Canad. Math. Bull. Vol. **29** (2), 1986

FIXED POINT THEOREMS FOR PROXIMATELY NONEXPANSIVE SEMIGROUPS

BY

MO TAK KIANG[1] AND KOK-KEONG TAN[2]

ABSTRACT. A commutative semigroup G of continuous, selfmappings on (X, d) is called proximately nonexpansive on X if for every x in X and every $\beta > 0$, there is a member g in G such that $d(fg(x), fg(y)) \leq (1 + \beta)\, d(x, y)$ for every f in G and y in X. For a uniformly convex Banach space $(X, \|\|)$, it is shown that if G is a commutative semigroup of continuous selfmappings on X which is proximately nonexpansive, then a common fixed point exists if there is an x_0 in X such that its orbit $G(x_0)$ is bounded. Furthermore, the asymptotic center of $G(x_0)$ is such a common fixed point.

1. **Introduction**. Let (X, d) be a metric space. A mapping $f{:}X \to X$ is called nonexpansive if for every $x, y \in X$, $d(f(x), f(y)) \leq d(x, y)$. A point x_0 with the property that $f(x_0) = x_0$ is called a fixed point of f. Many interesting results concerning conditions on X and f which guarantee the existence of a fixed point of f have been obtained in recent years (see e.g. [7], [8], [9]).

When $G{:}X \to X$ is a semigroup of nonexpansive mappings, a point x_0 with the property that $f(x_0) = x_0$ for every $f \in G$ is called a common fixed point of G. In [4], a semigroup G of continuous selfmappings on X is called ultimately nonexpansive if for every $x, y \in X$ and $\beta > 0$, there exists $g \in G$ such that $d(fg(x), fg(y)) \leq (1 + \beta)$ $d(x, y)$, for all $f \in G$. It was shown there, among other things, that if X is a reflexive, locally uniformly convex Banach space having a point with a precompact orbit, then G has a common fixed point.

The question arises as to whether the requirement that X has a point with a precompact orbit can be weakened to that it has a point with just a bounded orbit. In [10], an example is constructed to show that a singly generated semigroup G (i.e. $G = \{f^n: n = 1, 2, \ldots\}$) of continuous selfmappings on a bounded closed convex subset of the Hilbert space l_2 which is ultimately nonexpansive need not have a fixed point. A recent result by Edelstein [3] shows that if G is singly generated by f with an f-closure point which has a bounded orbit on a reflexive, locally uniformly convex Banach space, then G has a unique fixed point.

---

[1]Research supported by NSERC grant no. A8756.

[2]Research partially supported by NSERC grant no. A8096.

Received by the editors June 1, 1984, and, in revised form, April 29, 1985.

AMS Subject Classification (1980): 47H10.

Keywords and phrases. Common fixed point, proximately nonexpansive, asymptotic center.

https://doi.org/10.4153/CMB-1986-027-2 Published online by Cambridge University Press

In this paper, it is shown that if $(X, \|\ \|)$ is a uniformly convex Banach space and G a commutative semigroup of continuous selfmappings on X which is furthermore proximately nonexpansive (a condition stronger than that of ultimately non-expansiveness) then the existence of a point x_0 with a bounded orbit $G(x_0)$ guarantees a common fixed point. Indeed, the asymptotic center of $G(x_0)$ is such a point. Other results generalizing some known fixed-point theorems for nonexpansive (asymptotically nonexpansive) mappings are also stated.

2. **Preliminaries**

DEFINITION. *A semigroup G of continuous selfmappings of a metric space (X, d) is called proximately nonexpansive if for every $x \in X$, and every $\beta > 0$, there exists $g \in G$ such that $d(fg(x), fg(y)) \leq (1 + \beta)\, d(x, y)$ for all $f \in G$, and for all $y \in X$.*

It is clear that if G is nonexpansive then it is proximately nonexpansive while if it is proximately nonexpansive then it is ultimately nonexpansive.

For each $g \in G$, let $G_g = \{fg : f \in G\}$.

