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Cyclic Groups and the Three Distance Theorem

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Abstract. We give a two dimensional extension of the three distance theorem. Let θ be in $\mathbf{R}^2$ and let q be in $\mathbf{N}$. There exists a triangulation of $\mathbf{R}^2$ invariant by $\mathbf{Z}^2$-translations, whose set of vertices is $\mathbf{Z}^2 + \{0, \theta, \dots, q\theta\}$, and whose number of different triangles, up to translations, is bounded above by a constant which does not depend on θ and q.

1 Introduction

Let θ be in $\mathbf{R}$ and let q be in $\mathbf{N}$. The points $\{0\}, \{\theta\}, \{2\theta\}, \dots, \{q\theta\}$ cut the unit interval $[0, 1[$ into $q + 1$ intervals having at most three lengths ($\{x\}$ denotes the fractional part of x). This property is known as the *three distance theorem* and was first proved by V. T. Sós in 1957; (see [32–35]). Closely related to this result is the *three gap theorem*: if ϕ is in $]0, 1[$, the gap between the successive integers n such that $\{n\theta\} < \phi$ takes at most three values as shown by N. B. Slater [29]. These two results can be deduced from the continued fraction expansion of θ, but there are also direct proofs. A survey of the different approaches can be found in [31] A relation with the combinatorics on words has recently been established by P. Alessandri and V. Berthé who provided a combinatorial proof of the three distance theorem (see [1]).

Extensions of the three distance theorem lead to many works. The first was found in 1976 by F. K. R. Chung and R. L. Graham; they showed that if $\theta_1, \dots, \theta_d$ are d real numbers, then the points $\{k_i\theta_i\}$, for $1 \le i \le d$ and $0 \le k_i \le q_i$, cut the interval $[0, 1[$ into intervals having at most $3d$ values (see [9, 22] for a very simple proof). Later in 1993, A. S. Geelen and R. J. Simpson found a two dimensional extension of the three distance theorem . Let θ_1, θ_2 be two real numbers and n_1, n_2 two positive integers. Then the points $\{k_1\theta_1 + k_2\theta_2\}$, $0 \le k_1 < n_1$, $0 \le k_2 < n_1$, cut the interval $[0, 1[$ into $n_1 n_2$ intervals having at most $\min\{n_1, n_2\} + 3$ lengths (see [7, 14] for a d-dimensional version of Geelen and Simpson's result).

There are more abstract ways to generalize the three distance theorem . Endow $\mathbf{R}^2$ with the lexicographic order ($(x, y) \prec (x', y')$ means $y < y'$ or $y = y', x < x'$). Let Λ be a lattice in $\mathbf{R}^2$, $I \subset \mathbf{R}$ a bounded interval and $\Lambda(I) = \{(x, y) \in \Lambda : x \in I\}$. M. Langevin [21] proved that there exists a basis u, v of Λ such that for all w in $\Lambda(I)$, the next point in $\Lambda(I)$ for the lexicographic order, is one of the three points $w + u$, $w + v$ or $w + u + v$. Both the three distance theorem and the three gap theorem can be recovered by Langevin's result. With a growing abstraction, E. Fried and V. T. Sós [12] generalize Langevin's theorem to some ordered abelian groups.

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Kronecker sequences are a natural generalization of the sequences $(n\theta \mod 1)_{n\geq 0}$. Let q be a positive integer and let Θ be in the d-dimensional torus $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$. Consider the finite sequence $k\Theta$, $0 \leq k \leq q$. The (classical) discrepancy of this finite sequence has been extensively studied (see [11, pp. 66–90]) but less is known about its local conformation. Extension of the three distance theorem should give more information about it. In order to formulate a $\mathbf{T}^d$-three distance theorem, we have to replace the intervals. In [6], Voronoï's regions were chosen instead of the intervals. It leads to a partial extension of the three distance theorem to the d-dimensional torus; the extension holds only for a subsequence $q = q_n - 1$ of the sequence of all positive integers. This subsequence $(q_n)_{n\geq 0}$ is the sequence of all best simultaneous Diophantine approximations of Θ with respect to the Euclidean norm (see the definition below). The appearance of best approximation is easy to understand. On the one hand, best simultaneous approximations of Θ can be seen as a multidimensional continued fraction expansion and on the other hand, the three distance theorem can be strengthened for some particular q: let θ be a real number and let $\theta = [a_0; a_1, \dots, a_n, \dots]$ be its continued fraction expansion. Let q_n be the denominator of the rational $[0; a_1, \dots, a_n]$. Then for all $a \in \{1, \dots, a_{n+1}\}$, the points $\{k\theta\}$, $0 \leq k < aq_n + q_{n-1}$, cut the interval $[0, 1[$ into intervals having at most two lengths. We shall refer to this result as the "two distance theorem".

In the present work, we are mainly interested in two-dimensional extensions of the three distance theorem. Intervals are replaced by triangles in a $\mathbf{Z}^2$-invariant triangulation. We give two main results. The first corresponds to the two distance theorem and holds only for the q of the shape $q = q_n - 1$ where q_n is a best approximation. It improves the previous result about Voronoï's regions [6]. The second corresponds to the three distance theorem and holds for all positive integers q.

These two main results need a very simple lemma on translations in cyclic groups (Lemma 3.1) similar to Liang's proof of Chung and Graham's theorem [22]. By the way, we use this lemma in order to give another proof of the one dimensional two distance theorem and also of one known property of length words associated with the two distance theorem (see [26, 28]). In fact, our first idea to prove Theorem 1.3, was to use this property of length words. But it does not seem to work, whereas the lemma on cyclic groups proved to be efficient.

Another important ingredient of the proofs of both Theorems 1.2 and 1.3 is an extension to best Diophantine approximations of a property of continued fractions (§7, Theorem 7.1 for $d = 2$ and Theorem 7.2 for $d \geq 3$), a result which may be of independent interest.

1.1 Statement of the Two Main Results

Endow $\mathbf{R}^d$ with the usual Euclidean norm, and for every x in $\mathbf{R}^k$ set

$$\|x\| = \inf\{|x - n| : n \in \mathbf{Z}^d\}.$$

Clearly $\|\cdot\|$ induces a distance on the d-dimensional torus, which we also denote $\|\cdot\|$.

Definition 1.1 Let Θ be in $\mathbf{T}^d$. A positive integer q is a best approximation of Θ if $\|k\Theta\| > \|q\Theta\|$ for every integer k with $0 < k < q$. Let θ be in $\mathbf{R}^d$. A positive integer q is a best approximation of θ if it is a best approximation of its projection Θ in the d-dimensional torus, that is, if $\|k\theta\| > \|q\theta\|$ for every integer k with $0 < k < q$.

Best (Diophantine) approximations were introduced by C. A. Rogers [27] and first studied as a multidimensional continued fraction expansion by J. C. Lagarias [15–20], (see also [5,8]). Let θ be in $\mathbf{R}^d \backslash \mathbf{Q}^d$. Arranging the set of best approximations of θ in ascending order, we get an increasing sequence $(q_n)_{n\in\mathbf{N}}$ of positive integers starting with $q_0 = 1$.

Theorem 1.2 *Let θ be in $\mathbf{R}^2\backslash\mathbf{Q}^2$ such that $\mathbf{Z}^2 + \mathbf{Z}\theta$ is dense in $\mathbf{R}^2$. For all n in $\mathbf{N}$, there exists a $\mathbf{Z}^2$-invariant triangulation of $\mathbf{R}^2$ whose set of vertices is $\mathbf{Z}^2+\{0, \theta, \dots, (q_n-1)\theta\}$ and with only 6 different triangles up to translations. Furthermore, the diameters of the triangles go to zero when n goes to infinity.*

Theorem 1.3 *There exists an absolute effective constant K such that for all θ in $\mathbf{R}^2\backslash\mathbf{Q}^2$ such that $\mathbf{Z}^2 + \mathbf{Z}\theta$ is dense in $\mathbf{R}^2$ and for all integers $q \geq 1$, there exists a $\mathbf{Z}^2$-invariant triangulation of $\mathbf{R}^2$ whose set of vertices is $\mathbf{Z}^2 + \{0, \theta, \dots, q\theta\}$ and with at most K different triangles up to translations. Furthermore, the diameters of the triangles go to zero when q goes to infinity.*

Let $\theta = (\theta_1, \theta_2)$ be in $\mathbf{R}^2$. The topological hypothesis, $\mathbf{Z}^2+\mathbf{Z}\theta$ is dense in $\mathbf{R}^2$, is equivalent by Kronecker's theorem to an algebraic hypothesis: θ_1, θ_2 and 1 are linearly independent over $\mathbf{Q}$.

The proof of Theorem 1.2 is given in Section 9. Sections 3, 6, 7 (Theorem 7.1 only) and Section 8 (Lemma 8.4 only) are required for its proof. The rather long proof of Theorem 1.3 is given in Section 10, again, Sections 3, 6, 7 (Theorem 7.1 only) and Section 8 (Lemmas 8.3 and 8.4) are required for its proof.

2 Notations

Numbers: We denote the fractional part of a real x by $\{x\}$ and the lowest integer greater or equal than x by $[x]$. Therefore, $x = [x] + \{x\}$.

Geometry: The usual Euclidean norm of an element x of $\mathbf{R}^d$ is denoted by $|x|$. The distance to the nearest lattice point of an element x of $\mathbf{R}^d$ is denoted by $\|x\| = \inf\{|x-n| : n \in \mathbf{Z}^d\}$. The scalar product of two elements x and y of $\mathbf{R}^d$ is denoted by $x \cdot y$. The angle of two elements x and y of $\mathbf{R}^2$ is denoted by $\angle(x, y)$. The segment joining to elements x and y of $\mathbf{R}^d$ is denoted by $[x, y]$, $[x, y] = \{tx + (1-t)y : t \in [0, 1]\}$. The open segment (semi-open) is denoted by $]x, y[$ ($[x, y[$ or $]x, y]$).

Torus: $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$. θ will always denote an element of $\mathbf{R}^d$ and Θ its projection in $\mathbf{T}^d$. Let X be in $\mathbf{T}^d$. The distance of X to 0 is denoted by $\|X\| = \inf_{x\in X} |x|$. Therefore, the distance between two elements X and Y of $\mathbf{T}^d$ is $\|X - Y\|$.

Sets and metric spaces: The cardinal number of a set E is denoted by $|E|$. Let (E, d) be a metric space, let F be a subset of E and let a be an element of E. The distance between a and F is denoted by $d(a, F)$. We define $r(F) = \inf\{d(x, y) : x, y \in F$ and $x \neq y\}$ and $e(F) = \sup\{d(x, F) : x \in E\}$.

Graphs: Let $\mathcal{G}$ be a planar graph. The set of vertices of $\mathcal{G}$ is denoted by $\mathcal{V}(\mathcal{G})$, the set of connected components of $\mathbf{R}^2 \setminus \bigcup_{[A,B]\in\mathcal{G}} [A, B]$ is denoted by $\mathcal{C}(\mathcal{G})$ and the set $\{\overrightarrow{AB} : A$ and B are vertices of $\mathcal{G}$, and there exists ω in $\mathcal{C}(\mathcal{G})$ such that $A, B \in \partial\omega\}$ is denoted by $\mathcal{E}(\mathcal{G})$.

3 Cyclic Groups

For a given irreducible fraction $r = p/q$, let us place the points $\{kr\}$, $0 \leq k < q$, in the unit interval. We get the sequence i/q, $0 \leq i < q$, and each fraction i/q is equal to a $\{kr\}$ where $k = k(i)$ is an integer in $\{0, \dots, q - 1\}$. In this particular case, the two distance theorem means that the differences $k(i + 1) - k(i)$, $0 \leq i < q$, take two values. We can restate this property in a cyclic group $(G, \cdot)$ generated by an element a. Each x in G is of the shape $x = a^k$ with $0 \leq k < |G|$, set $\log(x) = k$. For b fixed in G, the differences $\log(bx) - \log(x)$, x in G, take two values. In the sequel, we shall use not only one translation $x \to bx$, but several simultaneously.

Lemma 3.1 *Let G be a cyclic group with generator a, and let n be the cardinal number of G.*

(i) *For all b in G, the number* $\log(bx) - \log(x)$, *$x \in G$, have at most two possible values (exactly two unless $b = e$).*

(ii) *For all m in $\mathbf{N}^*$ and for all $(b_1, \dots, b_m)$ in G^m, there exists a partition of $\{0, \dots, n - 1\}$ into $m + 1$ intervals $I_0, \dots, I_m$ such that for all $j \in \{0, \dots, m\}$, the m-tuple* $(\log(b_1 a^q) - \log(a^q), \dots, \log(b_m a^q) - \log(a^q))$ *does not depend on q in I_j.*

Proof (i) If $\log(b) + \log(x) < n$, then $\log(bx) = \log(b) + \log(x)$ and $\log(bx) - \log(x) = \log(b)$. If $\log(b) + \log(x) \geq n$, then $\log(bx) = \log(b) + \log(x) - n$ and $\log(bx) - \log(x) = \log(b) - n$.

(ii) Arrange the points $n_i = n - \log(b_i)$ in increasing order, $n_{i_1} \leq n_{i_2} \leq \cdots \leq n_{i_m}$. Set $n_{i_0} = 0$ and $n_{i_{m+1}} = n$. Let p be in $\{0, \dots, m\}$ and x be in G such that $\log(x) \in [n_{i_p}, n_{i_{p+1}}[$. For $q \in \{1, \dots, m\}$, $\log(b_{i_q}) + \log(x) < n \Leftrightarrow \log(x) < n - \log(b_{i_q}) = n_{i_q}$, therefore

$$\log(b_{i_q} x) - \log(x) = \begin{cases} \log(b_{i_q}) & \text{for } q > p, \\ \log(b_{i_q}) - n & \text{for } q \leq p. \end{cases}$$

It follows that $(\log(b_1 x) - \log(x), \dots, \log(b_m x) - \log(x))$ is constant on each interval $I_p = [n_{i_p}, n_{i_{p+1}}[$ and hence takes at most $m + 1$ different values. ■

We shall need a slightly more sophisticated result.

Lemma 3.2 *Let G be a commutative group, H a subgroup of G and a in G. Suppose G/H is a finite cyclic group generated by aH. Let n be the cardinal number of G/H.*

(i) *For each x in G there exist a unique integer* $\log(x)$ *in* $\{0,\dots,n-1\}$ *and a unique* $h(x)$ *in H such that* $x = a^{\log x}h(x)$.

(ii) *For all m in $\mathbf{N}^*$ and for all $(b_1,\dots,b_m)$ in G^m, there exists a partition of $\{0,\dots,n-1\}$ into $m+1$ intervals $I_0,\dots,I_m$ such that for all j in $\{0,\dots,m\}$ and all i in $\{1,\dots,m\}$, there exist $n_{i,j}$ in $\mathbf{Z}$ and $h_{i,j}$ in H such that for all $x \in G$ and all $i \in \{1,\dots,m\}$,* $\log(x) \in I_j$ *implies* $\log(b_i x) - \log(x) = n_{i,j}$ *and* $h(b_i x)h(x)^{-1} = h_{i,j}$.

Proof (i) is straightforward. To prove (ii), we use the previous lemma with the group G/H which is generated by aH. There exist intervals $I_0,\dots,I_m$ such that for all i, $\log((b_iH)(a^qH)) - \log(a^qH)$ is independent of q in I_j. Since for all x in G, $\log(xH) = \log(x)$, $\log(b_i a^q) - \log(a^q)$ is independent of q in I_j. Let $n_{i,j}$ be the value of $\log(b_i a^q) - \log(a^q)$ on I_j. For x in G, we have $b_i x = b_i a^{\log x}h(x) = a^{\log(b_i x)}h(b_i x)$, therefore

$$h(b_i x)h(x)^{-1} = b_i a^{-(\log(b_i x) - \log x)} = b_i a^{-n_{i,j}} = h_{i,j}$$

for all x with $\log(x) \in I_j$. ■

4 Proof of the Two Distance Theorem

This paragraph is not needed to prove Theorem 1.2 and 1.3; its aim is to give simple consequences of previous lemmas. The two distance theorem can be stated without using the continued fraction expansion:

Proposition 4.1 *Let θ be in $\mathbf{R}\backslash\mathbf{Z}$ and let q be a positive integer such that $\{k\theta\} \neq 0$, $0 < k \le q$. Suppose either*

(i) *the interval $]0, \{(q+1)\theta\}[$ does not contain any of the points $\{k\theta\}$, $1 \le k \le q$ (first case), or*

(ii) *the interval $]\{(q+1)\theta\}, 1[$ does not contain any of the points $\{k\theta\}$, $1 \le k \le q$ (second case).*

Let q_1, q_2 be the integers in $\{1,\dots,q\}$ such that $\{q_1\theta\} = \min\{\{k\theta\} : 1 \le k \le q\}$ and $\{q_2\theta\} = \max\{\{k\theta\} : 1 \le k \le q\}$. If $\{k_1\theta\} < \{k_2\theta\}$ are two consecutive points of $\{\{k\theta\} : 0 \le k \le q\}$, then $k_2 - k_1 = q_1$ or $-q_2$.

The usual statement of the two distance theorem follows from the proposition because the integers of the shape $q = aq_n + q_{n-1}$ are exactly those who satisfy the hypothesis of the proposition. Let us prove the proposition. In order to use Lemma 3.1 we replace θ by a rational approximation θ'. The key fact is the following lemma.

Lemma 4.2 *Let θ and q be as above. Set $\theta' = \theta - \frac{1}{q+1}\varepsilon$ where $\varepsilon = \{(q+1)\theta\}$ in the first case and $\varepsilon = \{(q+1)\theta\} - 1$ in the second case. Then θ' belongs to $\frac{1}{q+1}\mathbf{Z}$ and the points $\{k\theta\}$, $0 \le k \le q$ are in the same order as the points $\{k\theta'\}$, $0 \le k \le q$,* i.e., *for all $k_1, k_2 \in \{0,\dots,q\}$, $\{k_1\theta\} < \{k_2\theta\} \Leftrightarrow \{k_1\theta'\} < \{k_2\theta'\}$.*

Notation Let θ be in $\mathbf{R}$. Set $E_q(\theta) = \{0, \{\theta\}, \dots, \{q\theta\}\}$.

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Proof of Lemma 4.2 Clearly $\theta' \in \frac{1}{q+1}\mathbf{Z}$. Let us see that the two cases reduce to the first. Indeed, suppose that $]\{(q+1)\theta\}, 1[\cap E_q(\theta) = \varnothing$. Note that $\{(q+1)\theta\} = 0$ is impossible, for it implies $\{\theta\} = 0$ and $\theta \in \mathbf{Z}$. Since for all integer k with $\{k\theta\} \neq 0$, $\{k\theta\} + \{-k\theta\} = 1$, the image of set $E_q(\theta)\backslash\{0\}$ by the symmetry $x \to 1-x$ is the set $E_q(-\theta)\backslash\{0\}$. It follows that $]0, \{(q+1)(-\theta)\}[\cap E_q(-\theta) = \varnothing$. Furthermore,

$$(-\theta)' = (-\theta) - \frac{\{(q+1)(-\theta)\}}{q+1} = -\Big(\theta + \frac{1-\{(q+1)\theta\}}{q+1}\Big) = -\theta'.$$

Thus it suffices to use the first case with $-\theta$.

Now, assume that $\big]0, \{(q+1)\theta\}\big[\cap E_q(\theta) = \varnothing$. The real number ε is non negative. Let us show that $k\theta - \{k\theta\} = k\theta' - \{k\theta'\}$, for $0 \leq k \leq q$. By definition of the fractional part, $k\theta - \{k\theta\}$ is an integer and

$$k\theta' - (k\theta - \{k\theta\}) = -\frac{k\varepsilon}{q+1} + \{k\theta\} \leq \{k\theta\} < 1.$$

Moreover, since $\{k\theta\} \geq \{(q+1)\theta\} = \varepsilon$,

$$k\theta' - (k\theta - \{k\theta\}) \geq -\frac{k\varepsilon}{q+1} + \varepsilon \geq 0.$$

Thus the integer $k\theta - \{k\theta\}$ is the integer part of $k\theta'$ and is equal to $k\theta' - \{k\theta'\}$. We must show that $\{k_1\theta\} < \{k_2\theta\} \Leftrightarrow \{k_1\theta'\} < \{k_2\theta'\}$. Suppose on the contrary that $\{k_1\theta\} - \{k_2\theta\} < 0$ and $\{k_1\theta'\} - \{k_2\theta'\} \geq 0$. On making use of the equality $k\theta - \{k\theta\} = k\theta' - \{k\theta'\}$ for $k = k_1$ and k_2, we get

$$0 \leq \{k_1\theta'\} - \{k_2\theta'\} = \{k_1\theta\} - \{k_2\theta\} - (k_1 - k_2)\frac{\varepsilon}{q+1} < (k_2 - k_1)\frac{\varepsilon}{q+1}.$$

Therefore, $k_2 > k_1$, and $-\varepsilon < (k_1 - k_2)\frac{\varepsilon}{q+1} \leq \{k_1\theta\} - \{k_2\theta\} < 0$. It follows that

$$\{k_2\theta\} - \{k_1\theta\} \in]0, \varepsilon[= \big]0, \{(q+1)\theta\}\big[,$$

but this is impossible, for it would imply that $\{(k_2 - k_1)\theta\} = \{\{k_2\theta\} - \{k_1\theta\}\} = \{k_2\theta\} - \{k_1\theta\} \in]0, \{(q+1)\theta\}[$. ∎

Proof of Proposition 4.1 By the lemma, we can prove the proposition with θ' instead of θ. The projection of $E_q(\theta')$ in $\mathbf{T}^1 = \mathbf{R}/\mathbf{Z}$ is a subgroup G of $\mathbf{T}^1$. On the one hand, G is generated by the projection Θ' of θ' and on the other hand, G is generated by the projection of β of $\frac{1}{q+1}$. If $\{k_1\theta'\}$ and $\{k_2\theta'\}$ are consecutive points of $E_q(\theta')$, then $k_2\Theta' - k_1\Theta' = \beta$. Therefore, by Lemma 3.1 with $a = \Theta'$ and $b = \beta$, $k_2 - k_1$ takes two values. To determine these values, we just have to consider the two cases $k_1 = 0, k_2 = q_1$ and $k_1 = q_2, k_2 = 0$. ∎

A natural question arises: given an irrational number θ, what are the denominators of the rationals r such that the points $\{k\theta\}$, $0 \leq k \leq q$ and the points $\{kr\}$, $0 \leq k \leq q$ are in the same order?

Proposition 4.3 *Let θ be in $\mathbf{R}\backslash\mathbf{Q}$ and let q be in $\mathbf{N}$. There exists a in $\mathbf{Z}$ with* $\gcd(a, q+1) = 1$ *such that the points of $E_q(\theta)$ and $E_q(\frac{a}{q+1})$ are in the same order if and only if one of the sets*

$$E_q(\theta) \cap \,]0, \{(q+1)\theta\}[\quad \textit{and} \quad E_q(\theta) \cap \,]\{(q+1)\theta\}, 1[$$

is empty.

Proof Set $\theta' = \frac{a}{q+1}$. Let q_1 and q_2 be the elements of $\{1, \dots, q\}$ such that the intervals of $\mathbf{T}^1$, $]0, q_1\Theta'[\cap E_q(\Theta')$ and $]q_2\Theta', 0[\cap E_q(\Theta')$ are empty. By our assumption, the sets $]0, q_1\Theta[\cap E_q(\Theta)$ and $]q_2\Theta, 0[\cap E_q(\Theta)$ are also empty. The integers q_1 and q_2 are one sided best approximations of both Θ and Θ' and since $(q+1)\Theta' = 0$, we have $q+1 = q_1 + q_2$, therefore $q+1$ is also a one sided best approximation of Θ. It follows that $E_q(\theta) \cap \,]0, \{(q+1)\theta\}[$ or $E_q(\theta) \cap \,]\{(q+1)\theta\}, 1[$ is empty.

