric on S^n and the standard metrics on P^nC, P^nH, and P^2Ca with curvature between 1/4 and 1 have all geodesics closed of length 2π and we will now show:

There exist Finsler metrics on S^{2n} and S^{2n-1} with only $2n$ closed geodesics and on P^nC, P^nH, and P^2Ca with only $n(n+1)$, $2n(n+1)$, and 24 closed geodesics respectively. These Finsler metrics exist in any neighbourhood of the standard Riemannian metrics.

For S^{2n-1} any closed one-parameter group of isometries is conjugate to a diagonal matrix

$$\phi_t = \operatorname{diag}(R(pt/p_1), \ldots, R(pt/p_n))$$

where $p_i \in Z$, $p = p_1 \cdots p_n$ and $R(\omega)$ is a rotation in R^2 with angle ω. For S^{2n} the same is true if the matrix is enlarged by one row and one column with a 1 in the diagonal. The closed geodesics on S^n invariant under ϕ_t are given by the intersection of two-dimensional invariant planes in R^{n+1} with S^n. If the p_i are relatively prime only the two-dimensional planes corresponding to the planes of rotation are invariant under ϕ_t. Each gives rise to two closed geodesics and hence N_α has $2n$ closed geodesics on S^{2n} and S^{2n-1} if α is irrational. By different choices of the p_i's one obtains Finsler metrics on S^n where the number of closed geodesics is any even number between n and $n(n+1)/2$.

For P^nC the isometry group is $\mathrm{SU}(n+1)/Z_{n+1}$ which can be realized as follows: If we take the Hopf fibration $S^1 \to S^{2n+1} \to P^nC$ then $\mathrm{U}(n+1)$ acts on $S^{2n+1} \subset C^{n+1}$ and only the elements in the centre of $\mathrm{U}(n+1)$ induce trivial maps on P^nC. Every closed one-parameter subgroup in $\mathrm{U}(n+1)$ is conjugate to

$$\phi_t = \operatorname{diag}(e^{ipt/p_1}, \ldots, e^{ipt/p_{n+1}})$$

for some p_i and $p = p_1 \cdots p_{n+1}$. Any closed geodesic on P^nC lifts to a closed geodesic on S^{2n+1} which is orthogonal to the fibres and vice versa. Thus the lift to S^{2n+1} of a closed geodesic in P^nC invariant under ϕ_t lies in a two-dimensional complex subspace of C^{n+1} invariant under ϕ_t. If the p_i are relatively prime these consist of two-dimensional planes spanned by any two coordinate vectors. This reduces the situation to $\mathrm{U}(2)$ acting on $S^1 \to S^3 \to P^1C = S^2$. But we already know that on S^2 there are only two closed geodesics invariant under a one-parameter group. One easily sees that the closed geodesics coming from distinct planes are distinct. Hence N_α has $n(n+1)$ closed geodesics if α is irrational. A similar argument works for P^nH using the Hopf fibration $S^3 \to S^{4n+3} \to P^nH$. For P^2Ca there exists no such fibration, but one can also argue intrinsically by observing that any closed geodesic on P^2Ca lies in a unique 8-dimensional totally geodesic sphere of constant curvature 1 which then has to be left invariant under ϕ_t. A general one-parameter group of isometries leaves only three such spheres invariant and on each there exist 8 invariant closed geodesics by the previous discussion.

There are other Riemannian metrics satisfying the hypothesis needed for the construction of such Finsler metrics. On S^n one has, for example, the generalizations of Zoll surfaces [5, p. 120], but there also exist metrics on S^n with all geodesics closed and admitting no isometries [5, p. 126]. Nevertheless one does not obtain any examples with less closed geodesics than in the ones above. Notice also that by a theorem of Bott and Samelson [5, p. 186], the only manifolds admitting Riemannian metrics with all geodesics closed of the same least period are either diffeomorphic to P^nR or have the same integral cohomology ring as S^n, P^nC, P^nH, P^2Ca. The same theorem is probably true if one does not assume that the closed geodesics have the same least period, see e.g. [5, p. 192]. Hence spheres and projective spaces are the only manifolds on which one obtains examples as above of Finsler metrics with only finitely many closed geodesics. This is complemented by a theorem of Gromoll–Meyer [10], which has been generalized to Finsler metrics [14], and states that if the Betti numbers $b_i(\Lambda(M), K)$ of the free loop space $\Lambda(M) = C^0(S^1, M)$ are unbounded as $i \to \infty$ for some field K, then any Finsler metric on M (symmetric or not) has infinitely many closed geodesics. The only known simply connected examples of manifolds where $b_i(\Lambda(M), K)$ is bounded for every field K are again the spheres and projective spaces.

Notice that the assumption that H_0 comes from a Riemannian metric was not essential in the construction of N_α. More generally if H_0 and H_1 are two commuting Hamiltonians such that all orbits of X_{H_0} are closed and non-trivial with a common period and such that the flow of X_{H_1} induces an S^1 action, then the periodic orbits of

$$H = H_0 + \alpha H_1$$

are, for α irrational, the periodic orbits of H_0 invariant under the flow of H_1. See [8] for an application of this idea to the problem of the existence of brake orbits.

We will now show that the number of closed geodesics in the examples on spheres and projective spaces have some topological significance and are optimal in a certain sense.

Let M be a manifold with a Riemannian metric such that all geodesics are closed of the same least period 2π. Then one has an S^1 fibration $S^1 \to T_1M \to C$ where S^1 acts on T_1M by linear reparametrization of the closed geodesics. C is a compact manifold and from [25] one knows that the Euler class $e \in H^2(C)$ of the S^1 fibration $T_1M \to C$ satisfies $e^{n-1} = [C] \neq 0$. Perturbation methods as in [27] or Lusternik–Schnirelmann theory (respectively Morse theory) on the free loop space show that any Finsler metric N on M sufficiently close to g has at least as many closed geodesics of length approximately 2π as a function on C has critical points. If all closed geodesics of N with lengths close to 2π are non-degenerate (which is satisfied for an open and dense set of Finsler metrics N) then N has at least $\sum_{i=0}^{i=\dim C} b_i(C)$ closed geodesics of length approximately 2π.

The number of critical points of a function on C can be estimated from below by the category of C, which in turn can be estimated by the cup length of C. The cup length is the largest number of (not necessarily distinct) cohomology classes whose cup product is non-zero, where we are allowed to count $1 \in H^0(C)$ once.

