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Weighted Mean Operators on l_p

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Abstract. The weighted mean matrix M_a is the triangular matrix $\{a_k/A_n\}$, where $a_n > 0$ and $A_n := a_1 + a_2 + \cdots + a_n$. It is proved that, subject to $n^c a_n$ being eventually monotonic for each constant c and to the existence of $\alpha := \lim \frac{A_n}{na_n}$, $M_a \in B(l_p)$ for $1 < p < \infty$ if and only if $\alpha < p$.

1 Introduction

Let $a := \{a_n\}$ be a sequence of positive numbers, and let $A_n := a_1 + a_2 + \cdots + a_n > 0$. The weighted mean matrix $M_a := \{a_{nk}\}$ is defined by

$$a_{nk} := \begin{cases} \frac{a_k}{A_n} & \text{for } 1 \le k \le n, \\ 0 & \text{for } k > n. \end{cases}$$

The M_a-transform $y = \{y_n\}$ of the sequence $x = \{x_n\}$ is given by

$$y_n := (M_a x)_n := \frac{1}{A_n} \sum_{k=1}^{n} a_k x_k.$$

Suppose throughout that

$$1 < p < \infty, \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

and define

$$\sigma_1(n) := \frac{1}{A_n} \sum_{k=1}^{n} a_k \left(\frac{n}{k}\right)^{1/p},$$

$$\sigma_2(k) := \sum_{n=k}^{\infty} \frac{a_k}{A_n} \left(\frac{k}{n}\right)^{1/q},$$

$$\sigma_1 := \sup_{n \ge 1} \sigma_1(n), \qquad \sigma_2 := \sup_{k \ge 1} \sigma_2(k).$$

Let

$$\|M_a\|_p := \sup_{\|x\|_p \le 1} \|M_a x\|_p,$$

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where

$$\|x\|_p := \Big(\sum_{n=1}^{\infty} |x_n|^p\Big)^{1/p},$$

so that $M_a \in B(l_p)$, the Banach algebra of bounded linear operators on l_p, exactly when $\|M_a\|_p$ is finite (in which case it is the norm of M_a).

Concerning criteria for $M_a \in B(l_p)$, Cass and Kratz [6, Theorem 2] dealt completely with the case when a_n is generated by a logarithmico-exponential function, that is when

(LE) $\quad a_n := \ell(n)$, where $\ell(x)$ is a positive logarithmico-exponential function

for all sufficiently large positive values of x. In [7] Hardy defined, and investigated the properties of, logarithmico-exponential functions. He described a logarithmico-exponential function on $[x_0, \infty)$ to be a real valued function defined by a finite combination of the ordinary algebraic symbols (viz. $+$, $-$, $\times$, $\div$, $\sqrt[n]{\ }$) and the functional symbols $\log(\cdot)$ and $e^{(\cdot)}$, operating on the variable x and on real constants. In [6] Cass and Kratz showed that $\frac{A_n}{na_n}$ tends to a finite or infinite limit when (LE) is satisfied, and proved essentially:

Theorem CK *Suppose that a_n is given by (LE), and that $\frac{A_n}{na_n} \to \alpha$. Then $M_a \in B(l_p)$ if and only if $\alpha < p$, in which case*

$$\frac{p}{p-\alpha} \le \|M_a\|_p \le \sigma_1^{1/q}\sigma_2^{1/p}$$

and $\lim_{n\to\infty} \sigma_1(n) = \lim_{k\to\infty} \sigma_2(k) = \frac{p}{p-\alpha}$.

The following theorem due to Cartlidge [5] (see also Borwein [3, Example 3] and [4, Theorem 2]) gives a particularly simple sufficent condition for $M_a \in B(l_p)$:

Theorem C *If a_n is eventually non-decreasing, then $M_a \in B(l_p)$.*

Bennett [1, Theorem 2] has established necessary and sufficient conditions for $M_a \in B(l_p)$. Though relatively simple in form, Bennett's conditions are more difficult to apply in specific cases than those in the above two theorems.

The aim of this paper is to show that the requirement in Theorem CK that a_n be generated by a logarithmico-exponential function can be replaced by a far less restrictive monotonicity condition. To this end we shall prove:

Theorem 1 *Suppose that $\frac{A_n}{na_n} \to \alpha$, and that, for every constant c, $n^c a_n$ is eventually monotonic. Then $M_a \in B(l_p)$ if and only if $\alpha < p$. Further, if $0 < \alpha < p$, then*

$$\frac{p}{p-\alpha} \le \|M_a\|_p \le \sigma_1^{1/q}\sigma_2^{1/p}$$

and $\lim_{n\to\infty} \sigma_1(n) = \lim_{k\to\infty} \sigma_2(k) = \frac{p}{p-\alpha}$.