As before, it suffices to consider the case $E_q(\theta) \cap \,]0, \{(q+1)\theta\}[\,= \varnothing$. Set $\varepsilon = \{(q+1)\theta\}$ and $\theta' = \theta - \frac{1}{q+1}\varepsilon$. We have $(q+1)\theta' = (q+1)\theta - \varepsilon \in \mathbf{Z}$, hence $\theta' = \frac{a}{q+1}$ with $\in \mathbf{Z}$. By the previous lemma, the points $\{k\theta'\}$, $0 \le k \le q$, are in the same order as the points $\{k\theta\}$, $0 \le k \le q$ and since the points $\{k\theta\}$, $0 \le k \le q$, are all distinct, by the previous lemma the points $\{k\theta'\}$, $0 \le k \le q$, are all distinct. Now all these points are in $[0, 1[$, therefore the points $k\Theta'$, $0 \le k \le q$ are distinct in $\mathbf{T}^1$ and $\gcd(a, q+1) = 1$. ■

5 Circular Length Words

Let $\mathcal{A}$ be a finite alphabet and let I be an interval of $\mathbf{Z}$. Let us recall the definition of a C-balanced word $w : I \to \mathcal{A}$. We say w is *C-balanced* if the difference between the numbers of occurrences of any letter in two subwords of w of the same length is at most C.

Let θ be in $\mathbf{R}\backslash\mathbf{Q}$ and let Θ be its projection in $\mathbf{T}^1$. Let $(a_n)_{n\ge1}$ be the partial quotients of θ and $(q_n)_{n\ge0}$ the denominators of the convergents to θ. Fix an integer q of the shape $q = q_{n-1} + aq_n - 1$, $0 \le a \le a_{n+1}$. We can look at the points $\{k\theta\}$, $0 \le k \le q$ or at the points $k\Theta$, $0 \le k \le q$. Either way, the two distance theorem means that we have $(q+1)$ intervals of two different lengths. We associate to this configuration a word $w = (w_i)_{0\le i\le q} = w(\theta, q)$ of $q+1$ letters in the $\{0, 1\}$ alphabet:

> Choose the usual orientation on $\mathbf{T}^1$. When $k_2\Theta$ is the successor of $k_1\Theta$, $k_2 - k_1$ takes two values δ_0 and δ_1. Describe $\mathbf{T}^1$ starting at 0, and set $w_i = 0$ if the i-th interval is of the shape $[k\Theta, (k+\delta_0)\Theta]$ and 1 otherwise. Identifying $\{0, \dots, q\}$ with $\mathbf{Z}/(q+1)\mathbf{Z}$, the word w may be seen as a circular word.

It is known that w is a Sturmian word, that is, a word with exactly $k+1$ different subwords of lengths k (see [25, Ch. 6]), and therefore a 1-balanced word (see [28]). This balance property can be recovered directly with our approach.

Proposition 5.1 *Let I and J be two intervals of $\mathbf{Z}/(q+1)\mathbf{Z}$ of the same length. Then*

$$\big| |\{i \in I : w_i = 1\}| - |\{i \in J : w_i = 1\}| \big| \le 1.$$

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Proof Let l be the length of I and J. By properties of continued fraction and Lemma 4.2, we know that $w(\theta, q) = w(\theta', q)$ where $\theta' = \frac{a}{q+1}$ with $\gcd(q+1, a) = 1$. Let β be the projection of $\frac{1}{q+1}$ in $\mathbf{T}^1$. We have $\{0, \Theta', \dots, q\Theta'\} = \{0, \beta, \dots, q\beta\}$. We are going to use Lemma 3.1, the first lemma on cyclic groups, with the additive group $G = \{0, \Theta', \dots, q\Theta'\}$, $a = \Theta$, $b = l\beta$. For each $i \in \mathbf{Z}/(q+1)\mathbf{Z}$ set $k(i) = \log(i\beta)$. Suppose $I = \{i, \dots, i+l-1\}$, then the sub-word $w_i \cdots w_{i+l-1}$ is determined by the difference $k(i+1) - k(i), \dots, k(i+l) - k(i+l-1)$. We have

$$\begin{aligned}\log(i\beta + b) - \log(i\beta) &= k(i+l) - k(i)\\ &= \bigl|\{j \in I : w_j = 0\}\bigr|\,\delta_0 + \bigl|\{j \in I : w_j = 1\}\bigr|\,\delta_1\end{aligned}$$

and by the first lemma on cyclic groups, it has only two possible values when i describes $\mathbf{Z}/(q+1)\mathbf{Z}$. Therefore $\bigl|\{j \in I : w_j = 1\}\bigr|$ has two possible values only. Furthermore, if $I = \{i, \dots, i+l-1\}$ and $J = \{i+1, \dots, i+l\}$, then

$$\Bigl|\,|\{j \in I : w_j = 1\}| - |\{j \in J : w_j = 1\}|\,\Bigr| \le 1,$$

hence $|\{j \in I : w_j = 1\}|$, $i \in \mathbf{Z}/(q+1)\mathbf{Z}$, takes two values whose difference is 1. ■

Remarks (i) Can we extend the previous proposition to the two dimensional case? Consider a lattice $\mathbf{L}$ of $\mathbf{R}^2$ with a basis (e_1, e_2) and Λ a sublattice of $\mathbf{L}$ such that $\mathbf{L}/\Lambda$ is a cyclic group of order q. Fix θ a generator of this group. For each basic parallelogram $P_a = (a, a+e_1, a+e_2, a+e_1+e_2)$, $a \in \mathbf{L}$, set

$$\delta(a) = (k(a+e_1) - k(a), k(a+e_2) - k(a), k(a+e_1+e_2) - k(a)),$$

where $k(x)$ is the unique integer in $\{0, \dots, q-1\}$ such that $x + \Lambda = k(x)\theta + \Lambda$. By Lemma 3.2, $\delta(a)$ has at most four different values, $\delta_1, \delta_2, \delta_3, \delta_4$. Therefore we can label each parallelogram P_a with one of the four symbols $\delta_1, \delta_2, \delta_3$ or δ_4. This gives a two dimensional Λ-periodic word $w\colon \mathbf{L} \to \{1, 2, 3, 4\}$, $w(a) = \delta(a)$. With the isomorphism $(u, v) \in \mathbf{Z}^2 \to ue_1 + ve_2 \in \mathbf{L}$, we can see w as word on $\mathbf{Z}^2$. Now a $n \times m$-subword of w is any restriction of w to a rectangle $(u, v) + \{0, \dots, m-1\} \times \{0, \dots, n-1\} \subset \mathbf{Z}^2$. This raises to the following question.

> Is the difference between the numbers of occurrences of δ_1 in two $n \times m$-subwords of w, bounded by an universal constant?

If we follow the proof of Proposition 5.1, we count the number of new letters when we move from an $n \times m$-subword to an adjacent $n \times m$-subword. However, this number can be n or m and not just 1 as in the one dimensional case. This means that a new idea is necessary to answer the previous question.

(ii) We can note that for a word $w\colon \{1, \dots, n\} \to \{0, 1\}$, C-balancedness for $C > 1$ without further assumption is a property far weaker than 1-balancedness: the number of 1-balanced words of length n is polynomial in n and the number of 2-balanced word of length n is exponential in n [23].

(iii) V. Berthé and F. Tijdeman have studied 1-balancedness for multidimensional words. They have showed that if $d \ge 2$, a 1-balanced word $w\colon \mathbf{Z}^d \to \{0, 1\}$ must satisfy a very strong condition on the frequencies of the letters 0 and 1: these frequencies belong to a finite subset of the rational numbers [3].

6 Best Approximation in $\mathbf{R}^d$ and $\mathbf{T}^d$

Let $\theta \in \mathbf{R}^d$. Arranging the set of best approximation of θ in ascending order, we get an increasing sequence $(q_n)_{n\geq 0}$ of positive integers starting with $q_0 = 1$. For any positive integer n, let ε_n be the vector of $\mathbf{R}^d$ and P_n be the integer d-tuple such that

$$q_n\theta = P_n + \varepsilon_n \quad \text{and} \quad |\varepsilon_n| = \|q_n\Theta\|.$$

Set

$$\theta_n = \theta - \frac{1}{q_n}\varepsilon_n = \frac{1}{q_n}P_n \quad \text{and} \quad r_n = |\varepsilon_n|.$$

Then θ_n is the rational approximation of θ corresponding to the best approximation q_n. We consider the lattice $\Lambda_n = \mathbf{Z}^d + \mathbf{Z}\theta_n$, which is a lattice included in $\mathbf{Q}^d$ since θ_n has rational coordinates.

The following lemma is easy; its proof can be found in [6].

Lemma 6.1 *The sub-group $\langle\Theta_n\rangle$ of $\mathbf{T}^d$ generated by Θ_n, has exactly q_n elements, that is, $k\Theta_n$ is non-zero for all $0 \leq k \leq q_n - 1$. Furthermore, $\|p\theta - p\theta_n\| \leq r_n$ for all $0 \leq p \leq q_n - 1$. Moreover, the lattice Λ_n has determinant $\frac{1}{q_n}$ and its first minimum $\lambda_{1,n}$ satisfies $2r_{n-1} \geq \lambda_{1,n} \geq r_{n-1}/2$.*

7 A Good Basis of Λ_n

Let $d = 1$ and let $q = q_n - 1$. By Lemma 4.2, the points $\{k\theta\}$ and the points $\{k\theta_n\}$, $0 \leq k \leq q$ are in the same order. In fact the points of $\{k\theta_n\}$ divide $[0, 1[$ in q_n intervals of length $1/q_n$. Each of these intervals contains the corresponding points of $E_q(\theta)$, More precisely,

$$\text{for all } k \text{ in } \{0, \dots, q\} \begin{cases} k\Theta \in [k\Theta_n, k\Theta_n + \frac{1}{q_n}[& \text{if } \varepsilon_n \geq 0, \\ k\Theta \in]k\Theta_n - \frac{1}{q_n}, k\Theta_n] & \text{if } \varepsilon_n \leq 0. \end{cases}$$

The next two theorems extend this property to $d \geq 2$, and the first is an important ingredient in the proofs of Theorems 1.2 and 1.3. When $d = 1$, the lattice Λ_n cuts $\mathbf{R}$ into intervals of length $\frac{1}{q_n}$, but when $d \geq 2$, there are many tilings associated with the lattice Λ_n. To each basis $e_{1,n}, \dots, e_{d,n}$ of Λ_n, corresponds a tiling

$$P + \{t_1e_{n,1} + \cdots + t_ne_{n,d} : t_1, \dots, t_d \in [0, 1[\}, \quad P \in \Lambda_n.$$

We are looking for a basis of Λ_n such that each piece of the tiling contains exactly one point of the shape $k\theta + P$, $0 \leq k < q_n$, $P \in \mathbf{Z}^d$. Moreover, we wish to find a basis whose vectors are as short as possible. We are able to find such a basis when $d = 2$ (Theorem 7.1). Nevertheless, when $d \geq 3$, we are only able to find a basis whose vectors are not too long (Theorem 7.2).

Theorem 7.1 *With the notations of the last section, suppose $d = 2$. There exists a basis $e_{n,1}, e_{n,2}$ of Λ_n such that*

(i) *$i = 1,2$, $|e_{n,i}| \le \lambda_{n,1} + \lambda_{n,2}$, where $\lambda_{n,1}$, $\lambda_{n,2}$ are the minima of Λ_n;*
(ii) $|\sin\angle(e_{n,1}, e_{n,2})| \ge \sqrt{3}/8$;
(iii) *for all k in $\{0,\dots,q_n-1\}$ and all P in $\mathbf{Z}^d$,*

$$k\theta + P = k\theta_n + P + \frac{k}{q_n}\varepsilon_n \in k\theta_n + P + \{t_1 e_{n,1} + t_n e_{n,2} : t_1, t_2 \in [0,1[\}.$$

Proof *Step 1:* Let k be in $\{0,\dots,q_n-1\}$ and $x = k\theta_n + P$ be in $\Lambda_n \backslash \{0\}$. We have $x.\varepsilon_n < |x|^2$. Indeed, if $x.\varepsilon_n \ge |x|^2$, then $|\varepsilon_n| \ge |x|$ and

$$|x - \varepsilon_n|^2 = |\varepsilon_n|^2 - 2x.\varepsilon_n + |x|^2 \le |\varepsilon_n|^2$$

and therefore,

$$d(\varepsilon_n, x) \le |\varepsilon_n| = r_n \quad d(\varepsilon_n, x + \varepsilon_n) = |x| \le r_n.$$

By convexity, it follows that $d(\varepsilon_n, x + \frac{k}{q_n}\varepsilon_n) \le r_n$, and projecting in $\mathbf{T}^1$, we get

$$\|(q_n - k)\Theta\| = \|q_n\Theta - k\Theta\| \le d(\varepsilon_n, k\theta + P) = d\big(\varepsilon_n, x + \frac{k}{q_n}\varepsilon_n\big) \le r_n.$$

By definition of the best approximation, we have $k = 0$ and $x = P$. But, by definition of ε_n, $|\varepsilon_n|^2 \le |\varepsilon_n - Q|^2$, for all $Q \in \mathbf{Z}^2$, hence we have $\varepsilon_n.Q \le \frac{1}{2}|Q|^2$, which is false for $Q = P = x$. Note that this part of the proof works in every dimension.

Step 2: By Gauss reduction of a two dimensional lattice, there is a basis $e_{n,1}$, $e_{n,2}$ of Λ_n such that

$$|e_{n,1}| = \lambda_{n,1}, \quad |e_{n,2}| = \lambda_{n,2}.$$

Let α_1 and α_2 be the coordinates of ε_n in this basis. We can suppose α_1 and $\alpha_2 \ge 0$. Furthermore, since $|e_{n,1} \pm e_{n,2}|^2 \ge |e_{n,2}|^2$, we have $|\sin\angle(e_{n,1}, e_{n,2})| \ge \sqrt{3}/2$.

Step 3: Suppose $e_{n,1} \cdot e_{n,2} \ge 0$. By Step 1,

$$e_{n,1}^2 > e_{n,1}.\varepsilon_n = \alpha_1 e_{n,1}^2 + \alpha_2 e_{n,1}.e_{n,2} \quad \text{and} \quad e_{n,2}^2 > e_{n,2}.\varepsilon_n = \alpha_1 e_{n,1}.e_{n,2} + \alpha_2 e_{n,2}^2.$$

Since α_1 and α_2 are non negative, we get α_1 and $\alpha_2 < 1$, which proves the theorem in this case.

Step 4: Suppose $e_{n,1}.e_{n,2} < 0$ and $|e_{n,2}| \le 3|e_{n,1}|$. The vector ε_n has nonnegative coordinates in one of the bases $(e_{n,1}, e_{n,1} + e_{n,2})$ or $(e_{n,1} + e_{n,2}, e_{n,2})$. By Step 3, the coordinates of ε_n in one of these bases are in $[0,1[$. It remains to estimate the sinus. For the first basis, we see easily that

$$|\sin\angle(e_{n,1}, e_{n,1} + e_{n,2})| \ge \frac{1}{\sqrt{2}},$$

for the second basis, we can use a well-known formula in the triangle whose vertices are 0, $e_{n,2}$ and $e_{n,2} + e_{n,1}$

$$\frac{|\sin\angle(e_{n,2}, e_{n,1} + e_{n,2})|}{|e_{n,1}|} = \frac{|\sin\angle(e_{n,1}, e_{n,2})|}{|e_{n,1} + e_{n,2}|} \ge \frac{\sqrt{3}}{2} \times \frac{1}{|e_{n,1} + e_{n,2}|}.$$

Hence,

$$|\sin\angle(e_{n,1}, e_{n,1}+e_{n,2})| \geq \frac{\sqrt{3}}{2}\times\frac{1}{4}.$$

Step 5: Suppose $e_{n,1}\cdot e_{n,2} < 0$ and $|e_{n,2}| \geq 3|e_{n,1}|$. Let us show that $\alpha_2 < 1$. By Lemma 6.1, $|\varepsilon_n| \leq 2\lambda_{n,1} = 2|e_{n,1}|$. Hence,

$$\alpha_1^2|e_{n,1}|^2+\alpha_2^2|e_{n,2}|^2+2\alpha_1\alpha_2 e_{n,1}\cdot e_{n,2} \leq 4|e_{n,1}|^2.$$

Since $|\cos\angle(e_{n,1}, e_{n,2})| \leq \frac{1}{2}$ and $e_{n,1}\cdot e_{n,2} \leq 0$,

$$\alpha_1^2|e_{n,1}|^2+\alpha_2^2|e_{n,2}|^2-\alpha_1\alpha_2|e_{n,1}||e_{n,2}| \leq 4|e_{n,1}|^2.$$

Hence,

$$\frac{1}{2}(\alpha_1^2|e_{n,1}|^2+\alpha_2^2|e_{n,2}|^2) \leq 4|e_{n,1}|^2$$

and

$$\alpha_2^2 \leq 8\frac{|e_{n,1}|^2}{|e_{n,2}|^2} \leq 8\times\frac{1}{9} < 1.$$

Now we have $\alpha_1 < 1+\alpha_2$, otherwise, by Step 1,

$$\varepsilon_n\cdot e_{n,1} \leq |e_{n,1}|^2$$

and

$$\alpha_1|e_{n,1}|^2+\alpha_2 e_{n,1}\cdot e_{n,2} \leq |e_{n,1}|^2,$$
$$\alpha_2(e_{n,1}\cdot e_{n,2}) \leq -\alpha_2|e_{n,1}|^2.$$

Hence,

$$|e_{n,2}+e_{n,1}|^2 = |e_{n,1}|^2+|e_{n,2}|^2+2e_{n,1}.e_{n,2} < |e_{n,2}|^2,$$

which contradicts the equality $\lambda_{n,2} = |e_{n,2}|$. If $\alpha_1 \geq 1$, then $\varepsilon_n = \alpha_1 e_{n,1}+\alpha_2 e_{n,2} = (\alpha_1-\alpha_2)e_{n,1}+\alpha_2(e_{n,1}+e_{n,2})$ has nonnegative coordinates in the basis $(e_{n,1},\ e_{n,1}+e_{n,2})$, and as in Step 4, we easily see that $|\sin\angle(e_{n,1}, e_{n,1}+e_{n,2})| \geq \frac{1}{\sqrt{2}}$. ■

The next theorem is not necessary to prove Theorems 1.2 and 1.3, but may be of independent interest.

Theorem 7.2 *Suppose $d \geq 3$. There exists a basis $e_{n,1},\dots,e_{n,d}$ of Λ_n such that*

(i) *for all i in $\{1,\dots,d\}$, $|e_{n,i}| \ll \lambda_{n,d}$, where $\lambda_{n,d}$ is the d-th minimum,*
(ii) *for all k in $\{0,\dots,q_n-1\}$ and all P in $\mathbf{Z}^d$,*

$$k\theta+P = k\theta_n+P+\frac{k}{q_n}\varepsilon_n \in k\theta_n+P+\{t_1e_{n,1}+\cdots+t_ne_{n,d} : t_1,\dots,t_d \in [0,1[\}.$$

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Proof We need to find a basis of Λ_n such that the coordinates of ε_n are all in $[0,1[$.

Step 1: Let Λ be a lattice of $\mathbf{R}^d$, $e_1,\dots,e_d$ a basis of Λ and $\varepsilon = \sum_{i=1}^d a_i e_i$ a vector of $\mathbf{R}^d$ with nonnegative coordinates. We now prove that if $a_i \le N$ for $i = 1,\dots,d$, where $N \in \mathbf{N}^*$, then there exists a basis $e'_1,\dots,e'_d$ of Λ such that

$$\max\{|e'_i|, i = 1,\dots,d\} \le d^N \max\{|e_i|, i = 1,\dots,d\}$$

and ε has nonnegative coordinates in the new basis $e'_1,\dots,e'_d$ with only one coordinate $a_{i_0} \ge 1$, this coordinate being $\le N$. We proceed by induction on N. There is nothing to prove if $N = 0$. Let $N \ge 1$. Let $I = \{i : a_i \ge 1\}$ and choose i_0 such that

$$a_{i_0} = \min\{a_i : i \in I\}.$$

Set $f_{i_0} = e_{i_0} + \sum_{i \in I\setminus\{i_0\}} e_i$ and $f_i = e_i$ for $i \ne i_0$. We have

$$\varepsilon = \sum_{i \notin I\setminus\{i_0\}} a_i f_i + \sum_{i \in I\setminus\{i_0\}} (a_i - a_{i_0}) f_i.$$

There are two cases.

Case 1 If $a_{i_0} > N-1$, then $a_i - a_{i_0} < 1$ for all $I\setminus\{i_0\}$. It suffices to take $e'_1 = f_1,\dots,e'_d = f_d$, and we have

$$\max\{|e'_i|, i = 1,\dots,d\} = \max\{|f_i|, i = 1,\dots,d\} \le d \max\{|e_i|, i = 1,\dots,d\}.$$

Case 2 If $a_{i_0} \le N-1$, we apply induction hypothesis to the basis $f_1,\dots,f_d$. We get a basis $e'_1,\dots,e'_d$ with

$$\begin{aligned}\max\{|e'_i|, i = 1,\dots,d\} &\le d^{N-1}\max\{|f_i|, i = 1,\dots,d\}\\ &\le d^N \max\{|e_i|, i = 1,\dots,d\}.\end{aligned}$$

Step 2: We would like to use Step 1 with Λ_n and ε_n. Since the new basis given by Step 1 can have vectors as long as those of the initial basis, we must start with a basis $(e_1,\dots,e_d)$ of Λ_n such that

$$\max\{|e_i|, i = 1,\dots,d\} \ll \lambda_{n,d}.$$

The LLL algorithm (*cf.* [13, Theorem 5.3.13, p. 143]) gives us a basis $e_1,\dots,e_d$ of Λ_n such that $|e_1|\dots|e_d| \ll \det\Lambda_n$. By the Minkowski theorem on successive minima we get

$$\max\{|e_i|, i = 1,\dots,d\} \ll \lambda_{n,d}$$

where $\lambda_{n,d}$ is the last minimum of Λ_n. Furthermore, Babai [2] has proved that this basis verifies that for all $k \in \{1,\dots,d\}$, the sinus of the angle between e_k and the sub-space generated by the other vectors of the basis, is $\ge (\sqrt{3}/2)^d$. Then if $x =$

$\sum_{i=1}^{d} a_i e_i$, $|x| \geq (\sqrt{3}/2)^d \max\{|a_i||e_i| : i = 1, \dots, d\}$. By Lemma 6.1, $r_n = |\varepsilon_n| \leq r_{n-1} \leq 2\lambda_{n,1}$. It follows that the absolute values of the coordinates $a_1, \dots, a_d$ of ε_n in the LLL basis $e_1, \dots, e_d$, are all $\leq 2 \times (2/\sqrt{3})^d$. With Step 1 we can find a new basis $e'_1, \dots, e'_d$ such that $\max\{|e'_i|, i = 1, \dots, d\} \ll \lambda_{n,d}$ and $\varepsilon_n = \sum_{i=1}^{d} \alpha_i e'_i$ with $\alpha_i \in [0, 1[$ $i \geq 2$ and $\alpha_1 \in [0, C_d]$, where C_d depends only on d.