Since $e^{n-1} \neq 0$ the cup length of C is at least $\dim M$. Hence any function on C has at least $\dim M$ critical points. To compute $\sum b_i(C)$ one can first use the Gysin sequence of $S^{n-1} \to T_1M \to M$ to compute the cohomology of T_1M and then the Gysin sequence of $S^1 \to T_1M \to C$ to compute the Betti numbers of C. Since the cohomology ring of M is isomorphic to the cohomology ring of a sphere or a projective space if M is simply connected, we can restrict ourselves to these spaces and obtain:

M	non-zero $b_i(C, Q)$
S^{2n}	$b_i = 1$, i even
S^{2n+1}	$b_i = 1$, i even except $b_{2n} = 2$
P^nC	$b_0 = 1, b_2 = 2, b_4 = 3, \ldots, b_{2n-2} = n = b_{2n}, b_{2n+2} = n-1, \ldots, b_{4n-2} = 1$
P^nH	$b_0 = b_2 = 1, b_4 = b_6 = 2, \ldots, b_{4n-4} = b_{4n-2} = n = b_{4n} = b_{4n+2}, \ldots, b_{8n-2} = 1$
P^2Ca	$b_0 = b_2 = b_4 = b_6 = 1, b_8 = b_{10} = \cdots = b_{22} = 2, b_{24} = b_{26} = \cdots = b_{30} = 1$

Thus we obtain for $\sum b_i(C)$: $2n$ for S^{2n} and S^{2n-1}; $n(n+1)$ for P^nC; $2n(n+1)$ for P^nH; and 24 for P^2Ca.

Any Finsler metric on $M = S^n, P^nC, P^nH, P^2Ca$ sufficiently C^2 close to the standard metric has at least $\dim M$ *closed geodesics of length approximately* 2π. *If all closed geodesics of length close to* 2π *are non-degenerate, then the Finsler metric has at least* $2n$ *such closed geodesics on* S^{2n} *or* S^{2n-1}, $n(n+1)$ *on* P^nC, $2n(n+1)$ *on* P^nH, *and* 24 *on* P^2Ca.

Thus among all Finsler metrics (C^2 close to the standard metric) only the examples N_α on S^{2n} are optimal, but all examples are optimal if one considers only Finsler metrics with non-degenerate closed geodesics. As we will see later, for the examples N_α with α irrational, the closed geodesics are always non-degenerate.

These examples of Finsler metrics with only finitely many closed geodesics are rather exceptional, as the following result shows:

Among the C^2 Finsler metrics on a compact manifold, it is a generic behaviour in the C^2 topology, that the initial vectors to closed geodesics are dense in the unit tangent bundle.

For Hamiltonian systems this is a well known theorem, if the unit tangent bundle is replaced by any compact energy surface, see [**20**] and [**21**]. The statement for Finsler metrics follows easily from the one for Hamiltonian systems as we will now see. Let $H: T^*M \to R$ be a Hamiltonian such that the energy surface $H = c$ intersected with each cotangent space T_q^*M is a compact strictly convex hypersurface containing the origin of T_q^*M in its interior. The set of such Hamiltonians is an open set in the set of all Hamiltonians on T^*M. For each such Hamiltonian we can define a function F on T^*M which is homogeneous of degree two in the fibre direction and such that $F = c$ agrees with $H = c$. F is then a Finsler metric on M and the geodesics of F on $F = c$ are just reparametrizations of orbits of H on $H = c$, see e.g. [**26**]. Now the closing lemma for Finsler metrics follows directly from the closing lemma for Hamiltonian systems. Similarly a Finsler metric with a

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closed geodesic can be perturbed so that the closed geodesic becomes non-degenerate (under the homogenizing procedure the Poincaré map stays the same). But these two facts are sufficient to prove the above generic statement just as in the Hamiltonian case, see [**21**, pp. 591–592].

It is not known if this generic behaviour is also true for C^k Finsler metrics (or C^k Hamiltonian) in the C^k topology, $2<k\leq\infty$. It is also not known if the above generic statement is true for symmetric Finsler metrics or Riemannian metrics, even in the C^2 topology.

Let us finally remark that the purpose of Katok's paper [**12**] was not to construct such Finsler metrics N_α. Katok wanted to construct Riemannian metrics or Finsler metrcs on S^n whose geodesic flow is ergodic. He showed that one can perturb the Finsler metrics N_α to obtain Finsler metrics N'_α such that the geodesic flow of N'_α is ergodic and such that N_α and N'_α have the same closed geodesics and N_α and N'_α and all of their higher derivatives agree along these closed geodesics. Hence one also obtains ergodic Finsler metrics with the same number of closed geodesics and all of the geometric properties in § 2 remain true for N'_α.

It seems that the proof in [**12**] of the existence of only finitely many closed geodesics for N_α is different from the one in this section. See also [**1**] and [**14**] for a discussion of the Katok examples, but the number of closed geodesics in [**14**] has been determined incorrectly.

2. *Some geometric properties*

We will now examine some of the geometric properties of the Finsler metrics

$$H_\alpha = H_0 + \alpha H_1$$

on $M = S^n$, P^nC, P^nH, P^2Ca where H_0 comes from the standard metric on these spaces. In most cases we carry out the computation only for $M = S^2$, the other cases being similar. Let us first determine the lengths of the closed geodesics. From § 1 it is clear that the length agrees with the period of the periodic orbits of X_{H_α}. Let

$$c(t) = \psi_t^{H_0} x$$

be a closed geodesic of H_0 (in T^*M) invariant under ϕ_t. For $M = S^2$ it is clear that

$$\psi_s^{H_1} c(t) = c(t \pm s)$$

and hence

$$\psi_t^{H_0} \circ \psi_{\alpha t}^{H_1} x = c(t \pm \alpha t).$$

Therefore the length of $c(t)$ under N_α is equal to $2\pi/(1\pm\alpha)$.

The length of the closed geodesics of N_α on S^2 is equal to $2\pi/(1+\alpha)$ and $2\pi/(1-\alpha)$ respectively depending on whether the closed geodesic is traversed in or opposite to the direction of the rotation.

For the metrics N_α on S^n, if ϕ_t is as in § 1, the lengths of the closed geodesics turn out to be $2\pi/(1+p\alpha/p_i)$ and $2\pi/(1-p\alpha/p_i)$ respectively and similarly for the other manifolds.

We will now examine for which α H_α is still a Finsler metric, i.e. for which α $L_{\frac{1}{2}H_\alpha^2}$ is still a diffeomorphism. Let $x \in T^*M$ and let $*: T^*M \to TM$ be the usual identification between T^*M and TM induced by the Riemannian metric H_0, and $\| \ \|$ the norm of H_0 on TM. Then

$$D_F H_0(x) = *x/\|*x\| \quad \text{and} \quad D_F H_1(x) = V$$

and hence

$$D_F H_\alpha(x) = *x/\|*x\| + \alpha V.$$

Thus

$$D_F(\tfrac{1}{2}H_\alpha^2)(x) = H_\alpha(x) \cdot D_F H_\alpha(x)$$
$$= (\|*x\| + \alpha\langle V, *x\rangle)(*x/\|*x\| + \alpha V).$$

Since $\|V\| \le 1$ it follows easily that $D_F(\frac{1}{2}H_\alpha^2)$ is a diffeomorphism iff $|\alpha| < 1$.