Remarks The monotonicity condition in Theorem 1 is satisfied by any sequence generated by a logarithmic-exponential function as in Theorem CK.

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An example of a family of sequences $\{a_n\}$ satisfying the conditions of Theorem 1 but not the logarithmico-exponential condition (LE) is afforded by setting

$$a_n := f(n) \text{ for } n \geq 3, \quad \text{where } f(x) := \int_2^x \frac{t^{a-1}}{\log^a t}\, dt \text{ with } a > 0.$$

This can easily be demonstrated. Though $t^{a-1}\log^{-a} t$ is a logarithmico-exponential function its integral $f(x)$ is not. The monotonicity condition holds since, by partial integration,

$$f(x) \sim \frac{x^a}{a\log^a x} \text{ as } x \to \infty, \quad \text{and} \quad f(x) > \frac{x^a}{a\log^a x} \text{ for large positive } x,$$

so that

$$\frac{d}{dx}x^c f(x) = x^{c-1}\left(\frac{x^a}{a\log^a x} + cf(x)\right)$$

is eventually positive if $c > -1$ and eventually negative if $c \leq -1$. Further

$$A_n \sim \sum_{k=2}^n \frac{k^a}{a\log^a k} \sim \int_2^n \frac{t^a}{a\log^a t}\, dt \sim \frac{n^{a+1}}{a(a+1)\log^a n}, \quad \text{whence } \frac{A_n}{na_n} \to \frac{1}{a+1}.$$

2 Preliminary Results

Lemma 1 *Suppose that* $\frac{na_n}{A_n} \to \beta$ *where* $0 \leq \beta \leq \infty$. *Then*

$$\lim_{n\to\infty} n^c A_n = \begin{cases} \infty & \text{if } c > -\beta, \\ 0 & \text{if } c < -\beta. \end{cases}$$

Proof Let

$$\beta_n := \frac{na_n}{A_n}.$$

Suppose first that $0 \leq \beta < \infty$. Then, as $n \to \infty$,

$$n(\log A_n - \log A_{n-1}) = -n\log\left(1 - \frac{\beta_n}{n}\right) \to \beta,$$

and hence

$$\log A_n - \log A_1 = -\sum_{k=2}^n \log\left(1 - \frac{\beta_k}{k}\right) = (\beta + \epsilon_n)\log n, \quad \text{where } \epsilon_n \to 0.$$

Consequently $A_n = A_1 n^{\beta+\epsilon_n}$, and the desired conclusion follows.

Suppose now that $\beta = \infty$. Then, for $n \geq 2$,

$$\log A_n - \log A_{n-1} = -\log\left(1 - \frac{\beta_n}{n}\right) \geq \frac{\beta_n}{n},$$

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and hence

$$\log A_n \geq \sum_{k=2}^{n} \frac{\beta_k}{k} = \gamma_n \log n, \quad \text{where } \gamma_n \to \infty.$$

It follows that, for any real number c, $n^c A_n \geq n^{c+\gamma_n} \to \infty$. ■

Lemma 2 *Suppose that $0 < \beta < \infty$, and that $n^c a_n$ is eventually positive and increasing when the constant $c > 1 - \beta$, and eventually decreasing when $c < 1 - \beta$. Let*

$$s_1(n) := \frac{1}{n} \sum_{k=1}^{n} \frac{a_k}{a_n} \left(\frac{k}{n}\right)^{-\delta} \quad \textit{with } \delta := \frac{1}{p},$$

$$s_2(k) := k^\nu \sum_{n=k}^{\infty} \frac{a_k}{a_n} \cdot \frac{1}{n^{\nu+1}} \quad \textit{with } \nu := \frac{1}{q}.$$

(i) If $\beta > \delta$, then $\lim_{n\to\infty} s_1(n) = \lim_{k\to\infty} s_2(k) = \frac{1}{\beta-\delta}$.
(ii) If $\beta \leq \delta$, then $\lim_{n\to\infty} s_1(n) = \infty$.