Step 3: We drop the primes and let $e_1, \dots, e_d$ be the basis found in Step 2. If $\alpha_1 < 1$ we have $\varepsilon_n \in \{t_1 e_1 + \cdots + t_n e_d : t_1, \dots, t_d \in [0, 1[\}$ which give (ii). So it remains to find a basis with $\alpha_1 < 1$. Take any $i \geq 2$ with $e_i \cdot e_1 < 0$. While $\alpha_1 > 1$ and $e_i \cdot e_1 < 0$, replace the vector e_i by $e_i + e_1$ and do not change the other vectors. After each step, the coordinates of ε_n in the new basis are $\alpha_1 - \alpha_i, \alpha_2, \dots, \alpha_d$ and the length of $e'_i = e_i + e_1$ is given by $e'_i \cdot e'_i = (e'_i + e_i) \cdot e_1 + e_i^2$. Since $e_i \cdot e_1 < 0$, we have $|e'_i| \leq |e_i|$ if $e'_i \cdot e_1 \leq 0$ and $|e'_i| \leq |e_i| + |e_1|$ if $e'_i \cdot e_1 \geq 0$. At the end, we get a basis $e'_1, \dots, e'_d$ such that for $i = 2, \dots, d$,

$$
\begin{gathered}
|e'_i| \leq |e_i| + |e_1|, \\
\alpha_i \in [0, 1[, \\
\alpha_1 \in [0, 1[\text{ or } e'_i \cdot e'_1 \geq 0,
\end{gathered}
$$

Step 4: If $\alpha_1 < 1$, the theorem is proved. In the other case, $e'_i \cdot e'_1 \geq 0$, $i = 2, \dots, d$. By Step 1 of the proof of Theorem 7.1, $e'_1 \cdot \varepsilon_n < e_1'^2$. Therefore,

$$
\alpha_1 e_1'^2 + \sum_{i=2}^{d} \alpha_i e'_i \cdot e'_1 < e_1'^2
$$

and $\alpha_1 < 1$. ∎

8 Graph and Triangulation

Definition 8.1 A triangulation $\mathcal{T}$ of $\mathbf{R}^2$ is a *countable set of triangles* (the convex hull of three points) whose union is $\mathbf{R}^2$ and such that the intersection of two distinct triangles T_1 and T_2 of $\mathcal{T}$ is either one edge of both T_1 and T_2, or one vertex of both T_1 and T_2. Furthermore, a bounded region of $\mathbf{R}^2$ meets only finitely many triangles of $\mathcal{T}$. The edges of $\mathcal{T}$ are the edges of all triangles of $\mathcal{T}$ and the vertices of $\mathcal{T}$ are the vertices of all triangles of $\mathcal{T}$. We say that a triangulation $\mathcal{T}$ is $\mathbf{Z}^2$-invariant if for all T in $\mathcal{T}$ and all v in $\mathbf{Z}^2$, $v + T$ is in $\mathcal{T}$. More generally if Λ is a lattice of $\mathbf{R}^2$, we say that $\mathcal{T}$ is Λ-invariant if for all T in $\mathcal{T}$ and all v in Λ, $v + T$ is in $\mathcal{T}$.

Definition 8.2 A planar graph is given by a countable set of nonoriented edges $\mathcal{G}$ verifying,

(i) all edges are segments of $\mathbf{R}^2$,
(ii) if two edges $[A, B]$ and $[C, D]$ of $\mathcal{G}$ meet *i.e.,* $[A, B] \cap [C, D] \neq \varnothing$, then $[A, B] \cap [C, D]$ is a common extremity of both $[A, B]$ and $[C, D]$ or $[A, B] = [C, D]$,
(iii) a bounded region of $\mathbf{R}^2$ meets only finitely many edges of $\mathcal{G}$.

The vertices of the graph $\mathcal{G}$ are the extremities of the edges of $\mathcal{G}$.

We say that a planar graph $\mathcal{G}$ is $\mathbf{Z}^2$-invariant if for all $[A, B]$ in $\mathcal{G}$ and all v in $\mathbf{Z}^2$, $[A, B] + v$ is in $\mathcal{G}$. More generally, if Λ is a lattice of $\mathbf{R}^2$, we say that $\mathcal{G}$ is Λ-invariant if for all $[A, B]$ in $\mathcal{G}$ and all v in Λ, $[A, B] + v$ is in $\mathcal{G}$.

Figures 1 and 3 below represent planar graphs whereas Figure 5 does not.

Obviously, if $\mathcal{T}$ is a triangulation, the set of edges of all triangles of $\mathcal{T}$ is a planar graph $\mathcal{G}$ and it is possible to get $\mathcal{T}$ back from $\mathcal{G}$.

We shall need two results on planar graphs and triangulations. The first lemma allows us to get a triangulation with the desired property of finiteness. The second lemma is on the homotopy of planar graph. Both lemmas will be proved in the appendix.

Notation Let $\mathcal{G}$ be a planar graph. Let us denote by $\mathcal{V}(\mathcal{G})$ the set of vertices of $\mathcal{G}$ and by $\mathcal{C}(\mathcal{G})$ the set of connected components of $\mathbf{R}^2 \setminus \bigcup_{[A,B]\in\mathcal{G}} [A, B]$. In the following, we shall always write $\mathbf{R}^2 \setminus \mathcal{G}$ instead of $\mathbf{R}^2 \setminus \bigcup_{[A,B]\in\mathcal{G}} [A, B]$.

Let us denote by $\mathcal{B}(\mathcal{G})$ the set of boundaries of elements of $\mathcal{C}(\mathcal{G})$, more precisely

$$\mathcal{B}(\mathcal{G}) = \{\mathcal{B} \subset \mathcal{G} : \exists \omega \in \mathcal{C}(\mathcal{G}),\ \partial\omega = \bigcup_{[A,B]\in\mathcal{B}} [A, B]\}.$$

Lemma 8.3 *Let $\mathcal{G}$ be a $\mathbf{Z}^2$-invariant planar graph. Suppose all elements of $\mathcal{C}(\mathcal{G})$ are bounded. Set*

$$\mathcal{E}(\mathcal{G}) = \{\overrightarrow{AB} : A \text{ and } B \text{ are vertices of } \mathcal{G}, \\ \text{and there exists } \omega \text{ in } \mathcal{C}(\mathcal{G}) \text{ such that } A, B \in \partial\omega\}.$$

If the set $\mathcal{E}(\mathcal{G})$ is finite, then there exists a $\mathbf{Z}^2$-invariant triangulation $\mathcal{T}$ of $\mathbf{R}^2$ whose set of vertices is the set of vertices of $\mathcal{G}$ and whose number of different triangles, up to translations, is less than a constant depending only on the cardinal number of $\mathcal{E}(\mathcal{G})$.

Lemma 8.4 *Let $\mathcal{G}$ and $\mathcal{G}'$ be two planar graphs with no edge reduced to one point. Suppose that all elements of $\mathcal{C}(\mathcal{G})$ are bounded and that there exists a bijection $A \in \mathcal{V}(\mathcal{G}) \to A' \in \mathcal{V}(\mathcal{G}')$ such that*

- $\{|A - A'| : A \in \mathcal{V}(\mathcal{G})\}$ *is bounded;*
- *for each $t \in [0, 1]$,*

$$\mathcal{G}(t) = \{[(1-t)A + tA', (1-t)B + tB'] : [A, B] \in \mathcal{G}\}$$

  is a planar graph with no edge reduced to one point;
- *the map $A \in \mathcal{V}(\mathcal{G}) \to (1-t)A + tA' \in \mathcal{V}(\mathcal{G}(t))$ is a bijection.*

Then the map $\mathcal{B} \in \mathcal{B}(\mathcal{G}) \to \{[A', B'] : [A, B] \in \mathcal{B}\} \in \mathcal{B}(\mathcal{G}')$ induced by the map $A \to A'$ is a bijection.

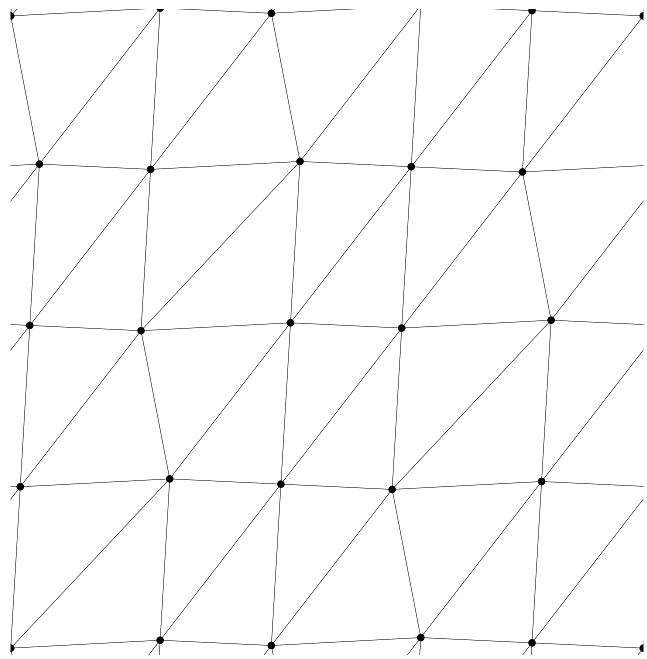

Figure 1: The triangulation given by Theorem 1.2

9 Two Distance Theorem in R^2, Proof of Theorem 1.2

Let θ be in $\mathbf{R}^2\backslash\mathbf{Q}^2$ such that $\mathbf{Z}^2+\mathbf{Z}\theta$ is dense in $\mathbf{R}^2$. We have to prove that for all n in $\mathbf{N}$, there exists a $\mathbf{Z}^2$-invariant triangulation of $\mathbf{R}^2$ whose set of vertices is $\mathbf{Z}^2+\{0,\theta,\dots,(q_n-1)\theta\}$, and with only six different triangles up to translations (see Figure 1). We also have to prove that the diameters of the triangles go to zero when n goes to infinity.

Let us outline the proof. The set of vertices $\mathbf{Z}^2+\{0,\theta,\dots,(q_n-1)\theta\}$ is very close to the lattice $\Lambda_n=\mathbf{Z}^2+\mathbf{Z}\theta_n$ where θ_n is the rational approximation of θ corresponding to the best approximation q_n (see §6). With the help of a basis (e_1,e_2) of Λ_n, it is not difficult to find a triangulation of $\mathbf{R}^2$ whose set of vertices is Λ_n; just take the collection of triangles

$$\mathrm{conv}(A,A+e_1,A+e_1+e_2),\quad \mathrm{conv}(A,A+e_2,A+e_1+e_2),\ A\in\Lambda_n.$$

There are only two kinds of triangles. Now each element A of Λ_n is of the shape $A=P+k\theta_n$ with $P\in\mathbf{Z}^2$ and is close to the element $A'=P+k\theta$ of $\mathbf{Z}^2+\{0,\theta,\dots,(q_n-1)\theta\}$. The map $A\to A'$ induces a map which sends each previous triangle T to a new triangle T'. A good choice of the basis (*cf.* Theorem 7.1) and Lemma 8.4 allow us to show that the set of new triangles is a triangulation. Next, Lemma 3.22 allows us to prove that there are at most six kinds of new triangles.

Proof Let θ be in $\mathbf{R}^2\backslash\mathbf{Q}^2$. Fix a best approximation q_n and choose a basis of Λ_n satisfying the conditions of Theorem 7.1. Consider the vectors $u_1 = e_{n,1}$, $u_2 = e_{n,2}$ and $u_3 = e_{n,1} + e_{n,2}$ of Λ_n. We define a first planar graph $\mathcal{G}$ by

$$\mathcal{G} = \{[A, A + u_i] : A \in \Lambda_n,\ i = 1, 2, 3\}.$$

Let $A = P + k\theta_n$ be in Λ_n where P is in $\mathbf{Z}^2$ and k in $\{0, \dots, q_n - 1\}$. Set $f(A) = \frac{k}{q_n}$, $A(t) = A + tf(A)\varepsilon_n$ for t in $[0, 1]$, $A' = P + k\theta = A(1)$. Set

$$\mathcal{G}' = \{[A', B'] : [A, B] \in \mathcal{G}\}, \quad \text{and} \quad \mathcal{G}(t) = \{[A(t), B(t)] : [A, B] \in \mathcal{G}, \}$$

for all t in $[0, 1]$. Note that

$$A' = A + \frac{k}{q_n}\varepsilon_n = P + k\theta_n + \frac{k}{q_n}\varepsilon_n = P + k(\theta_n + \frac{1}{q_n}\varepsilon_n) = P + k\theta.$$

Our aim is to prove that $\mathcal{G}'$ is a planar graph such that for all ω' in $\mathcal{C}(\mathcal{G}')$, $\partial\omega'$ is a triangle and the number of these triangles, up to translations, is at most 6.

It is obvious that for all ω in $\mathcal{C}(\mathcal{G})$, $\partial\omega$ is a triangle. Moreover, the map $A \in \mathcal{V}(\mathcal{G}) \to A' \in \mathcal{V}(\mathcal{G}')$ is clearly a bijection. So, by Lemma 8.4 on the homotopy of graphs, if for all t in $[0, 1]$, $\mathcal{G}(t)$ is a planar graph, then $\mathcal{G}(1) = \mathcal{G}'$ is a planar graph such that all elements of $\mathcal{C}(\mathcal{G}')$ are triangles. Now, we show that $\mathcal{G}(t)$ is a planar graph.

Step 1: For all $[A, B]$ in $\mathcal{G}$ and all C in $\mathcal{V}(\mathcal{G})$, if $C \neq A$ and if $C \neq B$, then $[C - \varepsilon_n, C + \varepsilon_n]$ does not meet $[A, B]$. In fact, $[C - \varepsilon_n, C + \varepsilon_n[$ is included in

$$\mathcal{R} = \{C + x_1e_1 + x_2e_2 : x_1, x_2 \in [0, 1[\} \cup \{C + x_1e_1 + x_2e_2 : -x_1, -x_2 \in [0, 1[\}$$

and $[A, B]$ does not meet $\mathcal{R}$, hence $[A, B] \cap [C - \varepsilon_n, C + \varepsilon_n] = \varnothing$.

Step 2: For all $[A, B]$ in $\mathcal{G}$ and for all C in $\mathcal{V}(\mathcal{G})$, if $C \neq A$ and if $C \neq B$, then for all t in $[0, 1]$, $C(t)$ is not in $[A(t), B(t)]$. Otherwise, there would exist λ in $[0, 1]$ such that

$$C + tf(C)\varepsilon_n = \lambda(A + tf(A)\varepsilon_n) + (1 - \lambda)(B + tf(B)\varepsilon_n).$$

This would mean that $C + t[f(C) - (\lambda f(A) + (1 - \lambda)f(B))]\varepsilon_n \in [A, B]$, but $t[f(C) - (\lambda f(A) + (1 - \lambda)f(B))] \in [-1, 1]$, and this contradicts Step 1.

Step 3: Let A, B and C be vertices of $\mathcal{G}$ such that $[A, B]$ and $[B, C]$ are in $\mathcal{G}$ and $A \neq C$. Then for all t in $[0, 1]$, $[A(t), B(t)[$ does not meet $]B(t), C(t)]$. Otherwise, $C(t)$ is in $]B(t), A(t)]$ or $A(t)$ is in $]B(t), C(t)]$ which contradicts Step 2.

Step 4: Let $[A, B]$ and $[C, D]$ be in $\mathcal{G}$. If they do not meet, then for all t in $[0, 1]$, $[A(t), B(t)]$ and $[C(t), D(t)]$ do not meet. Indeed, set

$$U = \{t \in [0, 1] : [A(t), B(t)] \cap [C(t), D(t)] = \varnothing\},$$

$$V = \{t \in [0, 1] :]A(t), B(t)[\cap]C(t), D(t)[\neq \varnothing \text{ and } \det(\overrightarrow{A(t)B(t)}, \overrightarrow{C(t)D(t)}) \neq 0\},$$

$$F = \{t \in [0, 1] : (A(t) \text{ or } B(t) \in [C(t), D(t)]) \text{ or } (C(t) \text{ or } D(t) \in [A(t), B(t)])\}.$$

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We have $[0,1] = U \cup V \cup F$. By Step 2, F is empty and by continuity, U and V are open and since U and V are disjoint, U or V is empty. Since 0 is in U, we have $U = [0,1]$.

Step 5: By Step 3 and Step 4, $\mathcal{G}(t)$ is a planar graph. Let ω' be a connected component of $\mathbf{R}^2 \backslash \mathcal{G}'$. By Lemma 8.4 on the homotopy of graphs, there exists ω in $\mathcal{C}(\mathcal{G})$ such that $\partial\omega' = \bigcup_{[A,B]\subset\partial\omega}[A',B']$. By definition of $\mathcal{G}$ we can find A in Λ_n such that

$$\partial\omega = [A, A+u_1] \cup [A+u_1, A+u_3] \cup [A, A+u_3]$$

or

$$\partial\omega = [A, A+u_2] \cup [A+u_2, A+u_3] \cup [A, A+u_3].$$

Suppose $\partial\omega = [A, A+u_1] \cup [A+u_1, A+u_3] \cup [A, A+u_3]$. In this case

$$\partial\omega' = [A', (A+u_1)'] \cup [(A+u_1)', (A+u_3)'] \cup [A', (A+u_3)']$$

is determined, up to translations, by the two vectors $(A+u_1)' - A'$ and $(A+u_3)' - A'$. Now we apply Lemma 3.2 on cyclic groups with $G = \Lambda_n$, $H = \mathbf{Z}^2$, $a = \theta_n$, $b_1 = u_1$ and $b_2 = u_3$. There exists a partition of $\{0, \ldots, q_n - 1\}$ into 3 intervals I_0, I_1, I_2 such that for all j in $\{0,1,2\}$ and all i in $\{1,2\}$ there exist $n_{i,j}$ in $\mathbf{Z}$ and $H_{i,j}$ in $\mathbf{Z}^2$ such that for all $A = P + k\theta_n \in \Lambda_n$,

$$k \in I_j \Rightarrow \begin{cases} A + b_1 = (k + n_{1,j})\theta_n + P + H_{1,j} \text{ and } k + n_{1,j} \in \{0, \ldots, q_n - 1\}, \\ A + b_2 = (k + n_{2,j})\theta_n + P + H_{2,j} \text{ and } k + n_{2,j} \in \{0, \ldots, q_n - 1\}. \end{cases}$$

It follows that for all $A = P + k\theta_n \in \Lambda_n$,

$$k \in I_j \Rightarrow \begin{cases} (A+u_1)' = A' + n_{1,j}\theta + H_{1,j}, \\ (A+u_3)' = A' + n_{2,j}\theta + H_{2,j}. \end{cases}$$

Therefore, up to translations, there are at most three triangles $(A', (A+u_1)', (A+u_3)')$. Finally, there are at most 6 triangles.

Step 6: It remains to prove that the diameters of the triangles go to zero when n goes to infinity. The length l of an edge of $\mathcal{G}$ is $|e_{1,n}|$, $|e_{2,n}|$ or $|e_{1,n} + e_{2,n}|$. By the choice of the basis (Theorem 7.1), we have $l \leq 2(\lambda_{1,n} + \lambda_{2,n})$. Furthermore, for every vertex A, the distance between A and A' is at most $|\varepsilon_n|$, therefore the diameter of each triangle of $\mathcal{G}'$ is at most $2(\lambda_{1,n} + \lambda_{2,n} + |\varepsilon_n|)$, which goes to zero when n goes to infinity. ■

10 Proof of Theorem 1.3

10.1 The Different Cases

Let θ be in $\mathbf{R}^2$ such that $\mathbf{Z}^2 + \mathbf{N}\theta$ is dense in $\mathbf{R}^2$ and let q be in $\mathbf{N}^*$. There exists n in $\mathbf{N}$ such that q is between $q_n - 1$ and $q_{n+1} - 1$ ($q_n - 1 \leq q < q_{n+1} - 1$). Furthermore, there exists $N \in \mathbf{N}^*$ such that $Nq_n - 1 \leq q < (N+1)q_n - 1$.

By Theorem 7.1, there is a basis (e_1, e_2) of Λ_n such that $\varepsilon_n = \alpha e_1 + \beta e_2$ with $0 \leq \beta \leq \alpha < 1$ and $|\sin\angle(e_1, e_2)| \geq \sqrt{3}/8$. Since $\mathbf{Z}^2 + \mathbf{Z}\theta$ is dense in $\mathbf{R}^2$, $\beta > 0$. We consider the four cases:

Case(1) $\beta \le 1/100$ and $N\beta < 1$.
Case(2) $\beta \le 1/100$ and $1 \le N\beta < 1 + 3\beta$.
Case(3) (Main case:) $\beta \le 1/100$ and $N\beta \ge 1 + 3\beta$.
Case(4) $\beta \ge 1/100$.

We will give a proof of Theorem 1.3 only in cases 1, 3 and 4. Case 2 is more difficult than Case 1, but far easier than Case 3 and we leave it to the reader. Case 4 is very different; we use Voronoï's diagram. Note that the inequality on $\angle(e_1, e_2)$ is only needed in Case 4.

Let us explain why we need to introduce four cases. We must construct a triangulation whose set of vertices is $\mathbf{Z}^2 + \{0, \theta, \dots, q\theta\}$. By Lemma 8.4, it is enough to construct a planar graph with the same set of vertices. Since $\mathbf{Z}^2 + \{0, \theta, \dots, q\theta\}$ is very close to $\Gamma = \mathbf{Z}^2 + \{0, \theta_n, \dots, q\theta_n\}$, we first define a planar graph whose set of vertices is Γ. If $A = P + k\theta_n$ is a vertex with $k + q_n \le q$, then the point $A + \varepsilon_n$ is another vertex and the segment $[A, A + \varepsilon_n]$ will always be an edge of our graph. All the vertices can be joined by a succession of such edges to the subset of vertices $\Lambda_n = \mathbf{Z}^2 + \{0, \theta_n, \dots, (q_n - 1)\theta_n\}$. Now we add some new edges which are not of the shape $[A, A+\varepsilon_n]$, in order to get a planar graph $\mathcal{G}$ such that all connected components of $\mathbf{R}^2 \backslash \mathcal{G}$ are small. The definition of these new edges will be different in cases 1–3. In the first case the new edges are easy to define because for any vertex A, the segment $[A, A + e_1]$ never meets an edge of the shape $[A', A' + \varepsilon_n]$. Therefore, we can add all the edges $[A, A + e_1]$, $A \in \Gamma$, to the graph $\mathcal{G}$ (see Figures 2 and 3). In the second case, a segment $[A, A + e_1]$ meets at most one edge of the shape $[A', A' + \varepsilon_n]$ and these segments are very few. In the main case, most of the segments $[A, A + e_1]$ meet one or more than one edge of the shape $[A', A' + \varepsilon_n]$ (see Figures 4 and 5). Therefore, we must find another kind of edge. This is the reason why the proof is rather technical in the main case.

The last case could probably be handled as the first three, but we find it more convenient to use Voronoï's diagram.

10.2 A Consequence of the Three Distance Theorem

The following proposition is an important argument of the proof in the main case. Given a lattice Λ and a segment $\mathbf{S}$ of $\mathbf{R}^2$, the subset $\Lambda + \mathbf{S}$ of $\mathbf{R}^2$ should not be very intricate even if the length of the segment $\mathbf{S}$ is far greater than the lengths of the vectors of a basis of Λ. To see it, consider the intersection of $\Lambda + \mathbf{S}$ with a straight line. We shall see that this intersection is of the shape $\mathbf{Z} + \{0, \dots, n\}\gamma$, therefore the three distance theorem allows us to count the number of different distances between consecutive points of this intersection. In fact, our real aim is not to study the distances, but some vectors closely related to these distances. Each point M of this intersection lies on a segment $A + \mathbf{S}$ with A in Λ. We want to count the number of vectors $A' - A$ corresponding to successive points M and M' of this intersection.

Proposition 10.1 *Let I be a bounded interval of $\mathbf{R}$, let $\Lambda = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$ be a lattice of $\mathbf{R}^2$ and let u be a vector of $\mathbf{R}^2$. Suppose u is not parallel to any lattice directions. Let $\mathbf{S}$ be the segment Iu and for any $P \in \mathbf{R}^2$ set $\mathcal{T}(P) = \{t \in \mathbf{R} : P + te_1 \in \Lambda + \mathbf{S}\}$. Then:*

(i) *For each $t \in \mathcal{T}(P)$ there exists a unique $A = A(P, t) \in \Lambda$ such that $P + te_1 \in A + \mathbf{S}$.*
(ii) *For $t \in \mathcal{T}(P)$, let $t' = \min\{s > t : s \in \mathcal{T}(P)\}$. The cardinal number of the set*

$$\{A(P, t') - A(P, t) : P \in \mathbf{R}^2 \text{ and } t \in \mathcal{T}(P)\}$$

is at most 6.

Proof (i) Suppose $P + te_1 = A + ru = B + su$ with $A, B \in \Lambda$. If $r \neq s$, then u is parallel to $B - A$ which is in Λ, so $r = s$ and $A = B$.