For every $g, h \in G$, define $g \leq h$ if and only if $G_g \supseteq G_h$.

Let $(X, \|\ \|)$ be a uniformly convex Banach space, G a commutative semigroup of continuous selfmappings on X. Let $x_0 \in X$ such that its orbit $G(x_0) = \{f(x_0) : f \in G\}$ is bounded. For every $g \in G$, define

(i) $\gamma_g(x_0)$ = the Chebyshev radius of $G_g(x_0)$ $= \inf_{y \in X} \sup_{f \in G} \{\|y - fg(x_0)\|\}$,

(ii) $c_g(x_0)$ = the Chebyshev center of $G_g(x_0)$ = the unique point $y \in X$ such that

$$\gamma_g(x_0) = \sup_{f \in G} \{\|y - fg(x_0)\|\};$$

thus

$$\gamma_g(x_0) = \sup_{f \in G} \{\|c_g(x_0) - fg(x_0)\|\}.$$

clearly if $g \leq h$, then for any $y \in X$,

$$\sup_{f \in G} \|y - fg(x_0)\| \geq \sup_{f \in G} \|y - fh(x_0)\|,$$

hence,

$$\inf_{y \in X} \sup_{f \in G} \|y - fg(x_0)\| \geq \inf_{y \in X} \sup_{f \in G} \|y - fh(x_0)\|,$$

i.e. $\gamma_g(x_0) \leq \gamma_h(x_0)$ which show that $\{\gamma_g(x_0)\}$ is a decreasing net.

Let $\gamma(x_0) = \inf_{g \in G} \gamma_g(x_0)$

LEMMA 1.

(a) $\gamma_g(x_0) \to \gamma(x_0)$

(b) *There exists $c(x_0) \in X$ such that $c_g(x_0) \to c(x_0)$.*

PROOF.

(a) Obvious

(b) CASE 1. *Suppose* $\gamma(x_0) = 0$. *Then* $\inf \gamma_g(x_0) = 0$, i.e. $\forall \epsilon > 0$, $\exists g \in G$ such that for $s, t \in G$, with $g \leq s$ and $g \leq t$, we have $\gamma_s(x_0) < \epsilon/2$ and $\gamma_t(x_0) < \epsilon/2$.

Choose any $z \in G_s(x_0) \cap G_t(x_0)$, then

$$\|c_s(x_0) - c_t(x_0)\| \leq \|c_s(x_0) - z\| + \|z - c_t(x_0)\| \leq \gamma_s(x_0) + \gamma_t(x_0) < \epsilon/2 + \epsilon/2 = \epsilon$$

which shows that $\{c_g(x_0)\}$ is a Cauchy net.

Hence, there exists $c(x_0) \in X$ such that $c_g(x_0) \to c(x_0)$.

CASE 2. *Suppose* $\gamma(x_0) > 0$. *If* $\{c_g(x_0)\}$ *is not a Cauchy net, then there exists* $\epsilon > 0$, *such that for every* $g \in G$, *there exists* $h, t \in G$ *with* $g < h$ *and* $g < t$ *such that* $\|c_h(x_0) - c_t(x_0)\| \geq \epsilon$. *Since* $\gamma_g(x_0) \to \gamma(x_0)$, *there exists* $g_0 \in G$ *such that* $\gamma_{g_0}(x_0) - \gamma(x_0) < (1/2)\gamma(x_0)\delta(\epsilon/D)$ *where* $D = \operatorname{diam} G(x_0)$.