N_α is a Finsler metric as long as $|\alpha| < 1$. As $\alpha \to 1$ the lengths of the closed geodesics on S^2 go to π and ∞ respectively.

Let us remark that the examples N_α are also very interesting when α is rational. Let $\alpha = k/m$ where k and m are relatively prime. The closed geodesics invariant under ϕ_t then have least period

$$2\pi/(1+\alpha) = 2\pi m/(k+m)$$

and

$$2\pi/(1-\alpha) = 2\pi m/(m-k)$$

respectively. But all other closed geodesics have least period $2\pi m$. Thus

There exist Finsler metrics on S^n, P^nC, P^nH, P^2Ca with all geodesics closed but not of the same least period.

If $\alpha = k/(k+1)$ for example then, after a renormalization, all geodesics are closed of length 2π except for one which has length $2\pi/(2k+1)$. This is remarkable in view of the following two facts.

For any Hamiltonian H on T^*M, homogeneous of degree one and $H(x) > 0$ for $x \neq 0$, and such that all orbits of X_H are closed, the orbits have a common period.

This follows by combining the result of Helton [**11**] with the proof of the theorem of Wadsley [**5**, p. 182], and the result of Moser [**19**, p. 622]. In particular this is always true for Finsler metrics. For Riemannian metrics there is a conjecture that if a metric on a simply connected manifold has all geodesics closed, then the closed geodesics have to have the same least period. Recently this was shown to be true for $M = S^2$ [**9**]. Their proof carries over to symmetric Finsler metrics and hence:

For any symmetric Finsler metric on S^2 such that all geodesics are closed, all geodesics have to have the same least period and no self-intersections.

We will now compute the Poincaré map of the closed geodesics. For a general Hamiltonian H with flow ψ_t, let $c(t)$ be a closed orbit of period T, $c(0) = c(T) = x$, and energy 1 i.e. $H(x)=1$. Choose a section Σ in $H^{-1}(1)$ transversal to c at x. Then there exist smaller neighbourhoods Σ_0 and Σ_1 of x in Σ and a function $\alpha:\Sigma_0\to R$ such that

$$\mathcal{P}(y)=\psi_{\alpha(y)}(y):\Sigma_0\to\Sigma_1$$

is a diffeomorphism which is symplectic with respect to ω/Σ. $\mathcal{P}$ is called the Poincaré map of c. $P=D_x\mathcal{P}$ is called the linearized Poincaré map of c. In general P differs from $D_x\psi_T$ since $T_x\Sigma$ is not necessarily invariant under $D_x\psi_T$. But if H is homogeneous, say of degree one, then one can find a Σ such that $T_x\Sigma$ is invariant. From the homogeneity it follows that

$$\psi_t(\lambda y)=\lambda\psi_{\lambda t}(y)$$

and differentiating with respect to λ and setting $\lambda=1$, $t=T$, $y=x$ we obtain

$$D_x\psi_T(\tilde{x})=T\cdot\dot{c}(0)+\tilde{x}$$

where $\tilde{x}$ is equal to x considered as a vertical vector at x. But we also have

$$D_x\psi_T(\dot{c}\,0))=\dot{c}(0).$$

Thus $\tilde{x}$ and $\dot{c}(0)$ form an invariant subspace which is non-degenerate with respect to ω since

$$\omega(\dot{c}(0),\tilde{x})=DH(\tilde{x})=H(x)=1.$$

Since $D_x\psi_T$ is symplectic, the ω orthogonal ccomplement E of this subspace is a non-degenerate invariant subspace. Choose Σ such that $T_x\,\Sigma=E$. Then

$$P=D_x\psi_T/E.$$

The linearized flow $D_x\psi_t$ can be described by choosing a variation of solution curves $c_s(t)$, $c_0=c$, and the variation vector

$$Y(t)=\left.\frac{\partial c_s}{\partial s}\right|_{s=0}$$

is a solution of the linearized flow:

$$D_x\psi_t(Y(0))=Y(t)$$

and hence $P(Y(0))=Y(T)$. To compute $Y(t)$ we choose a coordinate system q_i on M such that $q_2=\cdots=q_n=0$ is $c(t)$, $t=q_1$, and q_1 is periodic, $q_1(t+T)=q_1(t)$ (possible if c is orientable). Let (q_i,p_i) be the induced coordinate system on T^*M. Since c_s is a solution of the Hamiltonian equations $\dot{q}_i=H_{p_i}$, $\dot{p}_i=-H_{q_i}$ we obtain for the vector field

$$Y(t)=(\xi_i(t),\eta_i(t))$$

the linear differential equation:

$$\dot{\xi}_i=H_{p_jp_i}\cdot\eta_j+H_{q_jp_i}\cdot\xi_j$$
$$\dot{\eta}_i=-H_{p_jq_i}\cdot\eta_j-H_{q_jq_i}\cdot\xi_j$$

where derivatives are evaluated along $c(t)$.

We apply this to

$$H=\tfrac{1}{2}H_\alpha^2$$

since the flow of X_H is then the geodesic flow of N_α. We carry out the calculation for $M=S^2$. Let (q_1, q_2) be as above such that $\partial/\partial q_1 = V$ and $\partial/\partial q_2$ orthogonal to V and of unit length with respect to H_0. Then for the geodesic of length $2\pi/(1+\alpha)$ we have $H_1(q_i, p_i) = p_1$ and

$$H = \tfrac{1}{2}H_\alpha^2 = \tfrac{1}{2}((g^{ij}(q_1, q_2)p_ip_j)^{1/2} + \alpha p_1)^2$$
$$= \tfrac{1}{2}(((\cos^2 q_2)p_1^2 + p_2^2)^{\frac{1}{2}} + \alpha p_1)^2.$$

The periodic orbit $c(t)$ of $X_{\frac{1}{2}H_\alpha^2}$ with base point curve $c_1(t)$ and $H_\alpha(c(t)) = 1$ is of the form

$$c(t) = ((1+\alpha)t, 0, 1/(1+\alpha), 0)$$

since

$$D_F(\tfrac{1}{2}H_\alpha^2)(c(t)) = (c_1(t), \dot{c}_1(t)).$$

Along $c(t)$ we have

$$H_{p_1} = 1+\alpha,$$
$$H_{p_2} = H_{q_1} = H_{q_2} = 0$$

and

$$H_{p_1p_1} = (1+\alpha)^2,$$
$$H_{p_2p_2} = 1+\alpha,$$
$$H_{p_1p_2} = H_{q_1q_1} = H_{q_1q_2} = 0,$$
$$H_{q_2q_2} = 1/(1+\alpha).$$

Hence the differential equation for $Y(t)$ is

$$\dot{\xi}_1 = (1+\alpha)^2 \cdot \eta_1, \quad \dot{\eta}_1 = 0,$$
$$\dot{\xi}_2 = (1+\alpha)\cdot \eta_2, \quad \dot{\eta}_2 = -1/(1+\alpha)\cdot \xi_2.$$

The first set of equations tells us that E is spanned by $\partial/\partial q_2$, $\partial/\partial p_2$ and the second set of equations tells us that with respect to the symplectic basis $(1+\alpha)^{\frac{1}{2}}\partial/\partial q_2$, $(1+\alpha)^{-\frac{1}{2}}\partial/\partial p_2$ of E, P is a rotation with angle $2\pi/(1+\alpha)$. Similarly, for the geodesic traversed in the direction opposite to the rotation the Poincaré map is a rotation with angle $2\pi/(1-\alpha)$.