(ii) Let us use coordinates in the basis (e_1, e_2); note (u_1, u_2) the coordinates of u. Fix P in $\mathbf{R}^2$. We can suppose $P = pe_2$ with $p \in \mathbf{R}$. For t in $\mathbf{R}$ we have

$$\begin{aligned} t \in \mathcal{T}(P) &\iff \text{there exist } A \in \Lambda \text{ and } s \in I \text{ such that } P + te_1 = A + su \\ &\iff \text{there exist } a_1, a_2 \in \mathbf{Z} \text{ and } s \in I \text{ such that } \begin{cases} t = a_1 + su_1, \\ p = a_2 + su_2. \end{cases} \end{aligned}$$

By hypothesis $u_2 \neq 0$, so $s = \frac{1}{u_2}(p - a_2)$ and $t = a_1 + p\frac{u_1}{u_2} - a_2\frac{u_1}{u_2} = (p\frac{u_1}{u_2} + a_1) - a_2\frac{u_1}{u_2})$. It follows with $\gamma = -\frac{u_1}{u_2}$ that

$$\mathcal{T}(P) = \left\{ \left(\frac{pu_1}{u_2} + a_1 \right) + a_2\gamma : a_1 \in \mathbf{Z} \text{ and } a_2 \in (p - u_2 I) \cap \mathbf{Z} \right\}.$$

There are integers $a(p)$ and b such that $(p - u_2 I) \cap \mathbf{Z} = \{a(p), \dots, a(p) + b - 1$ or $a(p) + b\}$ (this interval may be empty). Set $I(p) = \{0, \dots, b - 1\}$ in the first case and $I(p) = \{0, \dots, b\}$ in the second case. So, with $a_2 = b_2 + a(p)$,

$$\mathcal{T}(P) = \left\{ \left(\frac{pu_1}{u_2} + a(p)\gamma \right) + (a_1 + b_2\gamma) : a_1 \in \mathbf{Z} \text{ and } b_2 \in I(p) \right\}.$$

Now, by the three distance theorem, $t' - t$ takes at most three different values in both cases, which give at most 6 different values. Furthermore, $\gamma = -\frac{u_1}{u_2}$ is an irrational number and $t' - t \in \mathbf{Z} + \mathbf{Z}\gamma$. Then there exist unique $n_1, n_2 \in \mathbf{Z}$ such that $t' - t = n_1 + n_2\gamma$. If $t' - t = n_1 + n_2\gamma$, we have

$$M(P, t') - M(P, t) = a_1' e_1 + a_2' e_2 - (a_1 e_1 + a_2 e_2) = n_1 e_1 + n_2 e_2$$

for

$$\begin{aligned} t' - t &= \left(\frac{pu_1}{u_2} + a(p)\gamma \right) + a_1' + b_2'\gamma - \left(\left(\frac{pu_1}{u_2} + a(p)\gamma \right) + a_1 + b_2\gamma \right) \\ &= (a_1' - a_1) + (a_2' - a_2)\gamma. \end{aligned}$$

Hence, $M(P, t') - M(P, t)$ takes at most six values. ∎

Remark

(1) A more careful use of the three distance theorem shows that the cardinal of the set $\{M(P, t') - M(P, t) : P \in \mathbf{R}^2 \text{ and } \in \mathcal{T}(P)\}$ is at most 4.

(2) On making use of Langevin's generalization of the three distance theorem (see [21]), it is possible to enlarge the definition of $\mathcal{T}(P)$ to $\mathcal{T}(P) = \{t \in \mathbf{R} : P + tv \in \Lambda + \mathbf{S}\}$ where v is an element of $\mathbf{R}^2$ independent with u. The same conclusion holds.

10.3 Proof of Theorem 1.3 in Case 1

This case is far easier than the main case, but the basic ideas of the proof are the same, geometric difficulties are avoided and the previous proposition is not necessary.

We define successively three planar graphs $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$, and thanks to Lemma 8.4, we shall see that $\mathcal{G}_3$ is the desired triangulation. The first graph is made from the rational approximation θ_n. Then we slightly move the vertices of $\mathcal{G}_1$ keeping the same edges; we get the second graph. Finally, we add some missing vertices to the graph $\mathcal{G}_2$ and get the graph $\mathcal{G}_3$.

Notation Let t be in $\mathbf{R}$. The lowest integer greater than or equal to t is denoted by $\lceil t \rceil$.

First graph: Assume $q = Nq_n - 1$ and consider the set

$$\Gamma_N = \mathbf{Z}^2 + \{0, \dots, (q_n - 1)\theta_n\} + \{0, \varepsilon_n, \dots, (N-1)\varepsilon_n\} = \bigcup_{k=0}^{N-1} (\Lambda_n + k\varepsilon_n).$$

The set Γ_N is an approximation of the set $\{0, \theta, \dots, q\theta\} + \mathbf{Z}^2$ which must be the set of vertices of our triangulation. Since the set Γ_N is simply deduced from the lattice Λ_n, it easy to find a triangulation $\mathcal{G}_1$ whose set of vertices is Γ_N. The graph $\mathcal{G}_1$ (Figure 2) is defined by the set of vertices Γ_N, and there are three kinds of edges. For all A in Γ_N but not in $\Lambda_n + (N-1)\varepsilon_n$, $[A, A + \varepsilon_n]$ is an edge (*vertical* edge). For all A in Γ_N, $[A, A + e_1]$ is an edge (*horizontal* edge). Set $m = \lceil (N-1)\alpha \rceil$ and for all A in $\Lambda_n + (N-1)\varepsilon_n$, set $A^* = (A - (N-1)\varepsilon_n) + me_1 + e_2$ (note that A^* is in Λ_n). For all A in $\Lambda_n + (N-1)\varepsilon_n$, $[A, A^*]$ is an edge (*exceptional* edge).

It is straightforward to prove that

(i) $\mathcal{G}_1$ is a planar graph;
(ii) all connected components of $\mathbf{R}^2 \backslash \mathcal{G}_1$ are parallelograms;
(iii) if $[A, B]$ is an edge of $\mathcal{G}_1$, $\pm\overrightarrow{AB}$ has only six possible values.

Each point of Γ_N is uniquely written as an element of $\mathbf{Z}^2$ plus an element of $\{0, \dots, (q_n - 1)\theta_n\}$ plus an element of $\{0, \varepsilon_n, \dots, (N-1)\varepsilon_n\}$ (see Lemma 10.3). Let $P = A + k\theta_n + l\varepsilon_n$ be such a point and set $P' = A + k\theta + l\varepsilon_n$. The point P' belongs to $\mathbf{Z}^2 + \{0, \theta, \dots, (Nq_n - 1)\theta\}$ for $P' = A + k\theta + l(q_n\theta - P_n) = A - lP_n + (k + lq_n)\theta$.

Second graph: To each edge $[P, Q]$ of $\mathcal{G}_1$, we associate the edge $[P', Q']$, which leads to a new graph $\mathcal{G}_2$ whose set of vertices is $\mathbf{Z}^2 + \{0, \theta, \dots, (Nq_n - 1)\theta\}$ (see Figure 3).

In order to show that the connected components $\mathbf{R}^2 \backslash \mathcal{G}_2$ are quadrilaterals, we use Lemma 8.4 on the homotopy of graphs. We need a continuous family $\mathcal{G}(t)$ with $\mathcal{G}(0) = \mathcal{G}_1$ and $\mathcal{G}(1) = \mathcal{G}_2$. For $t \in [0, 1]$, let $\mathcal{G}(t)$ be the graph whose edges are $[(1-t)A + tA', (1-t)B + tB']$ with $[A, B]$ in $\mathcal{G}_1$. We must show that these are planar graphs.

Lemma 10.2 *For all t in $[0, 1]$, $\mathcal{G}(t)$ is a planar graph.*

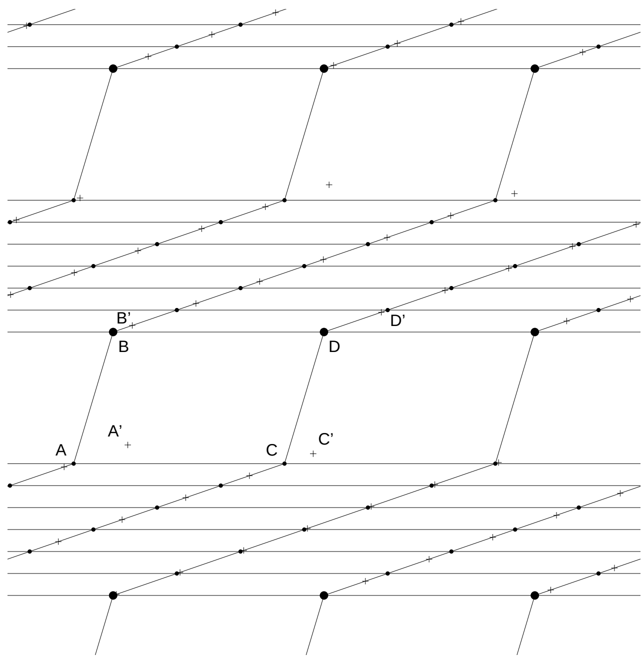

Figure 2: Graph $\mathcal{G}_1$. The points of Λ_n are represented by the big black dots, the other points of Γ_N are represented by the small black dots and the points of $\{0, \theta, \dots, q\theta\} + \mathbf{Z}^2$ are represented in cross-shaped.

Proof Since the proof is the same for all $t \in]0, 1]$, we do it only for $t = 1$. Let us show that $\mathcal{G}_2$ is a planar graph. We must study the intersections $[A', B'] \cap [C', D']$ where $[A, B]$ and $[C, D]$ are edges of $\mathcal{G}_1$. Since $N\beta < 1$, the only case which is not obvious is the following:

$$A \in \Lambda_n + (N-1)\varepsilon_n, \ [A, A^*] = [A, B] \quad \text{and} \quad [C, D] = [A + e_1, (A + e_1)^*].$$

We have $\overrightarrow{AA^*} = \alpha' e_1 + \beta' e_2$. By the choice of A^*, we have $\alpha', \beta' \geq 0$, and since $N\beta < 1$, we have $\beta' > \beta$. First note that the strip $A + \mathbf{R}\overrightarrow{AA^*} + [-1, 1]\varepsilon_n$ does not meet the line $A + e_1 + \mathbf{R}\overrightarrow{AA^*}$. Indeed, $A + \lambda\varepsilon_n \in A + e_1 + \mathbf{R}\overrightarrow{AA^*}$ if and only if there exists μ such that $\lambda\alpha = 1 + \mu\alpha'$ and $\lambda\beta = \mu\beta'$. Hence, $\lambda = (\alpha - \frac{\beta}{\beta'}\alpha')^{-1}$. Since $\alpha, \alpha' \in [0, 1[$ and $\frac{\beta}{\beta'} < 1$, we have $|\lambda| > 1$ and the point $A + \lambda\varepsilon_n$ does not belong to the line $A + e_1 + \mathbf{R}\overrightarrow{AA^*}$. Now, if there is a point P in $[A', {A^*}'] \cap [C', {C^*}']$, there exist $s, t, \lambda, \mu \in [0, 1]$ such that

$$P = sA + (1-s)A^* + \lambda\varepsilon_n = tC + (1-t)C^* + \mu\varepsilon_n,$$

therefore,

$$tC + (1-t)C^* = sA + (1-s)A^* + (\lambda - \mu)\varepsilon_n \in A + \mathbf{R}\overrightarrow{AA^*} + [-1, 1]\varepsilon_n.$$

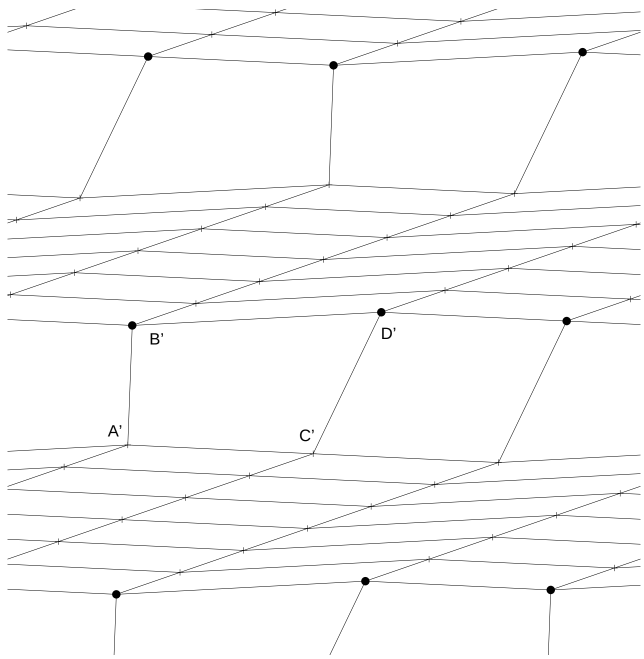

Figure 3: The graph $\mathcal{G}_2$ deduced from the graph $\mathcal{G}_1$ represented in Figure 2.

It follows that $[A', A^{*'}] \cap [C', C^{*'}] = \varnothing$.

Since all connected components of $\mathcal{G}_1$ are quadrilaterals, by Lemma 8.4 on the homotopy of planar graphs, the connected components of $\mathbf{R}^2 \backslash \mathcal{G}_2$ are also quadrilaterals. Now we use Lemma 3.2 on cyclic groups to count the number of edges of $\mathcal{G}_2$.

Let $A = P + k\varepsilon_n$ be in Γ_N with P in Λ_n and $k \in \{0, \dots, N-1\}$. We have $P = Q + i\theta_n$ and $P + e_1 = R + j\theta_n$ with Q, R in $\mathbf{Z}^2$ and i, j in $\{0, \dots, q_{n-1}\}$. By Lemma 3.2, the couple $(j - i, R - Q)$ has only two possible values, hence the vector

$$(A + e_1)' - A' = (j - i)\theta + R - Q$$

has only two possible values.

Suppose that $A = P + (N-1)\varepsilon_n = Q + i\theta_n + (N-1)\varepsilon_n$ is in $\Lambda_n + (N-1)\varepsilon_n$. We have

$$A^* = A - (N-1)\varepsilon_n + me_1 + e_2 = Q + i\theta_n + me_1 + e_2 = R + j\theta_n$$

where m does not depend on A. Hence, $R + j\theta_n - (Q + i\theta_n) = me_1 + e_2$. It follows that $(j - i, R - Q)$ takes only two distinct values, and so

$$A^{*\prime} - A' = R + j\theta - (Q + i\theta) - (N-1)\varepsilon_n = R - Q + (j - i)\theta + (N-1)\varepsilon_n$$

takes only two distinct values. The last kind of edge $[A, A + \varepsilon_n]$ gives only one possible value for $(A + \varepsilon_n)' = A' + \varepsilon_n$. Finally, we get $5 + 5$ possible values for the vectors $\overrightarrow{A'B'}$ where $[A', B']$ is an edge of $\mathcal{G}_2$.

Third graph: For q between $Nq_n - 1$ and $(N+1)q_n - 1$, we add some edges to $\mathcal{G}_2$ or cut some edges, to get a new graph $\mathcal{G}_3$ whose set of vertices is $\mathbf{Z}^2 + \{0, \theta, \dots, q\theta\}$. In fact, each new point P of $\mathbf{Z}^2 + \{Nq_n\theta, \dots, q\theta\}$ belongs to a connected component of $\mathbf{R}^2 \backslash \mathcal{G}_2$ or to an edge $[A, B]$ of $\mathcal{G}_2$. In the first case, we add the singular edge $[P, P]$. In the second case, we split the edge $[A, B]$ into two edges $[A, P]$ and $[P, B]$.

Notation $\mathcal{E}(\mathcal{G}_3) = \{\overrightarrow{AB} : A$ and B are vertices of $\mathcal{G}_3$, and there exists ω in $\mathcal{C}(\mathcal{G}_3)$ such that $A, B \in \partial\omega\}$.

By Lemma 8.3 on graphs, the following property implies Theorem 1.3.

$(*)$ There exists an absolute constant K independent of θ and q such that $|\mathcal{E}(\mathcal{G}_3)| \leq K$.

Therefore, we now prove property $(*)$. Since the connected components of $\mathbf{R}^2 \backslash \mathcal{G}_2$ are quadrilaterals and since $\mathcal{G}_2$ has only 5 kinds of edges, property $(*)$ is true for $\mathcal{G}_2$: the number of elements of

$$\mathcal{E}(\mathcal{G}_2) = \Big\{\overrightarrow{AB} : \ A \text{ and } B \text{ are vertices of } \mathcal{G}_2, \text{ and there exists } \omega \text{ in } \mathcal{C}(\mathcal{G}_2) \\ \text{such that } A, B \in \partial\omega\Big\}$$

is less than $(2 \times 5 \times 5 + 5) \times 2$. Let A be in $\Lambda_n + (N-1)\varepsilon_n$. The point $A' + \varepsilon_n$ is in the connected component of $\mathbf{R}^2 \backslash \mathcal{G}_2$ which contains at least one of the points A', $(A + e_1)'$, $A^{*\prime}$ and $(A^* + e_1)'$. It follows that $\mathcal{E}(\mathcal{G}_3) \subset \mathcal{E}(\mathcal{G}_2) \cup (\mathcal{E}(\mathcal{G}_2) + \mathcal{E}(\mathcal{G}_2) + \varepsilon_n)$, and $(*)$ is true for $\mathcal{G}_3$. ■

10.4 The Map $A \to A'$

In Section 10.3, we used the approximation

$$\begin{aligned}\Gamma_N &= \Lambda_n + \{0, \varepsilon_n, \dots, (N-1)\varepsilon_n\} \\ &= \mathbf{Z}^2 + \{0, \theta_n, \dots, (q_n - 1)\theta_n\} + \{0, \varepsilon_n, \dots, (N-1)\varepsilon_n\}\end{aligned}$$

of the set

$$\mathbf{Z}^2 + \{0, \theta, \dots, (Nq_n - 1)\theta\} = \mathbf{Z}^2 + \{0, \theta, \dots, (q_n - 1)\theta\} + \{0, \varepsilon_n, \dots, (N-1)\varepsilon_n\}.$$

This is an important idea of the proof of the first three cases. We have already used the properties of Γ_N and of the map $A \to A'$ which are given by the next lemma.

Lemma 10.3 *Let N be in $\mathbf{N}^*$ and $q = Nq_n - 1$.*

(i) *Every element of the set Γ_N is uniquely written as the sum of an element of $\mathbf{Z}^2$, an element of $\{0, \theta_n, \dots, (q_n - 1)\theta_n\}$ and an element of $\{0, \varepsilon_n, \dots, (N-1)\varepsilon_n\}$.*

(ii) *The map* $A \in \Gamma_N \to A' \in \mathbf{Z}^2 + \{0, \theta, \dots, (Nq_n - 1)\theta\}$ *defined by*

$$(P + k\theta_n + i\varepsilon_n)' = P + k\theta + i\varepsilon_n$$

where $P \in \mathbf{Z}^2$, $k \in \{0, \dots, q_n - 1\}$ *and* $i \in \{0, \dots, N-1\}$, *is one-to-one and onto.*

Proof (i) Suppose $P + k\theta_n + i\varepsilon_n = Q + l\theta_n + j\varepsilon_n$ with $P, Q \in \mathbf{Z}^2$, $k, l \in \{0, \dots, q_{n-1}\}$ and $i, j \in \{0, \dots, N-1\}$. If $i \neq j$, then $\varepsilon_n \in \frac{1}{i-j}\Lambda_n \subset \mathbf{Q}^2$ and this is impossible for $\theta \notin \mathbf{Q}^2$. Hence $i = j$. $P + k\theta_n = Q + l\theta_n$ means that $k\Theta_n = l\Theta_n$ in $\mathbf{R}^2/\mathbf{Z}^2$, and by Lemma 6.1 we get $k = l$ and $P = Q$.

(ii) Let $A = P + k\theta_n + i\varepsilon_n$ and $B = Q + l\theta_n + j\varepsilon_n$ be in Γ_N. If $A' = B'$, then

$$A' = P + k\theta + i\varepsilon_n = B' = Q + l\theta + j\varepsilon_n.$$

It follows that

$$\begin{aligned} P + k\theta + i\varepsilon_n - (Q + l\theta + j\varepsilon_n) &= \Big(P + k(\theta_n + \frac{1}{q_n}\varepsilon_n) + i\varepsilon_n\Big) \\ &\qquad - \Big(Q + l\Big(\theta_n + \frac{1}{q_n}\varepsilon_n\Big) + j\varepsilon_n\Big) \\ &= (P - Q) + (k - l)\theta_n + \Big(i - j + \frac{k-l}{q_n}\Big)\varepsilon_n. \end{aligned}$$

Again, $(i - j + \frac{k-l}{q_n}) \neq 0$ implies $\varepsilon_n \in \Lambda_n \subset \mathbf{Q}^2$. Now $|k - l| < q_n$, hence $i = j$ and $k = l$. It follows that $P = Q$.

It remains to show that the map is onto. Let $C = P + k\theta$ be in $\mathbf{Z}^2 + \{0, \theta, \dots, (Nq_n - 1)\theta\}$. There exist $l \in \{0, \dots, q_n - 1\}$ and $m \leq N - 1$ such that $C = P + (l + mq_n)\theta$. We have

$$\begin{aligned} C = P + (l + mq_n)\theta &= P + l\theta + mq_n\Big(\theta_n + \frac{1}{q_n}\varepsilon_n\Big) \\ &= (P + mq_n\theta_n) + l\theta + m\varepsilon_n. \end{aligned}$$

Since $q_n\theta_n \in \mathbf{Z}^2$, we get $C = ((P + mq_n\theta_n) + l\theta_n + m\varepsilon_n)'$. ■

10.5 Outline of the Proof in the Main Case: $\beta \leq 1/100$ and $N\beta \geq 1 + 3\beta$

We proceed as in first case. We successively define four $\mathbf{Z}^2$-invariant graphs. The set of vertices of the first graph is still Γ_N, but the edges are more difficult to define for $N\beta \geq 1$.

First graph:

$$\begin{aligned} \mathcal{V}(\mathcal{G}_1) = \Gamma_N &= \mathbf{Z}^2 + \{0, \dots, (q_n - 1)\theta_n\} + \{0, \varepsilon_n, \dots, (N-1)\varepsilon_n\} \\ &= \bigcup_{k=0}^{N-1} (\Lambda_n + k\varepsilon_n). \end{aligned}$$

In Section 10.7 we define the edges of the graph $\mathcal{G}_1$. An example is given Figure 4.

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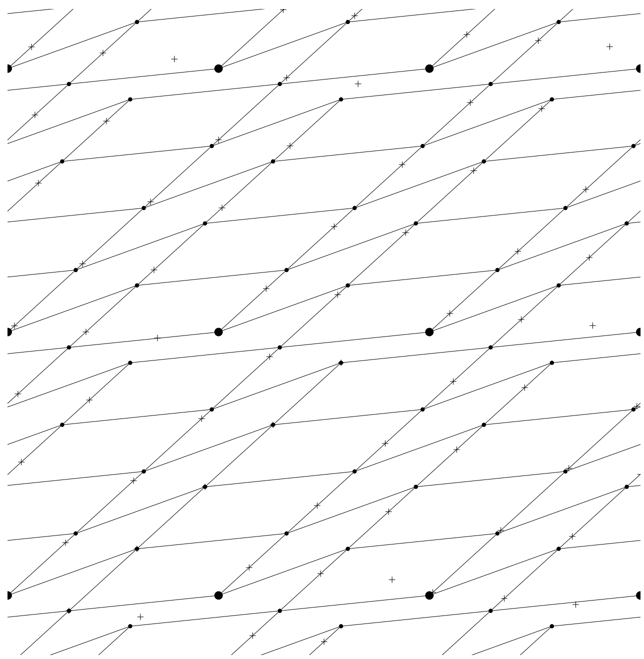

Figure 4: The graph $\mathcal{G}_1$ is constructed with rules given in Section 10.7.

Second graph: We remove some edges of $\mathcal{G}_1$ to get a second graph $\mathcal{G}_2$.

Third graph: To each edge $[P, Q]$ of $\mathcal{G}_2$, we associate the edge $[P', Q']$ (see Lemma 10.3 for the definition of the map $A \longrightarrow A'$) which leads to a new graph $\mathcal{G}_3$ whose set of vertices is $\mathbf{Z}^2 + \{0, \theta, \dots, (Nq_n - 1)\theta\}$.

We need the second graph to be sure that $\mathcal{G}_3$ is a planar graph. Figure 5 shows the kind of pictures we get without removing some edges first. The graph represented is the graph deduced from the graph $\mathcal{G}_1$ by the map $A \longrightarrow A'$. It is not a planar graph.

Fourth graph: For q between $Nq_n - 1$ and $(N + 1)q_n - 1$, we add some edges to $\mathcal{G}_3$ or cut some edges to get a new graph $\mathcal{G}_4$ whose set of vertices is $\mathbf{Z}^2 + \{0, \theta, \dots, q\theta\}$. In fact, each new point P of $\mathbf{Z}^2 + \{Nq_n\theta, \dots, q\theta\}$ belongs to a connected component of $\mathbf{R}^2 \backslash \mathcal{G}_3$ or to an edge $[A, B]$ of $\mathcal{G}_3$. In the first case, we add the singular edge $[P, P]$. In the second case, we split the edge $[A, B]$ into two edges $[A, P]$ and $[P, B]$.