Hence, for any $g, h \in G$ with $g > g_0$ and $h > g_0$,

$$(*) \qquad |\gamma_g(x_0) - \gamma_h(x_0)| \leq |\gamma_g(x_0) - \gamma(x_0)| + |\gamma_h(x_0) - \gamma(x_0)| < \gamma(x_0)\delta\left(\frac{\epsilon}{D}\right)$$

However, for this g_0 there exist $k, t \in G$ with $k > g_0$, $t > g_0$ such that $\|c_k(x_0) - c_t(x_0)\| \geq \epsilon$. Let $s \in G_k \cap G_t$ then $G_s \subseteq G_k \cap G_t$ which implies $\gamma_s(x_0) \leq \gamma_k(x_0)$ and $\gamma_s(x_0) \leq \gamma_t(x_0)$. Assume $\gamma_k(x_0) \leq \gamma_t(x_0)$. For every $f \in G$, then

$$\|c_k(x_0) - fs(x_0)\| \leq \gamma_k(x_0) \leq \gamma_t(x_0),$$

$$\|c_t(x_0) - fs(x_0)\| \leq \gamma_t(x_0)$$

and hence

$$\left\|\frac{c_k(x_0) + c_t(x_0)}{2} - fs(x_0)\right\| \leq \gamma_t(x_0)\left[1 - \delta\left(\frac{\epsilon}{\gamma_t(x_0)}\right)\right].$$

Therefore,

$$\gamma_s(x_0) \leq \gamma_t(x_0)\left[1 - \delta\left(\frac{\epsilon}{\gamma_t(x_0)}\right)\right] \leq \gamma_t(x_0)\left[1 - \delta\left(\frac{\epsilon}{D}\right)\right]$$

which implies that

$$\gamma_t(x_0) - \gamma_s(x_0) \geq \gamma_t(x_0)\left[\delta\left(\frac{\epsilon}{D}\right)\right] \geq \gamma(x_0)\left[\delta\left(\frac{\epsilon}{D}\right)\right].$$

This contradicts inequality (*) above as $t \geq g_0$ and $s \geq t \geq g_0$. Hence, $\{c_g(x_0)\}$ is a Cauchy net, which implies that there exists $c(x_0) \in X$ such that $c_g(x_0) \to c(x_0)$.

https://doi.org/10.4153/CMB-1986-027-2 Published online by Cambridge University Press

REMARK. In lemma 1(b), $c(x_0)$ is called the asymptotic center of $G(x_0)$. In the case when $G(x_0)$ is a sequence, the notion of the asymptotic center was first introduced by Edelstein in [2].

LEMMA 2. *If* $\gamma(x_0) = 0$, *then* $f(c(x_0)) = c(x_0)$, *for all* $f \in G$.

PROOF. We first observe that if $x_g \in G_g(x_0)$ for $g \in G$, then $x_g \to c(x_0)$. Indeed, let $x_g = f_g g(x_0)$ where $f_g \in G$. Then for each $\epsilon > 0$, there exists $g_1 \in G$ such that $\|c_g(x_0) - c(x_0)\| < \epsilon/2$ for every $g > g_1$, and there exists $g_2 \in G$ such that $\gamma_g(x_0) < \epsilon/2$ for every $g > g_2$. Now if $g_0 \in G$ is such that $g_0 > g_1$ and $g_0 > g_2$ then for each $g \in G$ with $g > g_0$ then we have

$$\begin{aligned}\|x_g - c(x_0)\| &= \|f_g g(x_0) - c(x_0)\| \leq \|f_g g(x_0) - c_g(x_0)\| + \|c_g(x_0) - c(x_0)\| \\ &\leq \gamma_g(x_0) + \|c_g(x_0) - c(x_0)\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\end{aligned}$$

which shows that $x_g \to c(x_0)$.

Next, for each g in G, choose any $x_g \in G_g(x_0)$. Then by the above observation $x_g \to c(x_0)$. Let $f \in G$, then $f(x_g) \in G_g(x_0)$ for each $g \in G$ which implies that $f(x_g) \to c(x_0)$ also. Since $x_g \to c(x_0)$, by continuity of each $f \in G$ we have $f(x_g) \to f(c(x_0))$. Hence $f(c(x_0)) = c(x_0)$, for every $f \in G$.

LEMMA 3. *For each* $\epsilon > 0$, *there exists* $g_0 \in G$, *such that for all* $g \in G$ *with* $g > g_0$, $\|c(x_0) - fg(x_0)\| \leq \gamma(x_0) + \epsilon$ *for each* $f \in G$.