Proof Let $1 > c_1 > 1 - \beta > c_2$. Then in either case there is a positive integer N such that

$$\left(\frac{k}{n}\right)^{-c_2} \leq \frac{a_k}{a_n} \leq \left(\frac{k}{n}\right)^{-c_1} \quad \text{for } n \geq k \geq N.$$

Suppose first that $\beta > \delta$. Then

$$\lim_{n\to\infty} n^{1-\delta} a_n = \infty,$$

and hence

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{N-1} \frac{a_k}{a_n} \left(\frac{k}{n}\right)^{-\delta} = 0.$$

Therefore

$$\limsup_{n\to\infty} s_1(n) \leq \lim_{n\to\infty} \frac{1}{n} \sum_{k=N}^{n} \left(\frac{k}{n}\right)^{-c_1-\delta} = \int_0^1 x^{-c_1-\delta}\, dx = \frac{1}{1-c_1-\delta}$$

and

$$\liminf_{n\to\infty} s_1(n) \geq \lim_{n\to\infty} \frac{1}{n} \sum_{k=N}^{n} \left(\frac{k}{n}\right)^{-c_2-\delta} = \int_0^1 x^{-c_2-\delta}\, dx = \frac{1}{1-c_2-\delta}.$$

Letting $c_1 \to 1-\beta$ from the right and $c_2 \to 1-\beta$ from the left, we get that $\lim_{n\to\infty} s_1(n) = \frac{1}{\beta-\delta}$. Also

$$\limsup_{k\to\infty} s_2(k) \leq \lim_{k\to\infty} k^\nu \sum_{n=k}^{\infty} \left(\frac{k}{n}\right)^{-c_1} \cdot \frac{1}{n^{\nu+1}} = \lim_{k\to\infty} k^{\nu-c_1} \sum_{n=k}^{\infty} \frac{1}{n^{1+\nu-c_1}} = \frac{1}{\nu-c_1},$$

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and similarly

$$\liminf_{k\to\infty} s_2(k) \ge \lim_{k\to\infty} k^{\nu-c_2} \sum_{n=k}^{\infty} \frac{1}{n^{1+\nu-c_2}} = \frac{1}{\nu - c_2}.$$

Again letting $c_1 \to 1-\beta$ from the right and $c_2 \to 1-\beta$ from the left, we get that $\lim_{k\to\infty} s_2(k) = \frac{1}{\beta-\delta}$, and this completes the proof of Part (i).

Suppose next that $\beta = \delta$. Then as above we get that

$$\liminf_{n\to\infty} s_1(n) \ge \frac{1}{1-c_2-\delta},$$

which tends to infinity as c_2 tends to $1-\beta$ from the left. Thus $s_1(n) \to \infty$ in this case.

Suppose finally that $\beta < \delta$. We can now assume that $1-\beta > c_2 > 1-\delta$ and obtain that, for $n > N$,

$$s_1(n) \ge \frac{1}{n} \sum_{k=N}^{n} \left(\frac{k}{n}\right)^{-c_2-\delta} \ge n^{c_2+\delta-1} N^{-c_2-\delta} \to \infty \quad \text{as } n \to \infty. \qquad \blacksquare$$

The following lemma is a special case of a known result [2, Theorem 2].

Lemma 3 *If $\{b_n\}$ is a sequence of positive numbers, and if*

$$\mu_1 := \sup_{n\ge1} \sum_{k=1}^{n} \frac{a_k}{A_n} \left(\frac{b_k}{b_n}\right)^{1/p} < \infty \quad \textit{and} \quad \mu_2 := \sup_{k\ge1} \sum_{n=k}^{\infty} \frac{a_k}{A_n} \left(\frac{b_n}{b_k}\right)^{1/q} < \infty,$$

then $M_a \in B(l_p)$ and $\|M_a\|_p \le \mu_1^{1/q} \mu_2^{1/p}$.

Lemma 4 *If $\{b_n\}$ is a bounded sequence of positive numbers such that $\sum b_n = \infty$, and if, as $n \to \infty$,*

$$\sum_{k=1}^{n} \frac{a_k}{A_n} \left(\frac{b_k}{b_n}\right)^{1/p} \to \sigma \quad \textit{(finite or infinite)},$$

then $\|M_a\|_p \ge \sigma$.