Notation Remember that $\mathcal{C}(\mathcal{G})$ denotes the set of connected components of $\mathbf{R}^2 \backslash \mathcal{G}$.

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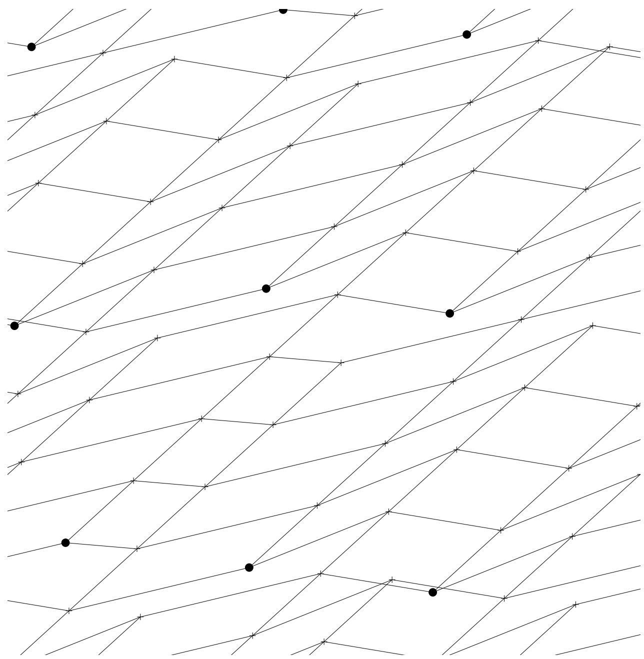

Figure 5: This picture is deduced from the graph of Figure 4 by the map $A \to A'$. It is no longer a planar graph. Some edges have to be removed.

Set

$$\mathcal{E}_i = \left\{ \overrightarrow{AB} : A \text{ and } B \text{ are vertices of } \mathcal{G}_i, \text{ and there exists } \omega \text{ in } \mathcal{C}(\mathcal{G}_i) \right.$$
$$\left. \text{such that } A, B \in \partial\omega \right\}.$$

Let us introduce a property of the graph $\mathcal{G}_i$.

P0 There exists an absolute constant K independent of θ and q such that $|\mathcal{E}_i| \le K$.

If **P0** holds for $\mathcal{G}_4$, then by Lemma 8.3 on planar graphs, Theorem 1.3 holds in the main case. In order to prove **P0**, we shall prove the following two properties of the graphs $\mathcal{G}_i$.

P1 The number of vectors $\overrightarrow{AB}$ with $[A, B]$ in $\mathcal{G}_i$ is finite and less than a number independent of θ and q.

P2 The number of edges of the boundary of any connected component of $\mathbf{R}^2 \backslash \mathcal{G}_i$ is less than a number independent of θ and q.

The proof of **P0** for $\mathcal{G}_4$ is organized as follows:

- $\mathcal{G}_1$ is defined such that **P1** is true for $\mathcal{G}_3$ (§10.6, Lemma 10.4) and that **P2** is true for $\mathcal{G}_1$ (§10.8, Proposition 10.17).

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- Next, we prove that the connected components of $\mathbf{R}^2\backslash\mathcal{G}_2$ are the union of at most 7 components of $\mathbf{R}^2\backslash\mathcal{G}_1$ (§10.9, Proposition 10.25); together with the property **P2** for $\mathcal{G}_1$, this shows that the property **P2** is true for $\mathcal{G}_2$.
- Lemma 8.4 on homotopy, property **P2** for $\mathcal{G}_2$ and property **P1** for $\mathcal{G}_3$ lead to property **P0** for $\mathcal{G}_3$ (§10.10, Proposition 10.35).
- Finally, we prove that **P0** is true for $\mathcal{G}_4$ (§10.11, Proposition 10.37).

10.6 Properties of $\mathcal{G}_1$

In this section we introduce three properties which imply that **P1** holds for $\mathcal{G}_3$. We shall define $\mathcal{G}_1$ in the next section.

P3 If $[A, B]$ is an edge of $\mathcal{G}_1$ parallel to ε_n, then $B = A \pm \varepsilon_n$.

P4 If $[P, Q]$ is an edge of $\mathcal{G}_1$ which is not parallel to ε_n, then $\overrightarrow{PQ} = \lambda e_1 + \gamma\varepsilon_n$ with $\lambda \neq 0$ and $\gamma \in\,]-1, 1[$.

P5 There exists a subset $\mathbf{V}$ of Λ_n with less than 8 elements such that if $[P, Q]$ is an edge of $\mathcal{G}_1$, then $\overrightarrow{RS} \in \mathbf{V}$ where $P = R + l\varepsilon_n$ and $Q = S + m\varepsilon_n$ with R, S in Λ_n and l, m in $\{0, \dots, N-1\}$.

Lemma 10.4 *If $\mathcal{G}_1$ satisfies* **P3, P4** *and* **P5**, *then there exists an absolute constant K_1 such that the number of vectors $\overrightarrow{A'B'}$ with $[A, B]$ in $\mathcal{G}_1$ is less than K_1. It follows that* **P1** *is true for $\mathcal{G}_3$.*

Proof The edges of $\mathcal{G}_1$ parallel to ε_n give only one vector, for if $[P, P + \varepsilon_n]$ is in $\mathcal{G}_1$ with $P = A + k\theta_n + l\varepsilon_n$, then

$$P' = A + k\theta + l\varepsilon_n, (P + \varepsilon_n)' = A + k\theta + (l + 1)\varepsilon_n \quad \text{and} \quad \overrightarrow{P'(P + \varepsilon_n)'} = \varepsilon_n.$$

For the other edges, we use **P5** and Lemma 3.2 with $G = \Lambda_n$, $H = \mathbf{Z}^2$, $a = \theta_n$ and $\{b_1, \dots, b_k\} = \mathbf{V}$. There exists a subset $\mathbf{E}$ of $\mathbf{Z} \times \mathbf{Z}^2$ whose cardinal number is less than $|\mathbf{V}|\,(|\mathbf{V}| + 1)$ and such that if $[P, Q]$ is an edge of $\mathcal{G}_1$ with $P = A + k\theta_n + l\varepsilon_n$ and $Q = B + j\theta_n + m\varepsilon_n$, then $(j - k, B - A) \in \mathbf{E}$. Set $R = A + k\theta_n$ and $S = B + j\theta_n$. By **P4**, $\overrightarrow{PQ} = \overrightarrow{RS} + (m - l)\varepsilon_n = \lambda e_1 + \gamma\varepsilon_n$ with $\gamma \in\,]-1, 1[$. Since $\overrightarrow{RS} \in \Lambda_n$ we have $\overrightarrow{RS} = a_1 e_1 + a_2 e_2$ with a_1 and a_2 in $\mathbf{Z}$, hence

$$a_1 e_1 + a_2 e_2 + (m - l)(\alpha e_1 + \beta e_2) = \lambda e_1 + \gamma(\alpha e_1 + \beta e_2),$$
$$a_2 e_2 + (m - l)\beta e_2 = \gamma\beta e_2, \quad (m - l) = -\frac{a_2}{\beta} + \gamma.$$

Since $\overrightarrow{RS}$ belongs to $\mathbf{V}$, a_2 has at most $|\mathbf{V}|$ different values and since $m - l$ is an integer and $\gamma \in\,]-1, 1[$, $m - l$ has at most $2|\mathbf{V}|$ different values. Finally, the number of vectors

$$\begin{aligned}\overrightarrow{P'Q'} &= (B + j\theta_n + m\varepsilon_n)' - (A + k\theta_n + l\varepsilon_n)' \\ &= (B - A) + (j - k)\theta + (m - l)\varepsilon_n\end{aligned}$$

is less than $2|\mathbf{V}|^2(|\mathbf{V}| + 1)$. ∎

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10.7 Definition of $\mathcal{G}_1$

In this section and the following we simply write Λ instead of Λ_n and ε instead of ε_n. We have $\varepsilon = \alpha e_1 + \beta e_2$ with $0 < \beta \leq \alpha < 1$ and $N\beta > 1 + 3\beta$, and since $\mathbf{Z}^2 + \mathbf{Z}\theta$ is dense in $\mathbf{R}^2$, ε is not parallel to any lattice direction of Λ. Set

$$\Gamma = \Lambda + \{0, \dots, N-1\}\varepsilon \quad \text{and} \quad \mathbf{S} = \Lambda + [0, N-1]\varepsilon.$$

We want to define $\mathcal{G}_1$ such that the set of vertices of $\mathcal{G}_1$ is Γ ($\Gamma = \Gamma_n$) and such that **P3**, **P4** and **P5** hold for $\mathcal{G}_1$.

Conventions

(1) For all points $A, B, \dots$ in $\mathbf{R}^2$, $(a_1, a_2), (b_1, b_2), \dots$ denote the coordinates of $A, B, \dots$ in the basis (e_1, ε).
(2) The vectors or edges parallel to ε will be called *vertical* and the others *horizontal*.
(3) Writing a segment $[A, B]$ of $\mathbf{R}^2$, we shall always suppose that $b_1 \geq a_1$.
(4) A point A is *above* (*under*) a horizontal segment $[C, D]$ means that $c_1 \leq a_1 \leq d_1$ and that there is $t < 0$ ($t > 0$) with $A + t\varepsilon \in [C, D]$.

Lemma 10.5 *For all A in Γ there exists B in Γ such that $b_2 \in [a_2, a_2+1[$ and $b_1 > a_1$.*

Proof Take $B = A + e_1$. ■

Lemma 10.6 *If A and B are in Γ, if $a_1 = b_1$ and if $A \neq B$, then $|a_2 - b_2| \geq 1$.*

Proof We have $A = pe_1 + qe_2 + n\varepsilon$ and $B = re_1 + se_2 + m\varepsilon$, therefore

$$\overrightarrow{AB} = (r-p)e_1 + (s-q)e_2 + (m-n)\varepsilon = (b_2 - a_2)\varepsilon.$$

Hence, if $|b_2 - a_2| \in]0, 1[$, ε is parallel to the lattice direction $(r-p)e_1 + (s-q)e_2$. ■

Definition 10.7 For all A in Γ, the *successor* of A is the point B of Γ such that $b_2 \in [a_2, a_2 + 1[$ and $b_1 > a_1$ with b_1 minimal. We denote the successor A by A^+. By Lemma 10.6, the point A^+ is unique.

Lemma 10.8 *$\forall A \in \Gamma$, $[A, A^+]$ is not vertical and $a_1^+ - a_1 \leq 1$.*

Proof By definition, $a_1^+ > a_1$. Hence $[A, A^+]$ is not vertical. Furthermore $a_1^+ \leq b_1$ where $B = A + e_1$. ■

Lemma 10.9 *$\forall A \in \Gamma$, $]A, A^+[\cap \mathbf{S} = \varnothing$.*

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Proof Otherwise, there exists P in Λ such that $(P + [0, N-1]\varepsilon) \cap [A, A^+[\neq \varnothing$. By Lemma 10.8, this intersection contains only one point B. All the points in $P + [0, N-1]\varepsilon$ have the same first coordinate in the (e_1, ε) basis. This coordinate is b_1 and is in $]a_1, a_1^+[$ for $B \in]A, A^+[$. Let n be in $\{0, \dots, N-2\}$ such that $B \in P + [n, n+1]\varepsilon$. The second coordinate of one of the two points $P + n\varepsilon$ and $P + (n+1)\varepsilon$ is in $[a_2, a_2 + 1[$ which contradicts the definition of A^+. ■

Lemma 10.10 $\forall A, B \in \Gamma, A \neq B \Rightarrow [A, A^+[\cap [B, B^+[= \varnothing$.

Proof We can suppose $a_1 \leq b_1$. We consider 6 cases.

Case 1 If $b_1 \geq a_1^+$, then $[a_1, a_1^+[\ \cap \ [b_1, b_1^+[= \varnothing$ and $[A, A^+[\ \cap \ [B, B^+[= \varnothing$.

Case 2 If $b_2 > a_2^+$, then $[a_2, a_2^+] \ \cap \ [b_2, b_2^+] = \varnothing$ and $[A, A^+[\ \cap \ [B, B^+[= \varnothing$.

Case 3 If $b_1 \in]a_1, a_1^+[$ and $b_2 \in [a_2, a_2^+]$, then $b_2 \in [a_2, a_2 + 1[$ and the point B contradicts the definition of A^+. Indeed, we have $b_2 \in [a_2, a_2 + 1[$ and $a_1 < b_1 < a_1^+$, therefore a_1^+ is not minimal.

Case 4 If $b_1 = a_1$ and $b_2 \in [a_2, a_2^+]$, then $|b_2 - a_2| < 1$, which contradicts Lemma 10.6.

Case 5 If $b_1 \in [a_1, a_1^+[$, $b_2 < a_2$ and $b_2^+ \geq a_2^+$, then $a_2^+ \in [b_2, b_2^+] \subset [b_2, b_2 + 1[$ and $b_1 < a_1^+$. Hence, by definition of B^+, $b_1^+ \leq a_1^+$. But $a_2 \leq a_2^+ \leq b_2^+ < b_2 + 1 \leq a_2 + 1$, hence $a_2 \leq b_2^+ < a_2 + 1$. Since $a_1 \leq b_1 < b_1^+$, we get $B^+ = A^+$. Therefore $[A, A^+[\ \cap \ [B, B^+[= \varnothing$.

Case 6 If $b_1 \in [a_1, a_1^+[$, $b_2 < a_2$ and $b_2^+ < a_2^+$, then $b_2^+ < a_2 + 1$. We consider three sub-cases.

Subcase 6.1 If $b_2^+ < a_2$, then $[b_2, b_2^+] \cap [a_2, a_2^+] = \varnothing$ and $[A, A^+[\ \cap \ [B, B^+[= \varnothing$.

Subcase 6.2 If $a_2 \leq b_2^+$ and $b_1^+ \leq a_1^+$, then B^+ contradicts the definition of A^+.

Subcase 6.3 If $a_1^+ < b_1^+$, then $B^+ \neq A^+$ and by definition of B^+, $b_2^+ < b_2 + 1 \leq a_2^+$. Consider $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ defined by $f(X) = (x_1 - a_1)(a_2^+ - a_2) - (x_2 - a_2)(a_1^+ - a_1)$. We have $f(A) = f(A^+) = 0$, furthermore, $f(B) = (b_1 - a_1)(a_2^+ - a_2) - (b_2 - a_2)(a_1^+ - a_1) \geq 0$ and

$$
\begin{aligned}
f(B^+) &= (b_1^+ - a_1)(a_2^+ - a_2) - (b_2^+ - a_2)(a_1^+ - a_1) \\
&= (b_1^+ - a_1^+)(a_2^+ - a_2) + (a_1^+ - a_1)(a_2^+ - a_2) - (b_2^+ - a_2)(a_1^+ - a_1) \\
&= (b_1^+ - a_1^+)(a_2^+ - a_2) + (a_1^+ - a_1)(a_2^+ - b_2^+) > 0,
\end{aligned}
$$

hence $]B, B^+[\cap [A, A^+[= \varnothing$.

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■

Lemma 10.11 *$\forall A, B \in \Gamma$, $A = B$ or $]A, A^+[\, \cap [B, B^+] = \varnothing$.*

Proof If $A \neq B$, then by the previous lemma, $]A, A^+[\, \cap [B, B^+[\, = \varnothing$. But, by Lemma 10.9, no point of $\mathbf{S}$ belongs to $]A, A^+[$, hence the point B^+ which is in $\Gamma \subset \mathbf{S}$, is not in $]A, A^+[$. ■

Now, we wonder whether, for A in Γ, there exists B in Γ with $B^+ = A$. As before it is easy to prove the next lemma.

Lemma 10.12 (*cf.* Lemma 10.5) *For each A in Γ there exists a unique B in Γ with $b_1 < a_1$, $b_2 \in\,]a_2 - 1, a_2]$ and b_1 maximal. We call this point A^-.*

Lemma 10.13 *Let A be in Γ. Suppose there is no P in Γ with $P^+ = A$. Let $B = A^-$ and $C = B^+$. Then $C \in \Lambda$, $c_1 < a_1$ and $c_2 \in\,]a_2, a_2 + 1[$.*

Proof Since $a_2 - 1 < b_2 \leq a_2$ and $b_2 \leq c_2 < b_2 + 1$, we have $c_2 \in\,]a_2 - 1, a_2 + 1[$. Therefore by Lemma 10.6, we have $c_1 \neq a_1$ or $C = A$ which is false by hypothesis. Hence $c_1 \neq a_1$. Since $a_2 - 1 < b_2 \leq a_2$, we have $b_2 \leq a_2 < b_2 + 1$, together with $b_1 < a_1$, this gives by definition of B^+, $c_1 = b_1^+ \leq a_1$ and $c_1 < a_1$. It follows by definition of $B = A^-$, that $c_2 > a_2$. The point $D = C - \varepsilon$ cannot belong to Γ, for $c_2 - 1 > a_2 - 1$ and $c_2 - 1 < (b_2 + 1) - 1 = b_2 \leq a_2$ which contradicts the definition of $B = A^-$. Finally $C \in \Gamma$ and $C - \varepsilon \notin \Gamma$ imply $C \in \Lambda$. ■

Lemma 10.14 *Let A be in Γ. Suppose there is no P in Γ with $P^+ = A$. Let $B = A^-$ and $C = B^+$. The segment $]C, A[$ does not meet $\mathbf{S}$, nor any segment $[D, D^+]$ with D in Γ.*

Proof By definition of A^- the rectangle

$$\mathcal{R} = \{X \in \mathbf{R}^2 : x_1 \in\,]a_1^-, a_1[\text{ and } x_2 \in\,]a_2 - 1, a_2]\}$$

contains no point of Γ.

Step 1: If a vertical segment $[D, D + \varepsilon]$ with D and $D + \varepsilon$ in Γ meets $]C, A[$, then $d_1 \in\,]c_1, a_1[$, hence $d_1 \in\,]b_1, a_1[$. If $d_2 \leq a_2$, then $d_2 \leq a_2 - 1$, for D is not in the rectangle $\mathcal{R}$. But $d_2 + 1 \leq a_2$ is the second coordinate of $D + \varepsilon$ which is not in $\mathcal{R}$, therefore $d_2 + 1 \leq a_2 - 1$ and $[D, D + \varepsilon] \cap\,]C, A[\, = \varnothing$. If $a_2 < d_2 \leq c_2$, then $a_2 - 1 < d_2 - 1 \leq c_2 - 1 \leq a_2$, therefore, $D - \varepsilon$ is in $\mathcal{R}$, hence $D \in \Lambda$. It follows that the rectangle

$$\mathcal{U} = \{X \in \mathbf{R}^2 : x_1 \in\,]a_1 - 1, a_1[\text{ and } x_2 \in\,]a_2, a_2 + 1]\}$$

contains the two points C and D of Λ, which is impossible for $\beta < 1$, and therefore $\mathcal{U}$ is included in a fundamental domain for the action of Λ. If $d_2 > c_2$, the result is obvious.

Step 2: Before studying the intersections $[D, D^+] \cap]C, A[$, let us note that the rectangle

$$\mathcal{V} = \{X \in \mathbf{R}^2 : x_1 \in]a_1^-, a_1[\text{ and } x_2 \in]a_2, c_2]\}$$

contains only one point of Γ which is C. Indeed, if P is in $\mathcal{V} \cap \Gamma$, then $Q = P - \varepsilon$ satisfies $a_1^- < q_1 < a_1$ and $a_2 - 1 < q_2 \le c_2 - 1 \le a_2$ which contradicts the definition of A^-. Furthermore, since $\beta < 1$, $\mathcal{V}$ cannot contain two points of Λ.

Let D be in Γ. We can suppose $D \neq B$.

- If $d_2 < a_2$ and $d_1 \in [b_1, a_1[$, then $d_2 \le a_2 - 1$ for $D \neq B$ and $D \notin \mathcal{R}$. Hence, $d_2^+ < d_2 + 1 \le a_2$, $[d_2, d_2^+[\cap]a_2, c_2[$ and $[D, D^+] \cap]C, A[= \varnothing$.
- If $d_2 < a_2$ and $a_1 \le d_1$, then $[d_1, d_1^+] \cap]c_1, a_1[= \varnothing$.
- If $d_2 < a_2$ and $d_1 < b_1$ and $d_2 < b_2$, then by definition of D^+, $d_1^+ \le b_1$ or $d_2 \le b_2 - 1$. It follows that $[d_1, d_1^+] \cap]c_1, a_1[= \varnothing$ or $[d_1, d_1^+] \cap]a_2, c_2[= \varnothing$. Hence $[D, D^+] \cap]C, A[= \varnothing$.
- If $d_2 < a_2$ and $d_1 < b_1$ and $b_2 \le d_2$, then $d_1^+ \le c_1$. Therefore, $[d_1, d_1^+] \cap]c_1, a_1[= \varnothing$.
- If $c_2 < d_2$, then $[d_2, d_2^+] \cap [a_2, c_2] = \varnothing$ and $[D, D^+] \cap [C, A[= \varnothing$.
- If $d_2 \in [a_2, c_2]$ and $d_1 < c_1$, then, by definition of D^+, $d_1^+ \le c_1$ and $[D, D^+] \cap]C, A[= \varnothing$.
- If $d_2 \in [a_2, c_2]$ and $a_1 \le d_1$, then $]c_1, a_1[\cap [d_1, d_1^+] = \varnothing$ and $[D, D^+] \cap]C, A[= \varnothing$.
- If $d_2 \in]a_2, c_2]$ and $d_1 \in [c_1, a_1[$, then $D \in \mathcal{V}$ and $D = C$.
- If $d_2 = a_2$ and $d_1 \in [c_1, a_1[$, then the point D contradicts the definition of B.

∎

Lemma 10.15 (*cf.* Lemma 10.14)

(i) *Let C be in Λ. There is at most one A in Γ with $(A^-)^+ = C$.*
(ii) *All segments $[C, A]$ with C in Λ and $C = (A^-)^+$ are disjoint.*

Proof For D in Λ, set

$$\mathcal{R}(D) = \{X \in \mathbf{R}^2 : x_1 \in [d_1, d_1 + 1[\text{ and } x_2 \in [d_2 - 1, d_2]\}.$$

Since $\beta < 1$, the rectangle $\mathcal{R}(D)$ contains exactly one point of Λ which is D. Furthermore, these rectangles are disjoint.

(i) Let A and B be in Γ such that $(A^-)^+ = (B^-)^+ = C$. Suppose that $A \neq B$. We have a_2 and $b_2 \in]c_2 - 1, c_2[$, hence, by Lemma 10.6, $a_1 \neq b_1$. Suppose $a_1 < b_1$. If A is above $[C, B]$, then by Lemma 10.14, A is in Λ for $[A - \varepsilon, A]$ meets $[C, B]$. Since $\mathcal{R}(C)$ contains only one point of Λ which is C, A is under $[C, B]$. It follows that A is in $\Lambda + (N - 1)\varepsilon$ and therefore $A - \varepsilon$ is in Γ. One of the points A or $A - \varepsilon$ is in

$$\{X \in \mathbf{R}^2 : b_1^- < x_1 < b_1 \text{ and } x_2 \in]b_2 - 1, b_2]\}$$

which contradicts the definition of B^-.

(ii) The segment $[C, A]$ is contained in $\mathcal{R}(C)$ and the rectangles, $\mathcal{R}(C)$, $C \in \Lambda$, are disjoint. ∎

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Definition of $\mathcal{G}_1$ $\mathcal{G}_1$ is the collection of all edges of one of the three types:

(i) vertical edges $[D, D+\varepsilon]$ with D and $D+\varepsilon$ in Γ,
(ii) horizontal edges $[A, A^+]$ with A in Γ,
(iii) exceptional horizontal edges $[C, A]$ where A is in Γ and there is no point P in Γ with $P^+ = A$ and $C = (A^-)^+$.