PROOF. For each $\epsilon > 0$, there exists $g_1 \in G$ such that $\|c_g(x_0) - c(x_0)\| < \epsilon/2$ for all $g \in G$ with $g > g_1$, and there exists $g_2 \in G$ such that $|\gamma_g(x_0) - \gamma(x_0)| < \epsilon/2$ for all $g \in G$ with $g > g_2$. Now, if $g_0 \in G$ is such that $g_0 > g_1$ and $g_0 > g_2$ then whenever $g \in G$ with $g > g_0$, we have

$$\begin{aligned}\|c(x_0) - fg(x_0)\| &\leq \|c(x_0) - c_g(x_0)\| + \|c_g(x_0) - fg(x_0)\| \\ &< \frac{\epsilon}{2} + \gamma_g(x_0) < \frac{\epsilon}{2} + \gamma(x_0) + \frac{\epsilon}{2} \\ &= \gamma(x_0) + \epsilon \text{ for all } f \in G.\end{aligned}$$

3. **Main Results**

THEOREM 1.*Let* $(X, \|\,\|)$ *be a uniformly convex Banach space, G be a commutative semigroup of continuous selfmappings on X. If G is proximately nonexpansive and if there is a point* x_0 *in X such that* $G(x_0)$ *is bounded, then the asymptotic center* $c(x_0)$ *of* $G(x_0)$ *is a common fixed point.*

PROOF. By Lemma 2, we may assume that $\gamma(x_0) > 0$. For each $x \in X$, define $d(x) = \inf_{g \in G} \sup_{f \in G} \{\|x - fg(x)\|\}$. Hence

$$d(c(x_0)) = \inf_{g \in G} \sup_{f \in G} \{\|c(x_0) - fg(c(x_0))\|\}$$

https://doi.org/10.4153/CMB-1986-027-2 Published online by Cambridge University Press

Clearly $d(c(x_0) < \infty$. Indeed, for any $\lambda > 0$, for the point x_0, there exists $g_0 \in G$ such that whenever $g \in G$ with $g > g_0$, we have $\|fg(x_0) - fg(c(x_0))\| \le (1 + \lambda)\|x_0 - c(x_0)\|$ for all $f \in G$.

Hence for any $f \in G$, we have

$$\begin{aligned}\|c(x_0) - fg(c(x_0)\| &\le \|c(x_0) - c_g(x_0)\| + \|c_g(x_0) - fg(x_0)\| \\ &\quad + \|fg(x_0) - fg(c(x_0))\|. \\ &\le \|c(x_0) - c_g(x_0)\| + \gamma_g(x_0) + (1 + \lambda)\|x_0 - c(x_0)\|,\end{aligned}$$

This implies that

$$\sup_{f \in G} \|c(x_0) - fg(c(x_0))\| \le \|c(x_0) - c_g(x_0)\| + \gamma_g(x_0) + (1 + \lambda)\|x_0 - c(x_0)\|$$

i.e.

$$d(c(x_0)) = \inf_{g \in G} \{\sup_{f \in G} \|c(x_0) - fg(c(x_0))\|\} < \infty$$

CASE 1. *Assume* $d(c(x_0) < 0$.

(1) *Let* ϵ^* *be such that* $0 < \epsilon^* < d(c(x_0))$.

Then for each $g \in G$, there exists $f_g \in G$ such that $\|c(x_0) - f_g g(x_0))\| > \epsilon^*$. Choose $\epsilon > 0$ so small such that

$$\left[1 - \delta\left(\frac{\epsilon^*}{\gamma(x_0) + \epsilon}\right)\right](\gamma(x_0) + \epsilon) < \gamma(x_0).$$

(2) *Next, choose* $\beta > 0$ *such that*

$$(1 + \beta)\left[\gamma(x_0) + \frac{\epsilon}{2}\right] < \gamma(x_0) + \epsilon$$

(3) *By proximately nonexpansiveness of* G*, for the point* $c(x_0)$ *and the above* $\beta > 0$*, there is* $g_0 \in G$ *such that*

$$\|fg_0(c(x_0)) - fg_0(y)\| \le (1 + \beta)\|c(x_0) - y\|,$$

for all $f \in G$ *and for all* $y \in X$.