The linearized Poincaré maps of the closed geodesics on S^2 are rotations with angles $2\pi/(1+\alpha)$ and $2\pi/(1-\alpha)$ respectively. In particular for α irrational all closed geodesics are elliptic with irrational exponents.

For the metrics on S^n the Poincaré maps turn out to split into rotations with angles $2\pi/(1\pm p\alpha/p_i)$.

We will now show that one can also give a description of the Morse theory of the energy functional

$$E(c) = \tfrac{1}{2}\int N^2(c)\,dt$$

on the free loop space $\Lambda = C^\infty(S^1, M)$. We do the computation again only for $M=S^2$, the other manifolds being similar. Denote by c_1 the short closed geodesic of length $2\pi/(1+\alpha)$ and by c_2 the long closed geodesic of length $2\pi/(1-\alpha)$.

The space $C^\infty(S^1, M)$ can be made into a Hilbert manifold by using H^1 curves and the Morse theory for closed geodesics for Riemannian metrics is usually done this way, see [**13**]. But for Finsler metrics this is not possible since the energy function E is only C^{1-} in this topology [**15**], which is sufficient for Lusternik–Schnirelmann theory but not for Morse theory. But the classical method of M. Morse of using finite dimensional approximations has already been used for Finsler metrics by him [**17**]. The basic fact is that for Finsler metrics one can join nearby points by unique minimal geodesics. In fact if M is compact there exists an $\eta > 0$ such that points at distance less than η can be joined by a unique minimal geodesic, see [**6**]. This is the only fact needed in order to construct finite dimensional approximations (compare also [**16**]).

Let

$$\Lambda^a = \{c \in \Lambda | E(c) \le a\}$$

and choose an integer $k > 0$ such that $2a < k \cdot \eta^2$. Define $\Lambda(k, a)$ as the subset of Λ^a such that $c|[i/k, (i+1)/k]$ is a geodesic for $i = 0, \ldots, k-1$. It follows from the Schwarz inequality that for any $c \in \Lambda^a$

$$L^2(c|[i/k, (i+1)/k]) \le 2a/k < \eta^2$$

and hence $c(i/k)$ and $c((i+1)/k)$ can be joined by a unique minimal geodesic. Therefore $\Lambda(k, a)$ is a deformation retract of Λ^a and by choosing k sufficiently large we can approximate all Λ^a by these subsets. But since an element of $\Lambda(k, a)$ is determined by the points $c(i/k)$, $\Lambda(k, a)$ is a finite dimensional manifold. The energy function $E|\Lambda(k, a)$ has the same critical points as E, namely the smooth geodesics c, and their Hessian has the same index and nullity. The index will be denoted by $\operatorname{ind}(c)$. Thus the Morse theory on Λ reduces to finite dimensional Morse theory since the direct limit of Λ^a as $a \to \infty$ is just Λ.

For Finsler metrics one defines Jacobi fields just as in the Riemannian case, namely as derivatives of variations of geodesics. If c_s is a one-parameter family of geodesics in M, then

$$Y(t) = \frac{d}{ds}\bigg|_{s=0} c_s(t) \in T_{c(t)}M$$

is a Jacobi field along c. It follows from the computation of the linearized flow $D\psi_t$ that the Jacobi fields in our example are the vector fields with components $\xi_i(t)$ with respect to the coordinate system q_i. Thus if $N_\alpha(\dot c) = 1$ the essential Jacobi fields (i.e. not tangent to the geodesic) are of the form

$$(b \cdot \cos t + d \cdot \sin t)\partial/\partial q_2.$$

The nullity of a closed geodesic, considered as a critical circle in Λ, is the dimension of the space of periodic Jacobi fields (modulo $\dot c$). Since α is irrational, the nullity of c_1 and c_2 is 0. With c all its iterates $c^k(t) = c(kt)$ are closed geodesics and represent different critical circles in Λ. Again, since α is irrational, all iterates c_1^k and c_2^k have nullity 0. Thus we have:

c_1 and c_2 and all of their iterates form non-degenerate critical submanifolds in Λ.

The critical manifold of point curves is always non-degenerate and we can therefore do Morse theory in Λ. A classical index theorem of M. Morse says that the index of a closed geodesic $c(t)$, $0 \le t \le T$, is equal to the number of conjugate points $c(t_0)$, $0 < t_0 < T$, counted with their multiplicity, plus the concavity of c. In general the concavity of c is defined to be the index + nullity of D^2E_c restricted to the Jacobi fields Y along c such that $Y(0) = Y(T)$ (but not necessarily smooth at 0) minus the nullity of c. If the endpoints are not conjugate this is an immediate consequence of the fixed endpoint index theorem. See [**18**] or [**3**] for a modern treatment. For surfaces the concavity has the following geometric interpretation: If the endpoints are not conjugate there exists exactly one Jacobi field Y with

$$Y(t) = f(t)\,\partial/\partial q_2$$

and

$$f(0) = f(T) = 1.$$

(Choose a coordinte system so that $q_2 = 0$ is c.) Then if $f'(T) < f'(0)$ the concavity is 1, and if $f'(T) \ge f'(0)$ the concavity is 0. In our example, the conjugate points occur at time $t = k \cdot \pi$, $k = 1, 2, \ldots$ and hence the endpoints of c_1^k or c_2^k are never conjugate. The Jacobi field

$$Y(t) = f(t)\,\partial/\partial q_2$$

with $f(0) = f(T) = 1$ is of the form

$$f(t) = \cos t + ((1 - \cos T)/\sin T)\sin t.$$

Since

$$f'(0) = (1 - \cos T)/\sin T \quad \text{and} \quad f'(T) = -(1 - \cos T)/\sin T$$

it follows that if

$$0 < T \bmod 2\pi < \pi \quad \text{then concavity} = 1$$

if

$$\pi < T \bmod 2\pi < 2\pi \quad \text{then concavity} = 0.$$

Thus for the short geodesic c_1:

$$\pi < T = 2\pi/(1+\alpha) < 2\pi \quad \text{and} \quad \operatorname{ind}(c_1) = 1$$