By Lemmas 10.6, 10.9, 10.11, 10.14 and 10.15, $\mathcal{G}_1$ is a planar graph. (Since Γ is discrete and the lengths of edges of $\mathcal{G}_1$ are bounded, a bounded region of $\mathbf{R}^2$ meets only finitely many edges of $\mathcal{G}_i$.) Clearly, $\mathcal{G}_1$ is invariant by Λ-translation, and $\mathcal{G}_1$ satisfies **P3** and **P4**.

Now we use Proposition 10.1 to prove the following.

Proposition 10.16 *Property **P5** is true for* $\mathcal{G}_1$.

Proof Let $[A, B]$ be an edge of $\mathcal{G}_1$. There exist unique R, S in Λ and m, l in $\{0, \dots, N-1\}$ such that $A = R + m\varepsilon$ and $B = S + l\varepsilon$. If $[A, B]$ is vertical, then $\overrightarrow{RS} = 0$ for ε is not parallel to any lattice direction. Suppose $[A, B]$ is a horizontal edge of $\mathcal{G}_1$ such that $B = A^+$. We would like to prove that $\overrightarrow{RS}$ takes at most 6 values. Set $\mathbf{U} =]-1, N-1]\varepsilon$ and for P in $\mathbf{R}^2$, $\mathcal{T}(P) = \{t \in \mathbf{R} : P + te_1 \in \Lambda + \mathbf{U}\}$. For $t \in \mathcal{T}(P)$ let $t' = \min\{s \in \mathcal{T}(P) : t < s\}$. By Proposition 10.1, for each t in $\mathcal{T}(P)$ there exists a unique point $M(P, t)$ such that $M(P, t) \in \Lambda$ and $P + te_1 \in M(P, t) + \mathbf{U}$. Furthermore, the difference $M(P, t') - M(P, t)$ takes at most 6 values when P runs through $\mathbf{R}^2$ and t runs through $\mathcal{T}(P)$. By definition of A^+,

$$a_1^+ = \min\{s > a_1 : \exists\, \delta \in [0, 1[, se_1 + a_2\varepsilon + \delta\varepsilon \in \Gamma\}.$$

Now $se_1 + a_2\varepsilon + \delta\varepsilon \in \Gamma$ with $\delta \in [0, 1[$ is equivalent to $A + (s - a_1)e_1 = se_1 + a_2\varepsilon \in \Lambda + \mathbf{U}$, therefore

$$a_1^+ - a_1 = \min\{s > 0 : A + se_1 \in \Lambda + \mathbf{U}\} = 0'.$$

Since $R = M(A, 0)$ and $S = M(A, a_1^+ - a_1)$, it follows by Proposition 10.1 that $\overrightarrow{RS}$ takes at most six values.

Since $\mathcal{G}_1$ is Λ-invariant, if $[A, B]$ is an exceptional edge, all other exceptional edges are of the shape $[A, B] + \overrightarrow{u}$ with $\overrightarrow{u} \in \Lambda$. Therefore, exceptional edges give rise to only one vector $\overrightarrow{RS}$. ∎

Remark The construction of the graph $\mathcal{G}_1$ works with $\mathbf{Z}^2$ instead of the lattice Λ_n and $\varepsilon = \theta$ instead of ε_n. But it leads to a graph whose edges have lengths which do not go to zero.

10.8 The Boundary of the Elements of $\mathcal{C}(\mathcal{G}_1)$

The aim of this paragraph is to prove **P2** for $\mathcal{G}_1$.

Proposition 10.17 *The connected components of* $\mathbf{R}^2 \backslash \mathcal{G}_1$ *are bounded and their boundaries are connected, bounded and made of at most six edges.*

Lemma 10.18 *If $[A, B]$ is an horizontal edge of $\mathcal{G}_1$, then A or B is not in Λ.*

Proof Suppose $A \in \Lambda$. The rectangle

$$\mathcal{R} = \{X \in \mathbf{R}^2 : x_1 \in [a_1, a_1 + 1[,\ x_2 \in [a_2, a_2 + 1/\beta[\}$$

is a fundamental domain for the action of Λ, therefore it contains exactly one point of $\Lambda + (N-1)\varepsilon$; let $C = D + (N-1)\varepsilon$ be this point. Since $(N-1)\beta \geq 1 + 2\beta$, we have $d_2 < a_2 - 2$. Since ε is not parallel to any directions of Λ, ε is not parallel to $\overrightarrow{DA}$ and $d_1 \in\ ]a_1, a_1 + 1[$. Since $[A, B]$ is a horizontal edge of $\mathcal{G}_1$, we have $b_2 \in\ ]a_2 - 1, a_2 + 1[$. Furthermore, by Lemma 10.9, $[A, B]$ does not cross $[D, C]$, therefore $b_1 \in\ ]a_1, d_1[$. It follows that $B \in \mathcal{R} - \varepsilon$. The rectangle $\mathcal{R} - \varepsilon$ contains already one point of Λ which is A, hence $B \notin \Lambda$. ■

A similar proof leads to:

Lemma 10.19 (*cf.* Lemma 10.18) *If $[A, B]$ is a horizontal edge of $\mathcal{G}_1$, then A or B is not in $\Lambda + (N-1)\varepsilon$.*

Lemma 10.20 *Let P be in $\mathbf{R}^2$. Then there is a horizontal edge $[A, B]$ of $\mathcal{G}_1$ and $\lambda \in [0, 3[$ such that $P - \lambda\varepsilon \in [A, B]$.*

Proof Since $(N-1)\beta \geq 1$, the set $E = \{A \in \Gamma : a_2 \in [p_2 - 2, p_2 - 1]$ and $a_1 < p_1\}$ is not empty. Let A be the point of E with a_1 maximal. If $p_1 \leq a_1^+$, we have $P - \lambda\varepsilon \in [A, A^+]$ with $\lambda \in [0, 2[$. If $a_1^+ < p_1$, by Lemma 10.18, one of the points A or A^+ is not in Λ; but A^+ must be in Λ for the point $A^+ - \varepsilon$ cannot be in E. Now, the line $[B = A - \varepsilon, B^+] \cup [B^+, B^{++}] \cup \cdots$ is under the point P. ■

Lemma 10.21 *Let $[A, B = A^+]$ be a horizontal edge of $\mathcal{G}_1$ with A and B not in $\Lambda + (N-1)\varepsilon$. If $(A + \varepsilon)^+ \neq B + \varepsilon$, then there exists D such that $[A + \varepsilon, D]$ and $[D, B + \varepsilon]$ are in $\mathcal{G}_1$.*

Proof For all point $P \in [A, B]$, let $[A_P, B_P]$ be the first horizontal edge of $\mathcal{G}_1$ meeting the half line $P + \lambda\varepsilon$, $\lambda > 0$. By Lemma 10.20 and the discreteness of $\mathcal{G}_1$ there are only a finite number of such edges, *i.e.*,

$$\{[A_P, B_P] : P \in [A, B]\} = \{[A_1, B_1], \ldots, [A_n, B_n]\}.$$

We can suppose that the first coordinates $a_{i,1}$ of A_i are in increasing order.

Step 1: Let us show that $a_1 \leq a_{i,1} \leq b_{i,1} \leq b_1$ for all i. Suppose $a_{i,1} < a_1$ for some i. Let P_i be a point of $[A, B]$ for which $[A_i, B_i]$ is the first edge to meet the half line $P + \mathbf{R}^+\varepsilon$. The edge $[A + \varepsilon, (A + \varepsilon)^+]$ cannot meet $[A_i, B_i[$, therefore, $A + \varepsilon$ is under $[A_i, B_i]$. It follows that the edge $[A + \varepsilon, (A + \varepsilon)^+]$ is entirely under $[A_i, B_i]$. Let $C = (A + \varepsilon)^+$. Since $N \geq 6$, the points $C + \varepsilon$, $C + 2\varepsilon$ and $C + 3\varepsilon$ are in Γ. But by previous remark, one of the edges $[C, C + \varepsilon]$, $[C + \varepsilon, C + 2\varepsilon]$ or $[C + 2\varepsilon, C + 3\varepsilon]$ meets

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$[A_i, B_i]$ which means that $C, C + \varepsilon$ or $C + 2\varepsilon$ is B_i and we can drop the edge $[A_i, B_i]$. By the same reasoning, we prove that $b_{i,1} \le b_1$. (In this case, we use the existence of a point C such that $C^+ = B + \varepsilon$ or such that $[C, B + \varepsilon]$ is an exceptional edge.)

Step 2: By 1, $a_{1,1} = a_1$. Furthermore, by definition of $\mathcal{G}_1$, there is no edge $[A, C]$ with $[A, C]$ above $[A, B]$, therefore $A_1 = A + \varepsilon$.

Step 3: Let us show that if a point A_i is such that $a_{i,1} > a_1$, then A_i is in Λ. Indeed, if $C = A_i - \varepsilon$ is in Γ, then the sequence of edges $[C_i, C_i^+]$, $[C_i^+, C_i^{++}]$, $[C_i^{++}, C_i^{+++}], \dots$ is under $[A_i, B_i]$, which contradicts the definition of the segment $[A_i, B_i]$.

Step 4: Since the rectangle $\mathcal{R} = \{X \in \mathbf{R}^2 : x_1 \in [a_1, b_1[$ and $x_2 \in [a_2, a_2 + 4]\}$ contains at most one point of Λ, it follows that $n = 1$ or 2.

Step 5: Suppose $n = 1$. Since $A + \varepsilon$ is not in Λ, the edge $[A_1, B_1]$ cannot be exceptional, hence $b_{1,2} \ge a_2 + 1 > b_2$. It follows that $B + \varepsilon$ is under $[A_1, B_1]$ and therefore $B_1 = B + \varepsilon$.

Step 6: Suppose $n = 2$. Considering the edge $[B_1, B_1^+]$, $[B_1^+, B_1^{++}]$, $[B_1^{++}, B_1^{+++}] \dots$, we see that A_2 must be under the edge $[A_1, B_1]$. Considering the predecessor of A_2, we see that $A_2 = B_1$. If B_2 is not B, then $B + \varepsilon$ is under $[A_2, B_2]$ and it follows that $B_2 = B + \varepsilon$. ∎

The proofs of Lemmas 10.22, 10.23 and 10.24 are similar to the proof of Lemma 10.21.

Lemma 10.22 *Suppose $[A, B = A^+]$ is in $\mathcal{G}_1$ and that B is in $\Lambda + (N-1)\varepsilon$. Then either $(A + \varepsilon)^+ = B^+ + \varepsilon$ either $[A + \varepsilon, (A + \varepsilon)^+]$ and $[(A + \varepsilon)^+, B^+ + \varepsilon]$ are in $\mathcal{G}_1$.*

Lemma 10.23 *Suppose $[A, B]$ is an exceptional edge of $\mathcal{G}_1$ and that B is not in $\Lambda + (N-1)\varepsilon$. Then $A^+ = B + \varepsilon$ is in $\mathcal{G}_1$.*

Lemma 10.24 *Suppose $[A, B]$ is an exceptional edge of $\mathcal{G}_1$ and that B is in $\Lambda + (N-1)\varepsilon$. Then either $A^+ = B^+ + \varepsilon$, or $[A^+, B^+ + \varepsilon]$ is in $\mathcal{G}_1$.*

Proof of Proposition 10.17 Let ω be in a connected component of $\mathbf{R}^2 \backslash \mathcal{G}_1$ and Q in ω. There exists a first horizontal edge to meet the half line $Q - t\varepsilon$, $t > 0$, let $[A_1, A_2]$ be this edge. By Lemmas 10.21, 10.22, 10.23 and 10.24, we can find a sequence of edges $[A_1, A_2], \dots, [A_{n-1}, A_n = A_1]$ with $n \le 6$, such that Q is inside the simple closed curve $\gamma = [A_1, A_2] \cup \cdots \cup [A_{n-1}, A_n]$. Now, it is easy to see that ω is bounded and that $\partial\omega = \gamma$. ∎

10.9 The Graph $\mathcal{G}_2$

Definition of $\mathcal{G}_2$ The vertices of $\mathcal{G}_2$ and the vertices of $\mathcal{G}_1$ are the same. We remove all horizontal edges $[A, B]$ of $\mathcal{G}_1$ such that there exist C in Λ and $t \in]0, 1[$ with $C - t\varepsilon \in]A, B[$ or such that there exist C in $\Lambda + (N-1)\varepsilon$ and $t \in]0, 1[$ with

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$C + t\varepsilon \in\,]A, B[$. In these cases, we shall say that the edge $[A, B]$ is *removed* by C. We keep all the other edges of $\mathcal{G}_1$. We get a new set of edges $\mathcal{G}_2$. Since $\mathcal{G}_1$ is Λ invariant, $\mathcal{G}_2$ is also Λ invariant. The only thing to prove about $\mathcal{G}_2$, is the property **P2**. It follows from the next proposition.

Proposition 10.25

(i) *If ω is a connected component of $\mathbf{R}^2\backslash\mathcal{G}_2$, there exist $\omega_1, \dots, \omega_m$ connected components of $\mathbf{R}^2\backslash\mathcal{G}_1$, with $m \le 7$, such that $\overline{\omega} = \overline{\omega_1} \cup \cdots \cup \overline{\omega_m}$.*

(ii) *The property **P2** holds for $\mathcal{G}_2$.*

Since **P2** holds for $\mathcal{G}_1$ (Proposition 10.17), (ii) follows from (i). The proof of (i) is rather long; we outline the proof:

Step 1: We study the number of removed edges by one element of Λ or $\Lambda + (N-1)\varepsilon$.

Step 2: Vertical curves. We construct a sequence $(\Gamma_n)_{n\in\mathbf{Z}}$ of simple curves such that

- $\Gamma_{n+1} = \Gamma_n + e_1$,
- all Γ_n are in the union of edges of $\mathcal{G}_2$,
- if $n \ne m$, then $\Gamma_n \cap \Gamma_m = \varnothing$,
- all Γ_n go from $+\infty$ to $-\infty$ in the ε direction, that is $\lim_{t\to+\infty} \Gamma_{n,2}(t) = +\infty$ and $\lim_{t\to-\infty} \Gamma_{n,2}(t) = -\infty$.

Step 3: Horizontal curves. For some $B \in \Gamma$ we construct a simple curve

$$\Delta_B = [B_0 = B, B_1] \cup [B_1, B_2] \cup \cdots \cup [B_{n-1}, B_n]$$

contained in $\mathcal{G}_2$ and in a strip of width 3, namely $\{X \in \mathbf{R}^2 : x_2 \in [t, t+3]\}$.

Step 4: Let ω be a connected component of $\mathbf{R}^2\backslash\mathcal{G}_2$. Since previous curves are in $\mathcal{G}_2$, they do not meet ω. Jordan's theorem allows us to enclose ω in a box $\mathcal{B}$ whose boundary is made of parts of previous curves. Furthermore, we construct the curves Δ_B sufficiently close to each other so that $\mathcal{B}$ is contained in a fundamental domain for the action of Λ on $\mathbf{R}^2$. By Step 1, we see that there are at most six removed edges of $\mathcal{G}_1$ in $\mathcal{B}$. This shows that ω is contained in the union of at most 7 connected components of $\mathcal{G}_1$.

10.9.1 Number of Removed Edges

Lemma 10.26 *For each C in Λ there exist at most three removed edges $[A_i, B_i]$ of the first kind,* i.e., *such that there exists $t_i \in\,]0, 1[$ with $C - t_i\varepsilon \in [A_i, B_i]$.*

For each C in $\Lambda + (N-1)\varepsilon$ there exist at most three removed edges $[A_i, B_i]$ of the second kind, i.e., *such that there exists $t_i \in\,]0, 1[$ with $C + t_i\varepsilon \in [A_i, B_i]$.*

It is enough to prove the following.

Lemma 10.27 *If P is in $\mathbf{R}^2$, then there are at most three edges $[A_i, B_i]$ such that*

$$]A_i, B_i[\, \cap [P, P+\varepsilon] \ne \varnothing, \quad i = 1, 2, 3.$$

Proof Suppose on the contrary that there are four edges $[A_i, B_i]$ such that $]A_i, B_i[\cap [P, P+\varepsilon] \neq \varnothing$, $i = 1, 2, 3, 4$. Let C_i be the intersection of $]A_i, B_i[$ and $[P, P+\varepsilon]$. We can assume that $c_{i,2}$ are in increasing order.

For all horizontal edge $[A, B]$, we have $b_1 - a_1 \leq 1$ and $|a_2 - b_2| < 1$, therefore, $a_{i,1} \in]p_1 - 1, p_1[$ and $a_{i,2} \in]p_2 - 1, p_2 + 1[$. Since the rectangle $\mathcal{R} = \{X \in \mathbf{R}^2 : x_1 \in]p_1 - 1, p_1[\text{ and } x_2 \in]p_2 - 1, p_2 + 2[\}$ contains at most one point of Λ and at most one point of $\Lambda + (N-1)\varepsilon$, there is at most one point of Λ among the A_i and one point of $\Lambda + (N-1)\varepsilon$. Remember that $N - 1 \geq 6$ and that the second coordinate is nondecreasing along all horizontal nonexceptional edges.

Case 1 The points A_i are all distinct.

If $a_{1,1} < a_{2,1}$, then A_2 or $A_2 - \varepsilon$ or $A_2 - 2\varepsilon$ is in Λ and hence $[A_2, A_2 + \varepsilon]$, $[A_2 + \varepsilon, A_2 + 2\varepsilon]$ and $[A_2 + 2\varepsilon, A_2 + 3\varepsilon]$ are in $\mathcal{G}_1$. It follows that $a_{3,1} \geq a_{2,1}$. But if $a_{3,1} > a_{2,1}$, then A_3 or $A_3 - \varepsilon$ or $A_3 - 2\varepsilon$ is in Λ and there would be two points of Λ in $\mathcal{R}$, which is not possible, so $a_{3,1} = a_{2,1}$. Therefore $A_3 = A_2 + \varepsilon$ or $A_2 + 2\varepsilon$. Again, we get $A_4 = A_3 + \varepsilon$, but this is impossible, for $\mathcal{T} = \{X \in \mathbf{R}^2 : x_1 \in]p_1 - 1, p_1[\text{ and } x_2 \in]p_2 - 1, p_2 + 1[\}$ cannot contain A_2, $A_2 + \varepsilon$ and $A_2 + 2\varepsilon$.

If $a_{1,1} > a_{2,1}$, then A_1 or $A_1 + \varepsilon$ or $A_1 + 2\varepsilon$ is in $\Lambda + (N-1)\varepsilon$. It follows that $[A_2, A_2 + \varepsilon]$, $[A_2 + \varepsilon, A_2 + 2\varepsilon]$ and $[A_2 + 2\varepsilon, A_2 + 3\varepsilon]$ are in $\mathcal{G}_1$. Therefore $a_{3,1}$ or $a_{4,1} < a_{2,1}$ is not possible.

If $a_{3,1} = a_{2,1}$, then $A_3 = A_2 + \varepsilon$ and $a_{4,1} > a_{2,1}$; now there is no room for A_4. Indeed, $a_{4,2} < a_{3,2}$ implies that $A_4 \in \Lambda$ and the edge $[A_3, B_3]$ cannot cross $[A_4, A_4 + \varepsilon] \cup [A_4 + \varepsilon, A_4 + 2\varepsilon]$. Furthermore, $a_{4,2} \geq a_{3,2}$ implies $b_{3,1} = a^+_{3,1} \leq a_{4,1} < p_1$ or $a_{4,2} \geq a_{3,2} + 1 = a_{2,2} + 2 > p_2 + 1$. It follows that $[A_4, B_4]$ is an exceptional edge. In this case the point $D = B_4^-$ verifies $d_2 < b_{4,2} < a_{3,2} + 1$, $d_2 > a_{4,2} - 1 \geq a_{3,2}$ and $a_{3,1} < d_1 < a_{4,1}$; therefore $a^+_{3,1} \leq d_1 < p_1$, which is impossible.

If $a_{3,1} > a_{2,1}$, then A_3 or $A_3 - \varepsilon$ is in Λ, hence $a_{4,1} \geq a_{3,1}$. Furthermore, $a_{3,2} \geq a_{2,2} + 1 > p_2$ for $A_2^+ \neq A_3$. Therefore $A_4 \neq A_3 + \varepsilon$ and $a_{4,1} > a_{3,1}$. Again A_4 has to be in Λ and there are too many points in Λ.

If $a_{1,1} = a_{2,1}$, then $A_2 = A_1 + \varepsilon$.

If $a_{3,1} > a_{2,1}$, then $A_3 \in \Lambda$ and $[A_3, A_3 + \varepsilon]$ is in $\mathcal{G}_1$. Now, there is no room for A_4.

If $a_{3,1} = a_{2,1}$, then $A_3 = A_2 + \varepsilon = A_1 + 2\varepsilon$ and $[A_3, B_3]$ cannot meet $[P, P + \varepsilon]$.

If $a_{3,1} < a_{2,1}$, then $A_2 \in \Lambda + (N-1)\varepsilon$. Therefore, $[A_3, A_3 + \varepsilon]$ is in $\mathcal{G}_1$. It follows that $a_{4,1} > a_{3,1}$ and $A_4 \in \Lambda$. Hence, $A_3 \notin \Lambda$ and $A_3^+ = B_3$. By definition of A_3^+, we have $a_{4,2} \geq a_{3,2} + 1$, for the same reason we have $a_{3,2} > a_{2,2}$, therefore $a_{4,2} \geq a_{2,2} + 1 \geq a_{1,1} + 2 > p_2 + 1$. It follows that $[A_4, B_4]$ is an exceptional edge. The point $D = B_3^-$ is not A_3 for $A_3^+ \neq A_4$. We have $d_2 > a_{4,2} - 1 \geq a_{3,2}$, $d_2 < p_2 \leq a_{1,2} + 2 \leq a_{2,2} + 1 \leq a_{3,2} + 1$, therefore, by definition of A_3^+, $d_1 \leq a_{3,1}$. If $d_1 = a_{3,1}$, then $D = A_3 + \varepsilon$ and $b_{4,2} > d_2 > p_2 + 1$, which is impossible, for $[A_4, B_4]$ is an exceptional edge. Finally, $d_1 < a_{3,1}$ is impossible for in this case, $[D, D^+ = A_4]$ must cross $[A_3, A_3 + \varepsilon]$ or $[A_3 + \varepsilon, A_3 + 2\varepsilon]$.

Case 2 The points A_i are not distinct.

Since there are four edges $[A_i, B_i]$, there must be at least one point A_i with two successors. Two A_i with two successors is impossible, for points with two successors

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are in Λ and $\mathcal{R}$ contains at most one point of Λ. It follows that one of the A_i is in Λ and has two successors, and that the other two are distinct and not in Λ.

If A_1 is in Λ, then $A_2 = A_1$, $a_{1,2} \in [p_2, p_2+1]$ and the edge $[A_1, A_1+\varepsilon]$ exists. The point A_3 must be over $[A_1, A_1^+]$, but this implies that A_3 is in Λ, which is impossible.

If $A_2 = A_3$ is in Λ, then $a_{2,2} \in [p_2, p_2+1]$ and the edge $[A_2, A_2+\varepsilon]$ exists. The point A_4 must be over $[A_2, A_2^+]$, but this implies that A_4 is in Λ, which is again impossible.