(4) *By Lemma 3, for this* $\epsilon > 0$*, there is* $g_1 \in G$ *such that for all* $g \in G$ *with* $g > g_1$*, we have* $\|c(x_0) - fg(x_0)\| \le \gamma(x_0) + \epsilon/2$*, for each* $f \in G$.

Let $g_\alpha \in G$ with $g_\alpha > g_1$ and f_α be the member in G corresponding to g_α such that $\|c(x_0) - f_\alpha g_\alpha(c(x_0)\| > \epsilon^*$; Then we also have $\|c(x_0) - hg_\alpha f_\alpha g_0(x_0)\| \le \gamma(x_0) + \epsilon/2$, for each $h \in G$.

(5) *In (3) above, replacing* f *by* f_0 *(where* f_0 *is the member corresponding to* g_0 *such that* $\|c(x_0) - f_0 g_0(c(x_0)\| > \epsilon^*$) and y by $f_\alpha g_0 h g_\alpha(x_0)$ *where* $h \in G$*, we have*

$$\|f_0 g_0(c(x_0)) - f_0 g_0 f_\alpha g_0 h g_\alpha(x_0)\| \le (1 + \beta)\|c(x_0) - f_\alpha g_0 h g_\alpha(x_0)\|,$$

for all $h \in G$.

https://doi.org/10.4153/CMB-1986-027-2 Published online by Cambridge University Press

(6) *Using results in (5), (4) and (2) above, we have*

$$\|f_0g_0(c(x_0)) - f_0g_0f_\alpha g_0hg_\alpha(x_0)\| \leq (1 + \beta)[\gamma(x_0) + \epsilon/2] < \gamma(x_0) + \epsilon, \qquad \forall h \in G.$$

(7) *Now from (5) above,* $\|c(x_0) - f_0g_0(c(x_0))\| > \epsilon^*$ *and from (6),*

$$\|f_0g_0(c(x_0)) - f_0g_0f_\alpha g_0hg_\alpha(x_0)\| < \gamma(x_0) + \epsilon$$

Also by (4) we have, by replacing h by f_0g_0h

$$\|c(x_0) - f_0g_0f_\alpha g_0hg_\alpha(x_0)\| \leq \gamma(x_0) + \frac{\epsilon}{2} < \gamma(x_0) + \epsilon, \qquad \forall h \in G$$

Hence, by uniform convexity of X,

$$\left\|\frac{c(x_0) + f_0g_0(c(x_0))}{2} - f_0g_0hg_\alpha f_\alpha g_0(x_0)\right\| \leq \left[1 - \delta\left(\frac{\epsilon^*}{\gamma(x_0) + \epsilon}\right)\right] \times (\gamma(x_0) + \epsilon) < \gamma(x_0),$$

for all $h \in G$.

Let t denote $f_0g_0g_\alpha f_\alpha g_0$, then clearly

$$\sup_{h\in G}\left\|\frac{c(x_0) + f_0g_0(c(x_0))}{2} - ht(x_0)\right\| < \gamma(x_0)$$

which implies

$$\gamma_t(x_0) = \inf_{y\in X}\{\sup_{h\in G}\|y - ht(x_0)\|\} < \gamma(x_0),$$

arriving at a contradiction. Therefore we must have

CASE 2. $d(c(x_0) = 0$ *that is* $\inf_{g\in G}\{\sup_{f\in G}\|x_0 - fg(x_0)\|\} = 0$

It follows that for each positive integer n there is $g_n \in G$ such that $\sup_{f\in g}\|x_0 - fg_n(x_0)\| < 1/n$, so that $fg_n(x_0) \to x_0$ for each $f \in G$. Now if $h \in G$, then $hfg_n(x_0) \to h(x_0)$ by continuity of h; but $hfg_n(x_0) \to x_0$ so that $h(x_0) = x_0$. Hence x_0 is a common fixed point of G proving the theorem.