for all α. But for c_2, $\operatorname{ind}(c_2)$ depends on α. If $0 < \alpha < \frac{1}{2}$ then $\operatorname{ind}(c_2) = 3$. To see what the Morse theory in Λ looks like, let α be small. Then N_α is close to the Riemannian metric g of constant curvature 1 and one can use the following argument. For g the simple closed geodesics form a non-degenerate critical submanifold in Λ diffeomorphic to $T_1S^2 = P^3R$ of index 1. If its energy value is a then $H_*(\Lambda^{a+\varepsilon}, \Lambda^{a-\varepsilon}, Z)$ can be computed using Morse theory:

$$H_0 = 0, \quad H_1 = Z, \quad H_2 = Z_2, \quad H_3 = 0, \quad H_4 = Z.$$

If α is small the closed geodesics c_1 and c_2 also lie in $\Lambda^{a+\varepsilon} - \Lambda^{a-\varepsilon}$ and are the only critical points there and thus have to generate all the homology. The picture of Morse theory then looks as in figure 1. c_1 generates a one- and two-dimensional and c_2 a three- and four-dimensional local homology class. Thus the one-dimensional class has to survive in $(\Lambda^{a+\varepsilon}, \Lambda^{a-\varepsilon})$, the two-dimensional one also, but

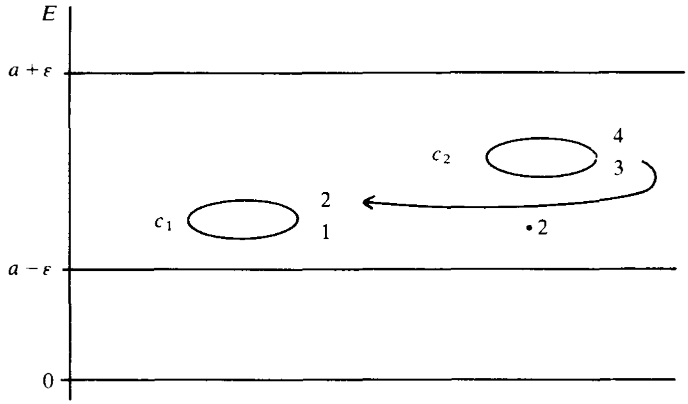

FIGURE 1

twice this class has to get killed by the three-dimensional class, whereas the four-dimensional class has to survive again. This is the only possible way to generate the homology. One can say that the boundary of the unstable manifold of c_2 (with respect to the gradient flow of E) has to wrap twice around the unstable manifold of the whole critical circle coming from c_1. The same thing happens to the iterates of c_1 and c_2, up to a certain energy level depending on α.

But as $\alpha \to 1$, the length of c_2 goes to ∞ and hence ind $(c_2) \to \infty$ whereas

$$\operatorname{ind}(c_1^k) = 2[k/2]+1$$

up to a certain energy level depending on α. Thus the picture of Morse theory looks as in figure 2 and only for large k does c_2 come into play. The critical circles

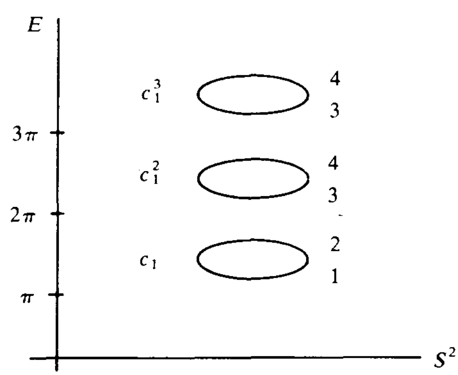

FIGURE 2

in Λ together with the point curves have to generate the homology of $\Lambda(S^2)$ which is [**28**]:

$$H_0 = Z, \quad H_{2k-1} = Z, \quad H_{2k} = Z \oplus Z_2, \quad k = 1, 2, 3, \ldots.$$

Since there exist two local homology classes in every dimension (except 1) some odd dimensional class has to kill twice some even dimensional class in order to

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generate this homology, although the precise boundary maps are not clear from the above picture.

Notice also that below any fixed dimension, the homology of $\Lambda(S^2)$ can be generated by only one closed geodesic. One could imagine that only one closed geodesic c could generate all of the homology of $\Lambda(S^2)$ as in figure 2. But at least in the non-degenerate case one can show that this is impossible by using the description of the index of the iterates in terms of an index function on S^1, due to Bott [7]. From this description it follows that if

$$\text{ind}\,(c^k) = 2[k/2]+1 \quad \text{for all } k,$$

then the Poincaré map of c would have to have an eigenvalue -1 which would mean that c^2 is degenerate.

It is rather seldom that one can give such a detailed picture of what the Morse theory on Λ has to look like, for a specific example. Usually the picture is trival as for globally symmetric spaces where all of the relative homology survives in all of Λ [**28**]. One of the few other cases where one can see what happens, at least for small energy values, is for the ellipsoid with three different axes close to 1 where some of the relative homology has to get killed [**17**].

Let us finally remark that it seems likely that all the examples N_α are completely integrable. For $M = S^n$ and $M = P^nC$ it follows from the methods in [**24**] that the examples N_α are completely integrable. For the other manifolds the problem is closely connected to the question of whether the standard metric on these spaces is completely integrable.

3. *Examples with few short closed geodesics*

We will now generalize the construction in § 1 to obtain some other interesting examples. Let M be a manifold with a Riemannian metric g such that all geodesics of g are closed with the same least period 2π. Then we have the S^1 fibration $S^1 \to T_1^*M \xrightarrow{\pi} C$ induced by the geodesic flow on T^*M. Here T_1^*M denotes the set of unit cotangent vectors. On C we also have the involution θ sending a closed geodesic into the one with opposite orientation. If ω is the symplectic two form on T^*M, then there exists a unique symplectic two form $\tilde{\omega}$ on C such that

$$\pi^*\tilde{\omega} = \omega \restriction T_1^*M$$

see [**25**]. θ satisfies

$$\theta^*\tilde{\omega} = -\tilde{\omega}.$$

Let $\tilde{f}: C \to R$ be a C^∞ function and $f = \tilde{f} \circ \pi$. Extending f to all of T^*M to become homogeneous of degree one, we obtain a function $H_1: T^*M \to R$. If H_0 is again the norm of covectors in the metric g, we will study Hamiltonians of the form

$$H_\alpha = H_0 + \alpha H_1.$$

For α sufficiently small $D_F(\frac{1}{2}H_\alpha^2)$ will be a diffeomorphism and hence

$$N_\alpha = H_\alpha \circ L_{\frac{1}{2}H_\alpha^2}$$

a Finsler metric. Notice that N_α is symmetric iff $\tilde{f}$ is invariant under θ.