If $A_3 = A_4$ is in Λ, then $[A_3, B_3]$ is an exceptional edge and $B_3 \notin \Lambda$. Since $[A_3, B_3]$ is exceptional, there is no point Q in Λ with $Q^+ = B_3$, hence A_1^+ and $A_2^+ \neq B_3$. Considering the polygonal line $[A_3, B_3] \cup [B_3, B_3 - \varepsilon]$, we see that $B_1 = A_1^+$ and $B_2 = A_2^+$ are in $\Lambda + (N-1)\varepsilon$ for the segments $[B_1, B_1+\varepsilon]$ and $[B_2, B_2+\varepsilon]$ meet $[A_3, B_3]$. Therefore $A_1^+ = A_2^+$, but this implies that $A_1 \in \Lambda + (N-1)\varepsilon$, which is impossible, for $\mathcal{R}$ contains only one point of $\Lambda + (N-1)\varepsilon$. ∎

Lemma 10.28 *Either $[A, A^+]$ is in $\mathcal{G}_2$ for all A in Λ, or $[A+\varepsilon, (A+\varepsilon)^+]$ is in $\mathcal{G}_2$ for all A in Λ.*

Proof These edges can only be removed by a point of $\Lambda + (N-1)\varepsilon$. Suppose that the edge $[A, A^+]$ is removed by a point C of $\Lambda + (N-1)\varepsilon$. Therefore $|a_1^+ - c_1| < 1$ and $|a_2^+ - c_2| \leq 2$. It follows that C and A^+ cannot be both in $\Lambda + (N-1)\varepsilon$, so A^+ is not in $\Lambda + (N-1)\varepsilon$. The interior of the parallelogram $\mathcal{P} = \operatorname{conv}(A, A^+, A^+ + \varepsilon, A+\varepsilon)$ cannot contain a point of Γ for such a point would be in Λ and would be too close to A which is already in Λ. It follows that the edge $[A+\varepsilon, (A+\varepsilon)^+]$ is above $\mathcal{P}$ and therefore can not be removed by C. Since $\mathcal{G}_1$ is invariant by Λ-translations, if $[A, A^+]$ is in $\mathcal{G}_2$ for A in Λ, it holds for all A in Λ and the same is true for $[A+\varepsilon, (A+\varepsilon)^+]$. ∎

10.9.2 Construction of Vertical Curves

Let A be in Λ. Let V be the vector of Λ such that (Case i) $A^+ \in A + V + \{0, N-1\}\varepsilon$ if $[A, A^+] \in \mathcal{G}_2$, and (Case ii) $(A+\varepsilon)^+ \in A + V + \{0, N-1\}\varepsilon$ if $[A, A^+] \notin \mathcal{G}_2$. By Λ-translation invariance the vector V does not depend on A.

Lemma 10.29 $v_2 \leq -1/\beta$.

Proof We prove the inequality in (Case ii); (Case i) is similar. Set $B = A + \varepsilon$. By definition of V we have $B^+ \in A + V + \{0, N-1\}\varepsilon$, therefore $b_2^+ \in [a_2 + v_2, a_2 + v_2 + (N-1)] \cap [a_2+1, a_2+2[$. Furthermore $v_2\beta \in \mathbf{Z}$ and $\beta < 1/2$, hence $v_2 \leq 0$. If $v_2 = 0$, then $v_1 \geq 1$ and the set $\{X \in \mathbf{R}^2 : x_1 \in \,]a_1, a_1+1[$ and $x_2 \in [a_2+1, a_2+2[\}$ contains no point of Γ, which is impossible for $(N-1)\beta \geq 1 + \beta$. Therefore $v_2\beta \leq -1$. ∎

We consider two cases.

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Case 1 The points A and $A+V$ are joined by the path

$$\gamma_A = [A, A^+] \cup [A^+, A^+ - \varepsilon] \cup [A^+ - \varepsilon, A^+ - 2\varepsilon] \cup \cdots \cup [A + V + \varepsilon, A + V].$$

Case 2 The points $A+\varepsilon$ and $A+V+\varepsilon$ are joined by the path

$$\gamma_A = [A+\varepsilon, B = (A+\varepsilon)^+] \cup [B, B-\varepsilon] \cup [B-\varepsilon, B-2\varepsilon] \cup \cdots \cup [A+V+2\varepsilon, A+V+\varepsilon].$$

Now we define the vertical path as the infinite union

$$\Gamma_A = \bigcup_{n\in\mathbf{Z}} \gamma_{A+nV}.$$

The curve Γ_A is clearly simple and goes from $+\infty$ to $-\infty$ in the ε direction, therefore, by Jordan's theorem, Γ_A splits $\mathbf{R}^2$ in exactly two connected components. Moreover $\Gamma_A \subset \mathcal{G}_2$ and $\Gamma_{A+e_1} = \Gamma_A + e_1$

Definition 10.30 The *vertical curves* are $\Gamma_n = \Gamma_{ne_1}$, $n \in \mathbf{Z}$.

Lemma 10.31 *Let A be in Λ and t in $\mathbf{R}$. Then* $\operatorname{diam} \Gamma_A \cap \{X \in \mathbf{R}^2 : x_2 = t\} < 1$.

Proof We give the proof in Case 1; the second case is similar. Let t be in $\mathbf{R}$ and set $\mathcal{D}_t = \{X \in \mathbf{R}^2 : x_2 = t\}$. For B in $\Lambda \cap \Gamma_A$, let I_B be the interval $\{x_2 : X \in \gamma_B\}$ and $J_B = [b_2, b_2^+]$. We have $I_B = [b_2 + v_2, b_2^+]$. Since $\beta < 1/2$ and $v_2 \le -1/\beta$, the intervals J_B, $B \in \Lambda \cap \Gamma_A$, are disjoint and an interval I_B only meets I_{B-V} and I_{B+V}. Furthermore, $I_B \cap I_{B-V} = J_B$. It follows that there exists B in $\Gamma_A \cap \Lambda$ such that $\mathcal{D}_t \cap \Gamma_A = \mathcal{D}_t \cap \gamma_B$ or $\mathcal{D}_t \cap (\gamma_B \cup \gamma_{B-V})$. Since $(N-1)\beta \ge 1$, we have $b_1^+ - b_1 < 1$ (use $(N-1)\beta \ge 1+\beta$ in Case 2). Therefore, $\operatorname{diam}(\mathcal{D}_t \cap \gamma_B) \le b_1^+ - b_1 < 1$. If $\mathcal{D}_t \cap \gamma_B \ne \varnothing$ and $\mathcal{D}_t \cap \gamma_{B-V} \ne \varnothing$, then $t \in J_B$ and $\operatorname{diam} \mathcal{D}_t \cap (\gamma_B \cup \gamma_{B-V}) \le b_1^+ - b_1 < 1$. ■

The following lemma is an obvious consequence of Lemma 10.31.

Lemma 10.32 *Let A be in Λ. Then $\Gamma_A \cap \Gamma_{A+e_1} = \varnothing$.*

10.9.3 Construction of Horizontal Curves.

Lemma 10.33 *Let A be in Γ. There exists a sequence $(A_n)_{n\in\mathbf{N}}$ of Γ such that $A_0 = A$, and for all n, $a_{n,2} \in [a_2, a_2 + 3[$ and either $A_{n+1} = A_n^+$ or $A_n - \varepsilon$ and $[A_n, A_{n+1}] \in \mathcal{G}_1$.*

Proof We proceed by induction. Take $A_0 = A$. Assume $A_0, \dots, A_n$ are constructed.

- If $a_{n,2} \le a_2 + 2$, take $A_{n+1} = A_n^+$.
- If $a_{n,2} > a_2 + 2$ and $[A_n, A_n - \varepsilon] \in \mathcal{G}_1$, take $A_{n+1} = A_n - \varepsilon$.
- If $a_{n,2} > a_2 + 2$ and $[A_n, A_n - \varepsilon] \notin \mathcal{G}_1$, then $A_{n-1} - \varepsilon \ne A_n$ and by Lemma 10.18, $[A_{n-1}, A_{n-1} - \varepsilon] \in \mathcal{G}_1$. We change A_n to $A_{n-1} - \varepsilon$ and take $A_{n+1} = (A_{n-1} - \varepsilon)^+$.

■

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Completion of Proof of Proposition 10.25 Let ω be a connected component of $\mathbf{R}^2\backslash\mathcal{G}_2$. Each curve Γ_n splits $\mathbf{R}^2$ into two connected components. Let Γ_n^+ be the connected component of $\mathbf{R}^2\backslash\Gamma_n$ which contains $\Gamma_{n+1} = \Gamma_n + e_1$ and Γ_n^- the other component. Set $\Omega_n = \Gamma_n^+\backslash\overline{\Gamma_{n+1}^+}$, so Ω_n is the region lying between Γ_n and Γ_{n+1}. Since, by definition of the curve Γ_n, ω does not meet the curve Γ_n, there exists $n_0 \in \mathbf{Z}$ such that ω is in $\Omega_{n_0} = \Omega$. For a point $A \in \Gamma\cap\Gamma_{n_0}$, let Δ_A be the horizontal path $[A_p, A_{p+1}] \cup [A_{p+1}, A_{p+2}] \cup \dots [A_{q-1}, A_q]$ where the sequence (A_n) is defined in Lemma 10.33 with $A_0 = A$, A_p is the last point of the sequence in Γ_{n_0}, and A_q is the first point of the sequence in Γ_{n_0+1} (the sequence of edges $[A_{n-1}, A_n]$ meets Γ_{n_0+1} since $a_n \in [a_2, a_2+3[$, $a_{n,1} \to \infty$, and there exists $X \in \Gamma_{n_0+1}$ with x_2 arbitrarily near to $+\infty$ or $-\infty$). If A and B are in $\Gamma\cap\Gamma_{n_0}$ with $a_2 - b_2 > 3$, then Δ_A does not meet Δ_B. We can find a sequence $(B_n)_{n\in\mathbf{Z}}$ in $\Gamma\cap\Gamma_{n_0}$ with

$$b_{n,2} + 3 < b_{n+1,2} \leq b_{n,2} + 4.$$

The seven paths $\Delta_{B_n}, \Delta_{B_{n+1}}, \dots, \Delta_{B_{n+6}}$ are all in the strip

$$\{X \in \mathbf{R}^2 : x_2 \in [b_{n,2}, b_{n,2} + 24[\},$$

hence in $\mathcal{R} = \Omega \cap \{X \in \mathbf{R}^2 : x_2 \in [b_{n,2}, b_{n,2} + 27[\}$.

In order to see that one of the paths $\Delta_{B_n}, \dots, \Delta_{B_{n+6}}$ is in $\mathcal{G}_2$, by Lemma 10.26, it suffices to show that no points of $\mathcal{R}$ are equivalent modulo Λ. Let X and Y be in $\mathcal{R}$ such that $X - Y \in \Lambda$. Since $\beta < 1/28$, we have $x_2 = y_2$ and $X - Y = me_1$ with $m \in \mathbf{Z}$. Suppose $m \geq 1$. Since $Y \in \Gamma_{n_0}^+$, $X \in \Gamma_{n_0}^+ + me_1$. Furthermore, $\Gamma_n^+ + e_1$ is included in Γ_{n+1}^+ for all n, hence $\Gamma_{n_0}^+ + me_1 = \Gamma_{n_0+1}^+ + (m-1)e_1 \subset \Gamma_{n_0+1}^+$ and $X \in \Gamma_{n_0+1}^+$. This contradicts $X \in \Omega_{n_0} = \Gamma_{n_0}^+\backslash\overline{\Gamma_{n_0+1}^+}$.

For each $n \in \mathbf{Z}$, choose C_n among $B_{7n}, \dots, B_{7n+6}$ such that Δ_{C_n} is in $\mathcal{G}_2$. The curve Δ_{C_n} splits Ω into two connected components and the sets

$$\mathcal{R}_n^+ = \{X \in \Omega : x_2 > c_{n,2} + 3\} \text{ and } \mathcal{R}_n^- = \{X \in \Omega : x_2 < c_{n,2}\}$$

are not in the same component. Since ω does not meet the curve Δ_{C_n}, ω cannot meet both $\mathcal{R}_n^-$ and $\mathcal{R}_n^+$. The choice of the sequence (C_n) shows that

$$c_{n,2} + 3 \leq c_{n+1,2} \leq c_{n,2} + 52.$$

It follows that $\operatorname{diam}\{x_2 : X \in \omega\} \leq 55$, hence there exists $t \in \mathbf{R}$ such that

$$\omega \subset \mathcal{B} = \{X \in \Omega : x_2 \in [t, t+55]\}.$$

As before, we see that $\mathcal{B}$ contains no equivalent points modulo Λ. Therefore, by Lemma 10.26, $\overline{\mathcal{B}}$ contains at most six removed edges and ω is the union of at most seven connected components of $\mathbf{R}^2\backslash\mathcal{G}_1$. ∎

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10.10 The Graph $\mathcal{G}_3$

Remember that $\mathcal{G}_3$ is the set of all edges $[A', B']$ with $[A, B]$ in $\mathcal{G}_2$ (see Section 10.4). We have to show that $\mathcal{G}_3$ a is $\mathbf{Z}^2$-invariant planar graph and that $\mathcal{G}_3$ satisfies **P0**. First, $\mathbf{Z}^2$-invariance is easy: observe that $(A+P)' = A' + P$ for all A in Γ and P in $\mathbf{Z}^2$. Now we show that $\mathcal{G}_3$ is a planar graph.

Lemma 10.34 *Let $[A, B] \neq [C, D]$ be in $\mathcal{G}_2$. Then $[A', B'] \cap [C', D'] = \varnothing$ or is one common extremity of $[A', B']$ and $[C', D']$.*

Proof

Step 1: We first suppose $\{A, B\} \cap \{C, D\} = \varnothing$.

Case 1 $[A, B = A + \varepsilon]$ and $[C, D = C + \varepsilon]$ are vertical. Since for each X in Γ, we have $x_1' = x_1$, the condition $a_1 \neq c_1$ implies $[A', B'] \cap [C', D'] = \varnothing$. In the other case we can suppose $c_2 > a_2$. By Lemma 10.6, we have $b_2 = a_2 + 1 \leq c_2 - 1$ and since for each X in Γ, we have $x_2' - x_2 \in]0, 1[$, $[A', B'] \cap [C', D'] = \varnothing$.

Case 2 $[A, B = A + \varepsilon]$ is vertical and $[C, D]$ horizontal. If $a_1 \notin [c_1, d_1]$, then the result is obvious. If $a_1 \in [c_1, d_1]$, then A is above $[C, D]$ or B is under $[C, D]$. If A is above $[C, D]$, by definition of removed edges we have $A + t\varepsilon \in [C, D]$ with $t \leq -1$. Therefore A is above $[C', D']$ and $[A', B'] \cap [C', D'] = \varnothing$. The same way of reasoning shows that $[A', B'] \cap [C', D'] = \varnothing$ if B is under $[C, D]$.

Case 3 $[A, B]$ and $[C, D]$ are horizontal. We can suppose $a_1 \leq c_1$. If $b_1 < c_1$, the result is obvious. If $b_1 = c_1$, then by Lemma 10.6, we have $|b_2 - c_2| \geq 1$ and therefore $B' \neq C'$ and $[A', B'] \cap [C', D'] = \varnothing$. If $c_1 < b_1 \leq d_1$, B is under or above $[C, D]$. Suppose B is above $[C, D]$. By definition of the removed edges, we have $B - t\varepsilon \in [C, D]$ with $t \geq 1$. Since $[A, B] \cap [C, D] = \varnothing$, C is under $[A, B]$ and $C + s\varepsilon \in [A, B]$ with $s \geq 1$. All the points Q of $[C', D']$ are of the shape $P + r\varepsilon$ with $P \in [C, D]$ and $r \in]0, 1[$, therefore $[C', D']$ is under $[A, B]$ and $[A', B'] \cap [C', D'] = \varnothing$. If $b_1 > d_1$, then C and D are both above $[A, B]$ or both under $[A, B]$. We see as before that $[A', B'] \cap [C', D'] = \varnothing$.

Step 2: Suppose $A = C$. If one of the edges is vertical, the result is obvious. Suppose that both edges are horizontal. One of the edges $[A, B]$ or $[C, D]$ must be exceptional. Suppose $[C, D]$ is exceptional. Therefore $A^+ = B$. If $d_1 \leq b_1$ by definition of removed edge, $D + t\varepsilon \in [A, B]$ with $t \geq 1$. Therefore, D' is under $[A, B]$ and $[A', B'] \cap [C', D'] = A' = C'$. If $b_1 > d_1$ by definition of removed edge, $B - t\varepsilon \in [A, B]$ with $t \geq 1$. Therefore B is above $[C', D']$ and $[A', B'] \cap [C', D'] = A' = C'$.

Step 3: The case $B = D$ is not possible. Indeed, if the point $B = D$ has two predecessors A and C, then by definition of exceptional edges, neither $[A, B]$ nor $[C, D]$ is exceptional and

$$[A, B] \text{ and } [C, D] \subset \{X \in \mathbf{R}^2 : x_2 \in]b_2 - 1, b_2]\}.$$

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Therefore one of the two edges $[A,B]$ and $[C,D]$ must be removed.

Step 4: The cases $B=C$ and $A=D$ are obvious. ■

It remains to prove that **P0** is true for $\mathcal{G}_3$:

Proposition 10.35 *Let $\mathcal{E}_3 = \{\overrightarrow{A'B'} : A'$ and B' are vertices of $\mathcal{G}_3$, and there exists ω' in $\mathcal{C}_3$ such that $A', B' \in \partial\omega'\}$. We have $|\mathcal{E}_3| \le K_2 = K_1^{42}$ where K_1 is defined in Lemma* 10.4*.*

Proof Let ω' be a connected component of $\mathbf{R}^2\backslash\mathcal{G}_3$, and C and D two vertices of $\mathcal{G}_3$ lying in $\partial\omega'$. By Lemma 8.4, there exists a connected component ω of $\mathbf{R}^2\backslash\mathcal{G}_2$ such that

$$\{[E,F] \subset \partial\omega' : [E,F] \in \mathcal{G}_3\} = \{[A',B'] : [A,B] \subset \partial\omega \text{ and } [A,B] \in \mathcal{G}_2\}.$$

Therefore, there are two vertices A and B of $\mathcal{G}_2$ which are in $\partial\omega$, such that $A' = C$ and $B' = D$. By Proposition 10.25, there are connected components $\omega_1, \dots, \omega_m$ of $\mathbf{R}^2\backslash\mathcal{G}_1$, with $m \le 7$, such that $\overline{\omega} = \overline{\omega_1} \cup \dots \cup \overline{\omega_m}$. It follows that there exists a sequence $A_0 = A$, $A_1 \dots, A_k = B$ of vertices of $\mathcal{G}_1$ such that

$$k \le 7 \text{ and } \forall i \in \{0,\dots,k-1\},\ \exists j \in \{1,\dots,m\},\ A_i \text{ and } A_{i+1} \in \partial\omega_j.$$

By Proposition 10.17, for each $i \in \{0,\dots,k-1\}$, there is a sequence

$$[B_{i,0}, B_{i,1}], \dots, [B_{i,k_i-1}, B_{i,k_i}]$$

of less than six edges of $\mathcal{G}_1$, lying in $\partial\omega_i$, such that $B_{i,0} = A_i$ and $B_{i,k_i} = A_{i+1}$. It follows that

$$\overrightarrow{CD} = \sum_{i=0}^{k-1} \overrightarrow{A'_iA'_{i+1}} = \sum_{i=0}^{k-1}\sum_{j=0}^{k_i-1} \overrightarrow{B'_{i,j}B'_{i,j+1}}$$

is the sum of at most 42 vectors of the shape $\overrightarrow{E'F'}$ with $[E,F]$ in $\mathcal{G}_1$. Finally, by Lemma 10.4, the number of such vectors is at most K_1^{42}. ■

10.11 The Graph $\mathcal{G}_4$

Let q be in $\{Nq_n - 1, \dots, (N+1)q_n - 2\}$. We recall the definition of $\mathcal{G}_4$: we add to $\mathcal{G}_3$ some new vertices and some new edges. The new vertices are the points of the set $\mathbf{Z}^2 + \{Nq_n\theta, \dots, q\theta\}$. Each new vertex belongs to a connected component of $\mathbf{R}^2\backslash\mathcal{G}_3$ or to an edge $[A,B]$ of $\mathcal{G}_3$. In the first case, we add the singular edge $[P,P]$. In the second case, we split the edge $[A,B]$ into two edges $[A,P]$ and $[P,B]$.

By Lemma 10.27, we have:

Lemma 10.36 *For all P in $\mathbf{R}^2$ the segment $[P, P+\varepsilon_n]$ meets at most six edges of $\mathcal{G}_3$.*

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(If $[P, P+\varepsilon_n]$ meets an edge $[A, B]$ of $\mathcal{G}_3$, then $[P-\varepsilon_n, P+\varepsilon_n]$ meets the edges $[C, D]$ of $\mathcal{G}_1$ where $C' = A$ and $D' = B$).

Remember that

$$\mathcal{E}_4 = \left\{ \overrightarrow{AB} : A \text{ and } B \text{ are vertices of } \mathcal{G}_4 \text{ and there exists} \right.$$
$$\left. \text{a connected component } \omega \text{ of } \mathbf{R}^2 \backslash \mathcal{G}_4 \text{ such that } A, B \in \partial\omega \right\}.$$

Proposition 10.37 *There is an absolute constant K_3 such that $\forall q \in \mathbf{N}$, $\forall \theta \in \mathbf{R}^2$, $|\mathcal{E}_4| \leq K_3$.*

Proof Set

$$\mathcal{E} = \left\{ \overrightarrow{AB} : A \text{ is a vertex of } \mathcal{G}_3, B \text{ is a vertex of } \mathcal{G}_4 \text{ but not of } \mathcal{G}_3 \text{ and} \right.$$
$$\left. \text{there exists a connected component } \omega \text{ of } \mathbf{R}^2 \backslash \mathcal{G}_4 \text{ with } A, B \in \partial\omega \right\}.$$

We have $\mathcal{E}_4 \subset \mathcal{E}_3 + \mathcal{E} \cup \{0\} + (-\mathcal{E}) \cup \{0\}$ (note that $0 \in \mathcal{E}_3$). Therefore by Proposition 10.35, it suffices to find an upper bound to $|\mathcal{E}|$. Let ω be in $\mathcal{C}(\mathcal{G}_4)$, A a vertex of $\mathcal{G}_3$ and B a vertex of $\mathcal{G}_4$ but not of $\mathcal{G}_3$, such that A and B are in the boundary of ω. By definition of $\mathcal{G}_4$, there is a vertex C of $\mathcal{G}_3$ such that $B = C + \varepsilon_n$. By Lemma 10.36, the segment $[C, B]$ cuts at most six edges of $\mathcal{G}_3$; let $([D_i, E_i])_{i=1,\ldots,m}$, $m \leq 6$, be the sequence of such edges. Let t_i be the element of $[0, 1]$ such that $t_i B + (1-t_i)C = [C, B] \cap [D_i, E_i]$. We can suppose that $t_1 \leq t_2 \leq \cdots \leq t_m$. The points C and D_1 are in the boundary of the same element of $C(\mathcal{G}_3)$, the same is true for D_i and D_{i+1}, $i = 1, \ldots, m-1$ and D_m and B are in the boundary of the same element $\mathcal{C}(\mathcal{G}_3)$. We have

$$-\overrightarrow{D_m B} = \overrightarrow{BC} + \overrightarrow{CD_1} + \sum_{i=1}^{m-1} \overrightarrow{D_i D_{i+1}}.$$

Hence,

$$\overrightarrow{D'_m B'} \in -(\varepsilon_n + \sum_{i=1}^{6} \mathcal{E}_3) \text{ and } \overrightarrow{A'B'} = \overrightarrow{A'D'_m} + \overrightarrow{D'_m B'} \in \mathcal{E}_3 - (\varepsilon_n + \sum_{i=1}^{6} \mathcal{E}_3),$$

finally, $\mathcal{E} \subset \mathcal{E}_3 - (\varepsilon_n + \sum_{i=1}^{6} \mathcal{E}_3)$. ∎

Completion of proof of Theorem 1.3 We now handle the main case. By Proposition 10.37, $\mathcal{G}_4$ satisfies **P0.** Furthermore, by definition, $\mathcal{G}_4$ is $\mathbf{Z}^2$-invariant. Now, Lemma 8.3 gives Theorem 1.3 in the main case. Furthermore, the lengths of the edges of $\mathcal{G}_4$ are smaller than $|e_{1,n}| + 2|\varepsilon_n|$. It follows that the diameters of the connected component of $\mathcal{G}_4$ go to 0 when n goes to infinity. ∎

10.12 Proof of Theorem 1.3 in Case 4

We are going to use Voronoï's diagram as in [5, 6]; it is probably not necessary but it seems more efficient. Nevertheless, we also need Delaunay's triangulation, for the vertices of Voronoï's diagram are not those we want. The basic definitions and properties about Voronoï's diagram and Delaunay's triangulation (see Definition 10.38, 10.39 and Proposition 10.40 below) can be found in [4, 24]; but it should be noticed that we use an infinite set instead of a finite set.

Definition 10.38 Let F be a subset of $\mathbf{R}^2$. For A in F, we call the region given by

$$V(F, A) = \{X \in \mathbf{R}^2 : d(X, A) \leq d(X, B) \text{ for } B \in F\}$$

the Voronoï polygon associated to A. The set $\mathcal{V}(F) = \{V(F, A) : A \in F\}$ is called the Voronoï diagram generated by F and the points in F are called generators.