With slight modifications the proof of Theorem 1 can be used to prove the following:

THEOREM 2. *Let* $(X, \|\,\|)$ *be a uniformly convex Banach space, K be a nonempty bounded closed convex subset of X. If G is a commutative semigroup of continuous selfmappings on K which is proximately nonexpansive, then for any* $x \in K$, *the asymptotic center* $c(x)$ of $G(x)$ *is a common fixed point of G.*

When G is singly generated by f the following result is an immediate consequence of Theorem 2:

COROLLARY 1. *Let* $(X, \|\,\|)$ *be a uniformly convex Banach space, K be a nonempty closed convex subset of X and* $f:K \to K$ *be continuous. If for each* $x \in K$ *and each* $\alpha > 0$ *there exists* $n \geq 1$ *such that*

https://doi.org/10.4153/CMB-1986-027-2 Published online by Cambridge University Press

$$\|f^m(x) - f^m(y)\| \leq (1 + \alpha)\|x - y\|$$

for all $m > n$ and all $y \in K$, then for each $x \in K$, the asymptotic center of its orbit $\{f^n(x): n = 1, 2, \ldots\}$ is a fixed point of f.

As a special case of corollary 1 we have the following strengthened version of a result due to Goebel and Kirk in [6]:

COROLLARY 2. *Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space X and let $f: K \to K$ be asymptotically nonexpansive, then for each $x \in K$, the asymptotic center of its orbit is a fixed point.*

An immediate consequence of Corollary 2 is the following strengthened version of a result due to Göhde in [5] and Browder in [1]:

COROLLARY 3. *Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space X, and let $f: K \to K$ be nonexpansive, then for each $x \in K$ the asymptotic center of its orbit is a fixed point.*

Acknowledgement. The authors wish to thank Professor Edelstein for his valuable suggestions.

REFERENCES

1. F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. U.S.A., **54** (1965), pp. 1041–1044.

2. M. Edelstein, *The construction of an asymptotic center with a fixed point property*, Bull. Amer. Math Soc., **78** (1972), pp. 206–208.

3. M. Edelstein, *On a fixed-point property of a reflexive, locally uniformly convex Banach space*, C.R. Math Rep. Acad. Sci. Canada, **4** (1982), pp. 111–116.

4. M. Edelstein and M. T. Kiang, *On ultimately nonexpansive semigroups*, Pac. J. of Math., **101** (1982), pp. 93–102.

5. D. Göhde, *Zum prinzip der kontraktiven Abbildung*, Math. Nachr., **30** (1965), pp. 251–258.

6. K. Goebel and W. A. Kirk, *A fixed point theorem for asymptotically-nonexpansive mappings*, Proc. Amer. Math Soc., **35** (1972), pp. 171–174.

7. K. Goebel and S. Reich, *"Uniform Convexity, Hyperbolic Geometry and Nonexpansive mappings"*, Marcel Dekker, New York, 1984.

8. W. A. Kirk, *Fixed point theory for nonexpansive mappings II*, Contemporary Mathematics, Vol. 18 (1983), pp. 121–140.

9. S. Reich, *The fixed point property for nonexpansive mappings I, II*, Amer. Math. Monthly, **83** (1976), pp. 266–268; **87** (1980), pp. 292–294.

10. D. Tingley, *An asymptotically nonexpansive commutative semigroup with no fixed points*, (to appear).

DEPARTMENT OF MATHEMATICS AND COMPUTING SCIENCE,
SAINT MARY'S UNIVERSITY
HALIFAX, N.S. CANADA B3H 3C3

DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTING SCIENCE
DALHOUSIE UNIVERSITY
HALIFAX, N.S. CANADA B3H 4H8