The function $\tilde{f}$ on C induces a Hamiltonian vector field $X_{\tilde{f}}$ with respect to $\tilde{\omega}$ and we denote its flow by $\psi_t^{\tilde{f}}$. Since

$$\pi^*\tilde{\omega}=\omega \quad \text{and} \quad H_1|T_1^*M=\tilde{f}\circ\pi$$

we have

$$\pi^*X_{H_1}=X_{\tilde{f}}$$

and hence

$$\pi\circ\psi_t^{H_1}=\psi_t^{\tilde{f}}.$$

Since H_1 is constant along the orbits of the S^1 action on T^*M, $\psi_t^{H_1}$ and $\psi_t^{H_0}$ commute, which implies that

$$\psi_t^{H_\alpha}=\psi_t^{H_0}\circ\psi_{\alpha t}^{H_1}.$$

Hence if $x\in T_1^*M$ is a periodic point of $\psi_t^{H_\alpha}$ of period T, then

$$\psi_{\alpha T}^{H_1}x=\psi_{-T}^{H_0}x$$

and hence $\tilde{x}=\pi(x)$ is a periodic point of $\psi_t^{\tilde{f}}$ of period αT. There are two possibilities. Either $\tilde{x}$ is a critical point of $\tilde{f}$ and hence

$$\psi_t^{\tilde{f}}\tilde{x}=\tilde{x} \quad \text{for all } t,$$

or $\psi_t^{\tilde{f}}\tilde{x}$ is a non-trivial periodic orbit of $X_{\tilde{f}}$ and αT is a multiple of its least period.

We first examine the case where $\tilde{x}$ is a critical point of $\tilde{f}$. It is clear that x is then a periodic point of X_{H_α}. To determine its period we can introduce a symplectic coordinate system (q_i, p_i) on T^*M such that $q_1=t$ is the time parameter along $\psi_t^{H_0}x$ and such that $p_1(x)=H_0(x)$. Then

$$H_1(x)=p_1\cdot\tilde{f}(q_2,\ldots,q_{n-1},p_2,\ldots,p_{n-1})$$

and

$$H_\alpha=p_1+\alpha p_1\tilde{f}.$$

The Hamilton equations for H_α are then

$$\dot{q}_1=\dot{t}=H_{p_1}=1+\alpha\tilde{f}, \quad \dot{p}_1=0,$$

and

$$q_i=\alpha p_1\tilde{f}_{p_i}, \quad \dot{p}_i=-\alpha p_1\tilde{f}_{q_i} \qquad \text{for } i>1.$$

$p_1=1$ if $x\in T_1^*M$ and hence the period of $\psi_t^{H_\alpha}x$ is $2\pi/(1+\alpha\cdot\tilde{f}(\tilde{x}))$.

To examine the periodic orbits of H_α coming from non-trivial periodic orbits of $X_{\tilde{f}}$ we observe:

If X is a C^1 vector field on R^n with $X(0)=0$, then there exists an $\varepsilon>0$ such that all non-trivial periodic orbits of X in a neighbourhood of 0 have period $>\varepsilon$.

Since we could not find a proof of this statement in the literature we give here a proof which was communicated to us by D. Epstein. Let U be a convex neighbourhood of 0 and let r be the maximum of $\|DX\|$ on U. If $c(t)$ is a non-trivial periodic orbit of X in U, let d be the diameter of c and let β, γ be such that

$$\|c(\beta)-c(\gamma)\|=d.$$

Then β and γ divide c into two parts c_1 and c_2. If we let

$$v = (c(\beta) - c(\gamma)/d,$$

then

$$d = \int_{c_1} \langle v, X(c(t)) \rangle \, dt$$

and hence

$$d \le T \cdot \max \langle v, X(c(t)) \rangle$$

where T is the period of c. Therefore there exists a t' such that

$$\langle v, X(c(t')) \rangle \ge d/T.$$

Similarly

$$\int_{c_2} \langle v, X(c(t)) \rangle \, dt = -d$$

and hence there exists a t'' such that

$$\langle v, X(c(t'')) \rangle \le -d/T.$$

This implies

$$\langle v, X(c(t')) - X(c(t'')) \rangle \ge 2d/T$$

and since

$$\|X(c(t')) - X(c(t''))\| \le r\|c(t') - c(t'')\| \le rd$$

we obtain $T \ge 2/r$ as desired.

We can apply this to the Hamiltonian vector field $X_{\tilde{f}}$ on C. Since C is compact there exists a lower bound for the period of any non-trivial periodic orbit of $X_{\tilde{f}}$. For the periodic orbits of H_α of the second kind this means that, as $\alpha \to 0$, their period goes to ∞. Hence:

Let N_α be a Finsler metric as above. Each critical point of $\tilde{f}$ gives rise to a closed geodesic whose length goes to 2π as $\alpha \to 0$. The length of all other close geodesics goes to ∞ as $\alpha \to 0$.

The same is of course true in the more general situation where we start with a Hamiltonian H_0 all of whose orbits are closed and non-trivial of least period 2π. A theorem of Weinstein [**27**] implies that any Hamiltonian H C^2 close to H_0 has at least as many periodic orbits of period close to 2π as a function on $C = H_0^{-1}(1)/S^1$ has critical points. The above statement for H_α then implies that this estimate is always optimal.

If M is a Riemannian manifold all of whose geodesics are closed of least period 2π, then C is simply connected. If M is simply connected this follows since T_1^*M is then simply connected if $\dim M > 2$ and from the homotopy sequence of π it follows that C is simply connected. But if $\dim M = 2$, we have $M = S^2$ and hence $C = S^2$. If M is not simply connected, then by [**5**, p. 187], M is diffeomorphic to P^nR and each closed geodesic is not null-homotopic. Hence the space of closed geodesics of the metric on P^nR is the same as the space of closed geodesics for the metric on the universal cover S^n and hence simply connected. Since C is simply connected, theorem 5.1 in [**23**] implies that there exists a function on C with only $\dim M$ critical points, at least if $\dim C \geq 6$. But if $\dim C < 6$, we have $M = S^2$, $P^2\mathbb{R}$, S^3, or $P^3\mathbb{R}$ (see [**5**, appendix D]) and hence $C = S^2$ or $C = S^2 \times S^2$ which implies again the existence of a function on C with only $\dim M$ critical points. In particular:

If $M = S^n$, P^nC, P^nH, or P^2Ca and g the standard metric, then for any $\varepsilon > 0$ there exists a Finsler metric on M, ε close to g, with $\dim M$ *closed geodesics with lengths in $(2\pi - \varepsilon, 2\pi + \varepsilon)$ and such that the length of all other closed geodesics is greater than $1/\varepsilon$.*

We do not know if one can find such Finsler metrics where these $\dim M$ closed geodesics are the only ones.