Definition 10.39 Let F be a subset of $\mathbf{R}^2$. Let D be a maximal subset of F such that all points of D are on the same circle and such that there is no point of F inside the circle. The convex hull of D is called a Delaunay polygon associated to F.

Remember the notations:

$$e(F) = \sup\{d(x, F) : x \in \mathbf{R}^2\}, \qquad r(F) = \inf\{d(x, y) : x, y \in F,\ x \neq y\}.$$

Proposition 10.40 *Let F be a discrete subset of $\mathbf{R}^2$ such that $e(F) < +\infty$. Then*

(i) *if A and B are two points of F on the same Delaunay polygon, then $V(F, A) \cap V(F, B) \neq \varnothing$;*
(ii) *if A and B are two points of F on the same Delaunay polygon and consecutive on this polygon, then $V(F, A) \cap V(F, B)$ is a segment which contains at least two points;*
(iii) *the set of all Delaunay polygons forms a tessellation of $\mathbf{R}^2$;*
(iv) *each point A in F is a vertex of a Delaunay polygon.*

Notation Let θ be in $\mathbf{R}^2$ and q be in $\mathbf{N}$. Set $F_q = \{0, \theta, \dots, q\theta\} + \mathbf{Z}^2$. In [5, 6] we used Voronoï's diagram in $\mathbf{T}^2$ instead $\mathbf{R}^2$. Some straightforward changes in results of [5] give the two following results we state without proofs.

Lemma 10.41 *Let θ be in $\mathbf{R}^2$, q in $\mathbf{N}$ and A in F_q.*

(i) *Let k be in $\{0, \dots, q-1\}$. If for all X in $V(F_q, A)$, $d(X, A) \leq d(X, -\theta + \mathbf{Z}^2)$, then $V(F_q, A) + \theta \subset V(F_q, A + \theta)$.*

(ii) *Let k be in $\{1, \dots, q\}$. If for all X in $V(F_q, A)$, $d(X, A) \leq d(X, (q+1)\theta + \mathbf{Z}^2)$, then $V(F_q, A) - \theta \subset V(F_q, A - \theta)$.*

Corollary 10.42 *Let θ be in $\mathbf{R}^2$ and q in $\mathbf{N}$. Set*

$$I^+ = \{k \in \{0, \dots, q-1\} : \exists X \in V(F_q, k\theta),\ d(X, k\theta) > d(X, -\theta + \mathbf{Z}^2),$$

$$I^- = \{k \in \{1, \dots, q\} : \exists X \in V(F_q, k\theta),\ d(X, k\theta) > d(X, (q+1)\theta + \mathbf{Z}^2)\}.$$

Then, up to translations, the number of distinct regions in $\mathcal{V}(F_q)$ is at most $1 + |I^+| + |I^-|$.

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Lemma 10.43

$$|I^+|, |I^-| \le \Big(\frac{2e(F_q)}{r(F_q)}\Big)^2.$$

Proof For all A in F_q and X in $V(F_q, A)$, the open ball $B^\circ(X, d(X, A))$ contains no point of F_q; therefore, $d(X, A) \le e(F_q)$. Let k be in I^+. There exist X in $V(F_q, k\theta)$ and $P \in -\theta + \mathbf{Z}^2$ such that $d(X, k\theta) > d(X, P)$. It follows that $d(k\theta, -\theta + \mathbf{Z}^2) \le 2e(F_q)$. Now, for all k, $d(k\theta, -\theta + \mathbf{Z}^2) = d(k\theta + \mathbf{Z}^2, -\theta)$, hence

$$|I^+| \le |\{A \in F_q : A \in B(-\theta, 2e(F_q))| \le \Big(\frac{2e(F_q)}{r(F_q)}\Big)^2.$$

The same argument shows the inequality for I^-. ∎

Proposition 10.44 *Let θ be in $\mathbf{R}^2$ and q be in $\mathbf{N}^*$ such that $q_n - 1 \le q < q_{n+1} - 1$. Suppose that (f_1, f_2) is a basis of Λ_n such that $|\sin\angle(f_1, f_2)| \ge \sqrt{3}/8$ and $\varepsilon_n = \alpha_1 f_1 + \alpha_2 f_2$ with $1/100 \le \alpha_1, \alpha_2 < 1$. Then*

(i) *up to translations, the number of different Voronoï regions in $\mathcal{V}(F_q)$ is at most 10^6,*
(ii) *the cardinal of the set*

$$\{\overrightarrow{AB} : A, B \in F_q \text{ and } V(F_q, A) \cap V(F_q, B) \text{ contains at least two points}\}$$

is at most 10^{12}.

Proof (i) By definition of the best approximation, for all q less than $q_{n+1} - 1$, we have $r(F_q) = r_n = |\varepsilon_n|$. The length of ε_n is greater than the length of its orthogonal projection on the line orthogonal to f_1 or f_2, therefore

$$\begin{aligned}|\varepsilon_n| &\ge \max(\alpha_1|\sin\angle(f_1, f_2)||f_1|,\ \alpha_2|\sin\angle(f_1, f_2)||f_2|)\\ &\ge \frac{1}{100} \times \frac{\sqrt{3}}{8}\max(|f_1|, |f_2|) \ge \frac{\sqrt{3}}{800} e(\Lambda_n).\end{aligned}$$

But $e(F_q) \le e(F_{q_n}) \le e(\Lambda_n) + r_n$, hence

$$\Big(\frac{2e(F_q)}{r(F_q)}\Big)^2 \le \Big(\frac{2e(\Lambda_n)}{\frac{\sqrt{3}}{800}e(\Lambda_n)} + 2\Big)^2 \le 10^6.$$

(ii) Let A be in F_q. If B is another point of F_q such that $V(F_q, A) \cap V(F_q, B)$ contains at least two points, then one edge of $V(F_q, A)$ is included in the bisector of $[A, B]$. It follows that for fixed A, the number of vectors $\overrightarrow{AB}$ such that B is in F_q and $V(F_q, A) \cap V(F_q, B)$ contains at least two points, is less than the number of edges of $V(F_q, A)$. Furthermore if $V(F_q, A_2) = V(F_q, A_1) + \overrightarrow{A_1A_2}$, then the two corresponding sets of vectors are the same. It follows that the cardinal of

$$\{\overrightarrow{AB} : A, B \in F_q \text{ and } V(F_q, A) \cap V(F_q, B) \text{ contains at least two points}\}$$

is less than the number of different Voronoï polygons up to translations multiplied by the maximal number of edges of a Voronoï polygon. As in the previous lemma, we see that if $V(F_q, A) \cap V(F_q, B) \neq \varnothing$, then $d(A, B) \leq 2e(F_q)$ and therefore the maximal number of edges is less than $\left(\frac{2e(F_q)}{r(F_q)}\right)^2$. Finally we get the upper bound $\left(\frac{2e(F_q)}{r(F_q)}\right)^4 \leq 10^{12}$. ∎

Completion of proof of Theorem 1.3 in Case 4 Let θ be in $\mathbf{R}^2$ and q be in $\mathbf{N}^*$ such that $q_n - 1 \leq q < q_{n+1} - 1$. Consider the set of all Delaunay polygons associated to F_q. It is obviously $\mathbf{Z}^2$-invariant. The chosen basis (e_1, e_2) of Λ_n is such that $|\sin\angle(e_1, e_2)| \geq \sqrt{3}/8$ and $\varepsilon_n = \alpha e_1 + \beta e_2$ with $0 \leq \beta \leq \alpha < 1$. By hypothesis we have $\beta \geq \frac{1}{100}$, so we can use the previous proposition:

$$\{\overrightarrow{AB} : A, B \in F_q \text{ and } V(F_q, A) \cap V(F_q, B) \text{ contains at least two points}\}$$

has less than 10^{12} elements. By Proposition 10.40, it means that the number of possible edges for a Delaunay polygon is less than 10^{12}. Furthermore, the number of edges of a Delaunay polygon is less than $\frac{2\pi e(F_q)}{r(F_q)}$ for the radius of the circle associated to a Delaunay polygon, is less than $e(F_q)$. As in the previous proof, we see that

$$\frac{e(F_q)}{r(F_q)} \leq \frac{e(\Lambda_n)}{\frac{\sqrt{3}}{800} e(\Lambda_n)} + 1 \leq 500.$$

Therefore, the maximal number of edges of a Delaunay polygon is at most 1000π. Finally, there is an absolute constant K such that the set

$$\{\overrightarrow{AB} : A, B \text{ are two points of the same Delaunay polygon}\}$$

has a cardinal less than K. It means that **P0** holds for the planar graph associated to Delaunay tessellation. We conclude with Lemma 8.3. ∎

A Appendix

Proof of Lemma 8.3 Consider the equivalence relation on the set $\mathcal{C}(\mathcal{G})$ of connected components of $\mathbf{R}^2 \backslash \mathcal{G}$ defined by $\omega_1 \sim \omega_2$ if and only if there exists $\overrightarrow{u}$ in $\mathbf{Z}^2$ such that $\omega_1 = \omega_2 + \overrightarrow{u}$. Choose a subset Ω of $\mathcal{C}(\mathcal{G})$ containing exactly one element of each equivalence class. Let ω be in Ω and set $\mathcal{E}(\omega) = \{[A, B] : A, B \text{ are vertices of } \mathcal{G}$ and $]A, B[\subset \omega\}$. For each ω in Ω, select a maximal subset $\mathcal{M}(\omega)$ of $\mathcal{E}(\omega)$ such that $\mathcal{G} \cup \mathcal{M}(\omega)$ is a planar graph. Set

$$\mathcal{G}' = \mathcal{G} \cup \Big(\bigcup_{\omega \in \Omega} \bigcup_{\overrightarrow{u} \in \mathbf{Z}^2} (\overrightarrow{u} + \mathcal{M}(\omega)) \Big).$$

Step 1: Let us show that $\mathcal{G}'$ is a planar graph. Let $[A, B]$ and $[C, D]$ be two edges of $\mathcal{G}'$ such that $[A, B] \cap [C, D] \neq \varnothing$.

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If $[A,B]$ and $[C,D]$ are in $\mathcal{G}$, there is nothing to prove.

If $[A,B]$ is in $\mathcal{G}'\backslash\mathcal{G}$ and $[C,D]$ in $\mathcal{G}$, then there exist ω in Ω and $\overrightarrow{u}$ in $\mathbf{Z}^2$ such that $[A,B]+\overrightarrow{u}\in\mathcal{M}(\omega)$, hence $[A,B]+\overrightarrow{u}$ and $[C,D]+\overrightarrow{u}$ are in $\mathcal{G}\cup\mathcal{M}(\omega)$. It follows that $([A,B]+\overrightarrow{u})\cap[C,D]+\overrightarrow{u})$ is a common extremity of $[A,B]+\overrightarrow{u}$ and $[C,D]+\overrightarrow{u}$.

If $[A,B]$ and $[C,D]$ are in $\mathcal{G}'\backslash\mathcal{G}$, then there exist ω_1, ω_2 in Ω and $\overrightarrow{u_1}$, $\overrightarrow{u_2}$ in $\mathbf{Z}^2$ such that $[A,B]+\overrightarrow{u_1}\in\mathcal{M}(\omega_1)$ and $[C,D]+\overrightarrow{u_2}\in\mathcal{M}(\omega_2)$. Suppose $[A,B]\cap[C,D]$ contains a point P of $]A,B[$. The point P is in $\omega_1-\overrightarrow{u_1}$ and in $\overline{\omega_2}-\overrightarrow{u_2}$, therefore $\omega_1-\overrightarrow{u_1}=\omega_2-\overrightarrow{u_2}$ and $\omega_1=\omega_2$. Moreover, since the connected components of $\mathbf{R}^2\backslash\mathcal{G}$ are bounded we have $\overrightarrow{u_1}=\overrightarrow{u_2}$. It follows that $[A,B]+\overrightarrow{u_1}$ and $[C,D]+\overrightarrow{u_1}$ are in $\mathcal{M}(\omega_1)\subset\mathcal{G}\cup\mathcal{M}(\omega_1)$, hence intersection $[A,B]\cap[C,D]$ cannot be inside $]A,B[$.

Step 2: Let U be a connected component of $\mathbf{R}^2\backslash\mathcal{G}'$. We have to prove that U is the interior of a triangle.

First we show that there exist $[A,B]$ and $[A,C]$ in ∂U such that $\{\lambda\overrightarrow{Ab}+\mu\overrightarrow{AC}:\lambda,\mu\in]0,r]\}$ is included in U for some positive r. Choose a point O in $\mathbf{R}^2$. There is a point A in $\overline{U}$ which is the furthest from O. This point is in ∂U and must be in $\mathcal{V}(\mathcal{G}')=\mathcal{V}(\mathcal{G})$. Set $r=\inf\{d(A,[Q,R]):[Q,R]\in\mathcal{G}'\text{ and }A\notin[Q,R]\}$. Then r is positive and the open ball $B(A,r)$ meets only edges of the shape $[A,P]\in\mathcal{G}'$. Let $P_0,\dots,P_{n-1}$ be the vertices such that $[A,P_i]$ is an edge $\mathcal{G}'$. We can order them such that the angles $\angle(\overrightarrow{AP_0},\overrightarrow{AP_i})$ increase. One of the sectors $(\overrightarrow{AP_i},\overrightarrow{AP_{i+1}})$ (with $P_n=P_0$) must meet U. Let $(\overrightarrow{AP_{i_0}},\overrightarrow{AP_{i_0+1}})$ be this sector, $B=P_{i_0}$ and $C=P_{i_0+1}$. The ball $B(O,d(A,O))$ contains this sector, hence $\angle(\overrightarrow{AP_{i_0}},\overrightarrow{AP_{i_0+1}})<\pi$.

Let ω be the connected component of $\mathbf{R}^2\backslash\mathcal{G}$ containing U. We can suppose that $\omega\in\Omega$. Let $\mathcal{V}$ be the set of vertices of $\mathcal{G}'$ which belong to $T=\{\lambda\overrightarrow{AB}+\mu\overrightarrow{AC}:\lambda,\mu\in]0,1]\}$ and let $\mathbf{C}$ be the convex hull of $\mathcal{V}\cup\{B,C\}$. Since there is no edge between A and a point of $\mathcal{V}$, $T\backslash\mathbf{C}$ does not meet $\mathcal{G}'$. Furthermore, if $\mathcal{V}\neq\varnothing$, there exists a point D in $\mathcal{V}$ such that $]A,D[\subset T\backslash\mathbf{C}$, but this contradicts the maximality of $\mathcal{M}(\omega)$. It follows that $\mathcal{V}\neq\varnothing$ and by maximality of $\mathcal{M}(\omega)$, $[B,C]\in\mathcal{G}'$.

Step 3: Finally, all triangles are made of edges belonging to $\mathcal{E}(\mathcal{G})$. By hypothesis, $|\mathcal{E}(\mathcal{G})|\leq K$, therefore the number of triangles up to translations is less than K^2. ■

Proof of Lemma 8.4 Set $\mathcal{O}(\mathcal{G})=\{(A,B):[A,B]\in\mathcal{G}\}$. Then $\mathcal{O}(\mathcal{G})$ is the set of oriented edges of $\mathcal{G}$; note that $(A,B)\in\mathcal{O}(\mathcal{G})\Leftrightarrow(B,A)\in\mathcal{O}(\mathcal{G})$.

We use $\mathcal{O}(\mathcal{G})$ to make the sides of the edges precise. For $\overrightarrow{u}$, a non zero vector, let $\overrightarrow{u}^{\perp}$ be the unit vector such that $(\overrightarrow{u},\overrightarrow{u}^{\perp})$ is a direct orthogonal basis of $\mathbf{R}^2$. We say that an element (A,B) of $\mathcal{O}(\mathcal{G})$ is a boundary edge of $\omega\in\mathcal{C}(\mathcal{G})$ if there exists $t_0>0$ such that $\frac{1}{2}(A+B)+t\overrightarrow{AB}^{\perp}\in\omega$ for all $t\in]0,t_0]$. Denote by $\mathcal{BO}(\mathcal{G})$ the set of "oriented" boundaries of elements of $\mathcal{C}(\mathcal{G})$, more precisely

$$\mathcal{BO}(\mathcal{G})=\{\mathcal{A}\subset\mathcal{O}(\mathcal{G}):\exists\omega\in\mathcal{C}(\mathcal{G}),\ (A,B)\in\mathcal{A}\Leftrightarrow(A,B)\text{ is a boundary edge of }\omega\}.$$

For each $t\in[0,1]$ and each vertex A of $\mathcal{G}$, set $A(t)=(1-t)A+tA'$.

We shall prove the more precise result. The map

$$\mathcal{A}\in\mathcal{BO}(\mathcal{G})\to\{(A',B'):(A,B)\in\mathcal{A}\}$$

is one-to-one and its image is $\mathcal{BO}(\mathcal{G}')$.

For each t, we define a binary relation on $\mathcal{O}(\mathcal{G}(t))$ by $(A(t), B(t)) \sim_t (C(t), D(t))$ if $(A(t), B(t))$ and $(C(t), D(t))$ are boundaries edges of the same element $\omega \in \mathcal{C}(\mathcal{G}(t))$. It is easy to show that $\sim_t$ is an equivalence relation.

By a simple continuity argument, it is possible to show that for all (A, B) and (C, D) in $\mathcal{O}(\mathcal{G})$ and all t_0 in $[0, 1]$, there exists $\delta > 0$ such that

$$(A(t_0), B(t_0)) \sim_{t_0} (C(t_0), D(t_0)) \Rightarrow \forall t \in [t_0 - \delta, t_0 + \delta],\ (A(t), B(t)) \sim_t (C(t), D(t)).$$

The converse

$$\begin{aligned}&(A(t_0), B(t_0)) \text{ is not equivalent to } (C(t_0), D(t_0))\\ &\qquad\qquad \Rightarrow \forall t \in [t_0 - \delta, t_0 + \delta],\ (A(t), B(t)) \text{ is not equivalent to } (C(t), D(t)).\end{aligned}$$

is "geometrically" obvious but is more difficult to prove. We need an auxiliary result (see [10, Appendix of Ch. IX]):

Let **U** be the unit circle of **C**. Let $K \subset \mathbf{C}$ be a compact set.

- A continuous map $f\colon K \to \mathbf{U}$ is not essential if there exits a continuous map $g\colon K \to \mathbf{R}$ such that $f(z) = \exp ig(z)$ for all z in K. If the map g does not exist we say that f is essential on K
- Let a and b be two points of $\mathbf{C}\backslash K$. The points a and b are in the same connected component of $\mathbf{C}\backslash K$ if and only if the map

  $$s_{a,b}(z) = \frac{z-a}{z-b} \times \frac{|z-b|}{|z-a|}$$

  is not essential.

Let (A, B) and (C, D) be in $\mathcal{O}(\mathcal{G})$ and t_0 in $[0, 1]$. Suppose that $(A(t_0), B(t_0))$ is not equivalent to $(C(t_0), D(t_0))$ (it may be that $[A, B] = [C, D]$). Then $(A(t_0), B(t_0))$ is in the boundary of a connected component ω_0 and $(C(t_0), D(t_0))$ is not in the boundary of ω_0. Therefore, there exists $\alpha_0 > 0$ such that for all $\alpha \in]0, \alpha_0]$,

$$\frac{1}{2}(A(t_0)+B(t_0))+\alpha\overrightarrow{A(t_0)B(t_0)}^{\perp} \in \omega_0 \quad \text{and} \quad \frac{1}{2}(C(t_0)+D(t_0))+\alpha\overrightarrow{C(t_0)D(t_0)}^{\perp} \notin \omega_0.$$

For $\alpha \geq 0$, set

$$I(\alpha) = \Big[\frac{1}{2}(A(t_0) + B(t_0)) - \alpha\overrightarrow{A(t_0)B(t_0)}^{\perp}, \frac{1}{2}(A(t_0) + B(t_0)) + \alpha\overrightarrow{A(t_0)B(t_0)}^{\perp}\Big],$$

$$J(\alpha) = \Big[\frac{1}{2}(C(t_0) + D(t_0)) - \alpha\overrightarrow{C(t_0)D(t_0)}^{\perp}, \frac{1}{2}(C(t_0) + D(t_0)) + \alpha\overrightarrow{C(t_0)D(t_0)}^{\perp}\Big].$$

By continuity, there exist α and $\delta > 0$ such that the following hold:

$$\forall t \in [t_0 - \delta, t_0 + \delta], \forall [E, F] \in \mathcal{G}\backslash\{[A, B]\},\ d([E(t), F(t)], I(\alpha)) > 0;$$

$$\forall t \in [t_0 - \delta, t_0 + \delta], \forall [E, F] \in \mathcal{G}\backslash\{[C, D]\},\ d([E(t), F(t)], J(\alpha)) > 0;$$

$]A(t), B(t)[\cap I(\alpha/2)$ contains exactly one point $G(t)$;

$]C(t), D(t)[\cap J(\alpha/2)$ contains exactly one point $H(t)$.

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Set $a = \frac{1}{2}(A(t_0) + B(t_0)) + \alpha\overrightarrow{A(t_0)B(t_0)}^{\perp}$ and $b = \frac{1}{2}(C(t_0) + D(t_0)) + \alpha\overrightarrow{C(t_0)D(t_0)}^{\perp}$. For all t in $[t_0 - \delta, t_0 + \delta]$, a and b do not belong to an edge of $\mathcal{G}(t)$. Let $\omega_1(t)$ (resp., $\omega_2(t)$) be the connected component of $\mathbf{R}^2 \backslash \mathcal{G}(t)$ containing a (resp., b). Note that $\omega_1(t_0) = \omega_0$. Set $K_0 = \partial\omega_1(t_0)$, K_0 is a finite union of distinct edges of $\mathcal{G}(t)$,

$$K_0 = \bigcup_{i=1}^{m} [A_i(t_0), B_i(t_0)].$$

For t in $[t_0 - \delta, t_0 + \delta]$, set $K_t = \bigcup_{i=1}^{m} [A_i(t), B_i(t)]$. Since the map $A \in \mathcal{V}(\mathcal{G}) \to A(t) \in \mathcal{V}(\mathcal{G}(t))$ is bijective and since $\mathcal{G}(t)$ is a planar graph, for all $t \in [t_0 - \delta, t_0 + \delta]$ the map

$$f_t \colon \lambda A_i(t_0) + (1 - \lambda)B_i(t_0) \in K_0 \to \lambda A_i(t) + (1 - \lambda)B_i(t) \in K_t$$

is a homeomorphism. Furthermore, the map

$$F \colon (t, P) \in [t_0 - \delta, t_0 + \delta] \times K_0 \longrightarrow f_t(P)$$

is continuous. By our choice of α and δ, neither a nor b are in K_t for t in $[t_0 - \delta, t_0 + \delta]$, therefore the map

$$G \colon (t, P) \in [t_0 - \delta, t_0 + \delta] \times K_0 \longrightarrow s_{a,b}(f_t(P)) \in \mathbf{U}$$

is continuous. Since by hypothesis a and b are not in the same connected component of $\mathbf{C} \backslash K_0$, the map $s_{a,b}$ is essential on K_0. Suppose that there exists t_1 in $[t_0 - \delta, t_0 + \delta]$ such that

$$s_{a,b} \colon P \in K_{t_1} \longrightarrow s_{a,b}(P)$$

is not essential on K_{t_1}. Since the map $f_{t_1} \colon K_0 \to K_{t_1}$ is continuous, the map $P \in K_0 \to s_{a,b}(f_{t_1}(P))$ is not essential, hence by homotopy, it follows that for all $t \in [t_0 - \delta, t_0 + \delta]$ the map $s_{a,b} \circ f_t$ is not essential on K_0 (see [10, Appendix of Ch. IX]) but for $t = t_0$ this means that a and b are in the same connected component of $\mathbf{C} \backslash K_0$ which is false. Finally, for all t in $[t_0 - \delta, t_0 + \delta]$, a and b are not in the same connected component of $\mathbf{C} \backslash K_t$ and $(A(t), B(t)) \sim_t (C(t), D(t))$ is false. Finally, since $[0, 1]$ is connected, we see that for all (A, B) and (C, D) in $\mathcal{O}(\mathcal{G})$,

$$(A(0), B(0)) \sim_0 (C(0), D(0)) \iff (A(1), B(1)) \sim_1 (C(1), D(1)). \qquad \blacksquare$$

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