Proof Observe that if

$$D_n := \prod_{k=1}^{n} \left(1 - \frac{b_k}{b}\right)^{-1} \quad \text{where } b > \sup_{k\ge1} b_k,$$

and $d_n := D_n - D_{n-1}$ for $n \ge 2$, then $b_n = b\frac{d_n}{D_n}$ and $D_n \to \infty$. The desired conclusion is now a consequence of a known result [2, Theorem 4]. $\blacksquare$

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3 Proof of Theorem 1

Case 1: $\alpha = \lim_{n\to\infty} \frac{A_n}{na_n} = 0$ By Lemma 1, we have that $\frac{A_n}{n} \to \infty$, and hence that $a_n = \frac{na_n}{A_n} \cdot \frac{A_n}{n} \to \infty$. The theorem's monotonicity hypothesis thus implies that a_n is eventually non-decreasing, and it follows, by Theorem C, that $M_a \in B(l_p)$.

Case 2: $0 < \alpha < p$ Since $A_n \sim \alpha n a_n$, it follows from Lemma 1, with $\beta := 1/\alpha$, that

$$\lim_{n\to\infty} n^c a_n = \begin{cases} \infty & \text{if } c > 1-\beta, \\ 0 & \text{if } c < 1-\beta. \end{cases}$$

Hence, by the theorem's monotonicity hypothesis, $n^c a_n$ is eventually increasing when $c > 1-\beta$ and evenually decreasing when $c < 1-\beta$. By Lemma 2(i), we get that, as $n \to \infty$,

$$\sigma_1(n) := \frac{1}{A_n}\sum_{k=1}^{n} a_k \left(\frac{n}{k}\right)^{1/p} \sim \beta s_1(n) \to \frac{\beta}{\beta-\delta} = \frac{p}{p-\alpha},$$

and, as $k \to \infty$,

$$\sigma_2(k) := \sum_{n=k}^{\infty} \frac{a_k}{A_n}\left(\frac{k}{n}\right)^{1/q} \sim \beta s_2(k) \to \frac{\beta}{\beta-\delta} = \frac{p}{p-\alpha}.$$

Hence, by Lemma 3, $M_a \in B(l_p)$ and $\|M_a\|_p \le \sigma_1^{1/q}\sigma_2^{1/p}$, and, by Lemma 4, $\|M_a\|_p \ge \frac{p}{p-\alpha}$. This completes the proof of the case $0 < \alpha < p$ of Theorem 1.

Case 3: $p \le \alpha < \infty$ As in Case 2 we have that a_n satisfies the monotonicity conditions of Lemma 2, and hence, by Part (ii) of that lemma,

$$\sigma_1(n) := \frac{1}{A_n}\sum_{k=1}^{n} a_k \left(\frac{n}{k}\right)^{1/p} \sim \beta s_1(n) \to \infty \quad \text{as } n \to \infty.$$

It follows, by Lemma 4, that $M_a \notin B(l_p)$.

Case 4: $\alpha = \lim_{n\to\infty} \frac{A_n}{na_n} = \infty$ Observe that a_n, which is assumed to be eventually monotonic, cannot be eventually non-decreasing, for if it were then $\frac{A_n}{na_n}$ would be bounded. Hence a_n must be eventually non-increasing. As before let $\delta := 1/p$. Suppose first that A_n tends to a finite limit as $n \to \infty$. Then

$$\sigma_1(n) := \frac{1}{A_n}\sum_{k=1}^{n} a_k \left(\frac{n}{k}\right)^{1/p} \ge \frac{1}{A_n}\sum_{k=1}^{[\sqrt{n}]} a_k \left(\frac{n}{k}\right)^{\delta} \ge n^{\delta/2}\frac{A_{[\sqrt{n}]}}{A_n} \to \infty \quad \text{as } n \to \infty.$$

Suppose finally that $A_n \to \infty$. Note that, by Lemma 1, for every $\epsilon > 0$, $n^{-\epsilon}A_n \to 0$ as $n \to \infty$. Let $M = \sup_{n\ge 1} \frac{na_n}{A_n}$. Then $0 < M < \infty$ and

$$S := \sum_{k=1}^{\infty} a_k k^{-\delta} \le M \sum_{k=1}^{\infty} A_k k^{-1-\delta} < \infty,$$

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since $k^{-\delta/2}A_k$ is bounded. Hence

$$\sigma_1(n) \sim \frac{Sn^\delta}{A_n} \to \infty \quad \text{as } n \to \infty.$$

Again it follows, by Lemma 4, that $M_a \notin B(l_p)$. ■

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