We will now examine the situation for symmetric Finsler metrics. The perturbation methods in [**27**] or Lusternik–Schnirelmann theory imply that any symmetric Finsler metric sufficiently C^2 close to g has at least as many closed geodesics of length close to 2π as a function on C/θ has critical points. Conversely if we have a function on C/θ with k critical points then it lifts to a function on C with $2k$ critical points and the above methods imply that the corresponding symmetric Finsler metric N_α has k closed geodesics with lengths close to 2π (since we count $c(t)$ and $c(-t)$ as only one closed geodesic now) and the length of all other closed geodesics goes to ∞ as $\alpha \to 0$.

But the problem of finding the smallest number of critical points for a function on C/θ is still open. We make some remarks now in the case $M = S^n$. Then C is the space $G^0_{2,n-1}$ of oriented two planes in R^{n+1} and C/θ is the space $G_{2,n-1}$ of unoriented two planes in R^{n+1}. The cup length of $G_{2,n-1}$ has been computed by Alber and turns out to be

$$g(n) = 2n - s - 1 \quad \text{where } n = 2^k + s < 2^{k+1}.$$

Notice that $(3n-1)/2 \leq g(n) \leq 2n - 1$ and if $n = 2^k$ then

$$g(n) = 2n - 1.$$

Since any manifold M has a function with $\dim M + 1$ critical points [**23**, proposition 2.9], there exists a function on $G_{2,n-1}$ with $2n-1$ critical points. Hence if $n = 2^k$, $g(n)$ is optimal. In general not even the category of $G_{2,n-1}$is known. One only knows that $\text{cat}\,(G_{2,n-1}) < 2n - 1$ if $n \neq 2^k$, see [**4**], and hence

$$\text{cat}\,(G_{2,n-1}) = g(n) \quad \text{if } n = 2^k + 1.$$

But we do not know if there exists a function on $G_{2,n-1}$ with $2n - 2 = g(n)$ critical points if $n = 2^k + 1$. But for $n = 3$, J. Milnor gave an example of a function on $G_{2,2}$ with only $g(3) = 4$ critical points. Since this example has not been published before we will describe it briefly here.

$G^0_{2,2}$ can be identified with $S^2 \times S^2$ as can be seen, e.g. by identifying the two planes in R^4 with the decomposable unit vectors in $\Lambda^2 R^4$. The $*$ operator induces a splitting

$$\Lambda^2 R^4 = \Lambda^+ \oplus \Lambda^-$$

and $\omega \varepsilon \Lambda^2 R^4$ is decomposable iff $\|\omega^+\| = \|\omega^-\| = 1/\sqrt{2}$. This also shows that the unoriented planes $G_{2,2}$ can be viewed as

$$S^2 \times S^2/(x, y) \sim (-x, -y).$$

If we choose coordinates (x_i, y_i), $i = 1, 2, 3$, on $S^2 \times S^2$ where $\sum x_i^2 = \sum y_i^2 = 1$ then

$$f(x_i, y_i) = f_1 + f_2, \quad f_1 = \sum_{i=1}^{i=3} ((y_i - x_i)/2)^2, \quad f_2 = x_1^2 - x_2^2$$

defines a function on $S^2 \times S^2$ invariant under $(x, y) \to (-x, -y)$. We claim that the induced function $\tilde{f}$ on $G_{2,2}$ has only 6 critical points with 4 critical levels. To see this observe that the critical points of $\tilde{f}_1$ are given by $y_i = x_i$ and $y_i = -x_i$ and are projective two planes with critical levels 0 and 1. It follows easily that the critical points of $\tilde{f}_1 + \tilde{f}_2$ must be contained in these projective two planes and hence consist of the critical points of $\tilde{f}_2$ restricted to them. But $\tilde{f}_2$ on $x_i = y_i$ or $x_i = -y_i$ has three critical points each with critical levels -1, 0, 1. Hence $\tilde{f}_1 + \tilde{f}_2$ has 6 critical points with critical levels -1, 0, 1, 2 and there are two critical points each on the critical levels 0 and 1. Since these critical levels are connected, proposition 2.9 in [**23**] now implies that one can find a new function $\tilde{f}'$ which collapses the critical points of $\tilde{f}$ which lie on the same level into one critical point. Hence $\tilde{f}'$ is the desired function with 4 critical points. We obtain:

There exist symmetric Finsler metrics on S^n, in any neighbourhood of the constant curvature metric, with only $2n-1$ closed geodesics of lengths close to 2π and such that the length of all other closed geodesics is larger than any prescribed number. On S^3 there exist such Finsler metrics with only $g(3) = 4$ closed geodesics of lengths close to 2π.

The example on S^3 suggests that theorem 5.1.1 in [**13**] is incorrect. There it was claimed that any Riemannian metric on S^n has at least $2n-1$ short closed geodesics (which, if the metric is close to the standard metric, has to have a length close to 2π). But the proof does not use any properties of Riemannian metrics which do not also hold for symmetric Finsler metrics. The mistake seems to lie in the concept of geometric subordination, introduced in [**13**, p. 174].

If one requires, as before, that in addition the closed geodesics of the symmetric Finsler metric are non-degenerate, then the perturbation methods imply that the Finsler metric has at least $\sum b_i(C/\theta)$ closed geodesics with length close to 2π. But for $M = S^n$

$$\sum b_i(G_{2,n-1}, Z_2) = n(n+1)/2.$$

On the other hand one knows that the Riemannian metric one obtains from an n-dimensional ellipsoid with distinct principal axes close to 1 has $n(n+1)/2$ closed geodesics with lengths close to 2π and all other closed geodesics are larger than any prescribed number if one chooses the axes sufficiently close to 1, see [**17**].

For Riemannian metrics it has been recently shown [**3**] that any metric with curvature $\frac{1}{4} \le K \le 1$ has at least $g(n)$ closed geodesics with lengths in $[2\pi, 4\pi]$, and if all closed geodesics are non-degenerate, at least $n(n+1)/2$ such closed geodesics.

But it is not known if a corresponding theorem is true for symmetric Finsler metrics.

Returning now to the general situation of an arbitrary M with all geodesics closed of the same least period 2π, let us make the following remarks, which shed some light on the examples in § 1. If we start with a function $\tilde{f}$ on C such that the flow of X_{H_1} on T^*M induces an S^1 action, as was the case for the examples in § 1, then the same proof as in § 1 implies that the only closed geodesics of H_α for α irrational are the closed geodesics of H_0 invariant under the flow of H_1. These correspond exactly to the critical points of $\tilde{f}$. But if $\tilde{f}$ has only finitely many critical points, then any zero of $X_{\tilde{f}}$ has index 1 since the flow of X_{H_1} and hence also the flow of $X_{\tilde{f}}$ induces an S^1 action. Hence, by the Hopf index theorem, the Euler characteristic of C is equal to the number of critical points of $\tilde{f}$ on C. But by the computation of $b_i(C)$ in § 1, the Euler characteristic of C is $2n$ for $M=S^{2n-1}$ or S^{2n} and $n(n+1)$, $2n(n+1)$, and 24 for $M=P^nC$, P^nH, and P^2Ca respectively. Hence we do not obtain any Finsler metrics with less closed geodesics by this method, although the closed geodesics can be degenerate in this more general situation. If $\tilde{f}$ is in addition invariant under θ, then the same argument on C/θ implies that

$$\tfrac{1}{2}\chi(C)=\chi(C/\theta)=\#\{\text{critical points of } \tilde{f} \text{ on } C/\theta\}.$$

But at least for $M=S^n$, P^2C, P^2H, P^2Ca, $\frac{1}{2}\chi(C)<\dim M$, which means that there exists no such function $\tilde{f}$ invariant under θ. Hence the method in § 1 cannot be used to produce symmetric Finsler metrics with only finitely many closed geodesics. Nevertheless, it still seems possible that if one considers functions $\tilde{f}$ on C invariant under θ, such that one has sufficient control over the periods of non-trivial periodic orbits of $X_{\tilde{f}}$, one could find a symmetric Finsler metric H_α which, for appropriate α, has only finitely many closed geodesics.

Finally we will show that one can easily compute the linearized Poincaré map of the short closed geodesics of H_α. If $\tilde{x}$ is a critical point of $\tilde{f}$ and $c(t)=\psi_t^{H_\alpha}x$ the corresponding periodic orbit of H_α of length $2\pi/(1+\alpha\tilde{f}(\tilde{x}))$, then we can introduce coordinates as in the beginning of this section and compute the differential equation for the linearized flow $D_x\psi_t^{H_\alpha}$ as in § 2. (Here we choose $x\in T_1^*M$ and hence $p_1=1$ along $c(t)$.)

$$\begin{aligned}\dot{\xi}_i&=H_{p_jp_i}\cdot\eta_j+H_{q_jp_i}\cdot\xi_j\\&=\alpha\tilde{f}_{p_jp_i}(\tilde{x})\cdot\eta_j+\alpha\tilde{f}_{q_jp_i}(\tilde{x})\cdot\xi_j\\\dot{\eta}_i&=-H_{p_jq_i}\cdot\eta_j-H_{q_jq_i}\cdot\xi_j\\&=-\alpha\tilde{f}_{p_jq_i}(\tilde{x})\cdot\eta_j-\alpha\tilde{f}_{q_jq_i}(\tilde{x})\cdot\xi_j.\end{aligned}$$

This differential equation has constant coefficient matrix $\alpha J\cdot A$ where

$$J=\begin{pmatrix}0 & \mathrm{I}\\ -\mathrm{I} & 0\end{pmatrix}\quad\text{and}\quad A=\operatorname{Hess}\tilde{f}(\tilde{x}).$$

Hence the linearized Poincaré map is

$$P=\exp(\alpha T\cdot J\cdot A)$$

where T is the period of c. Using the classification of normal forms of a quadratic polynomial with respect to a symplectic coordinate system (see e.g. [**2**, p. 381]) one can compute for each normal form of A the corresponding matrix

$$P = \exp(\alpha TJA)$$

and one can easily verify the following claims: (see [**3**] for some of the notation)

(1) If $\tilde{x}$ corresponds to a maximum or minimum of $\tilde{f}$ (possibly degenerate) then, with respect to a symplectic basis, P splits into 2×2 blocks

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} - \pi \le \phi \le \pi, \quad \text{and/or} \quad \begin{pmatrix} 1 & 0 \\ \sigma & 1 \end{pmatrix}$$

where $\phi \ge 0$ and $\sigma = 1$ if $\tilde{x}$ is a minimum and $\phi \le 0$ and $\sigma = -1$ if $\tilde{x}$ is a maximum. The blocks

$$\begin{pmatrix} 1 & 0 \\ \sigma & 1 \end{pmatrix} \quad \text{and} \quad \phi = 0$$

only occur if $\tilde{x}$ is degenerate.

(2) If the index of $A \le k < \frac{1}{2}\dim C$, then P has at least k Jordan blocks $J_R(e^{i\phi}, m, \sigma)$ with $\phi \ge 0$ and $\sigma \ge 0$. If $\phi \ne 0$, then m is odd and $\sigma > 0$.

(3) If the index + nullity of $A \ge k > \frac{1}{2}\dim C$, then P has at least k Jordan blocks $J_R(e^{i\phi}, m, \sigma)$ with $\phi \ge 0$ and $\sigma \le 0$. If $\phi \ne 0$ m is odd and $\sigma < 0$.

Notice that (1)–(3) are completely analogous with the results in [**3**]. The assumptions on the index and nullity of A in (2) and (3) are quite natural, since Lusternik–Schnirelmann theory implies that if $H_k(C) \ne 0$, then there exists a critical point $\tilde{x}$ of $\tilde{f}$ such that the index of $A \le k \le$ index + nullity of A, where $A = \text{Hess}\, \tilde{f}(\tilde{x})$.

Notice also that (1)–(3) remain true in the more general situation where H_0 is a Hamiltonian function on an arbitrary symplectic manifold with all orbits closed (and non-trivial) of the same least period.

Using the above remarks one can now easily construct functions $\tilde{f}$ such that the corresponding Finsler metric H_α has only two short closed geodesics of elliptic–parabolic type, the one corresponding to the maximum and minimum of $\tilde{f}$. One can also construct such Finsler metrics which are symmetric. Furthermore one can show that the above relationship between the index of A (and hence the index of c) and the number of Jordan blocks that P has with eigenvalues on the unit circle (and their signs) are the only possible ones in general. Hence the results in [**3**] are optimal, as far as the estimates for the number of Jordan blocks and their signs are concerned.

We would like to close by asking the following questions, which seem to be the most important unsolved problems concerning the existence of closed extremals in variational calculus:

(1) Do there exist symmetric Finsler metrics with only finitely many closed geodesics?

(2) Do there exist non-symmetric Finsler metrics on $M = S^n$, P^nC, P^nH, P^2Ca with only $\dim M$ closed geodesics?

(3) Do there exist symmetric Finsler metrics on S^n with only $g(n)$ short closed geodesics? Or equivalently, do there exist functions on $G_{2,n-1}$ with only $g(n)$ critical points?

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(4) Does every Finsler metric on $M = S^n$, P^nC, P^nH, P^2Ca have $\dim M$ closed geodesics?

(5) Does every symmetric Finsler metric on S^n have $g(n)$ closed geodesics?

(6) If, for a simply connected manifold M, $b_i(\Lambda(M), K)$ is bounded for every field K, is then $H^*(M, Z)$ isomorphic to the cohomology ring of $M = S^n$, P^nC, P^nH, or P^2Ca? This would imply that any Finsler metric on a compact simply connected manifold whose cohomology ring differs from these, has infinitely many closed geodesics.

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