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IMPROVING AN INEQUALITY FOR THE DIVISOR FUNCTION

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Abstract

Using elementary means, we improve an explicit bound on the divisor function due to Friedlander and Iwaniec [*Opera de Cribro*, American Mathematical Society, Providence, RI, 2010]. Consequently, we modestly improve a result regarding a sieving inequality for Gaussian sequences.

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1. Introduction

Let $\tau(n)$ be the number of divisors of n. While asymptotic estimates for weighted sums $\sum \tau(n)a_n$ are generally difficult to obtain, explicit bounds often suffice in applications.

We shall consider the relationship between $\tau(n)$ and averages of $\tau(d)$ for small divisors d of n. Landreau [4] showed that for any integer $k \geq 2$ there exists a constant $C_k > 0$ such that

$$\tau(n) \leq C_k \sum_{\substack{d|n \\ d \leq n^{1/k}}} (2^{\omega(d)}\tau(d))^k \quad \text{for } n \geq 1, \tag{1.1}$$

where $\omega(n)$ counts the number of distinct primes dividing n. We wish to make the constants C_k effective. Friedlander and Iwaniec [2] considered, *inter alia*, a weakened version of (1.1) for $k = 4$, making use of the trivial bound $2^{\omega(n)} \leq \tau(n)$. They showed that

$$\tau(n) \leq C \sum_{\substack{d|n \\ d \leq n^{1/4}}} \tau(d)^8 \quad \text{for } n \geq 1, \tag{1.2}$$

holds for $C = 256$. Numerical evidence suggests that this constant is far from optimal. In fact, it can be verified easily that (1.2) holds with $C = 8$ for all $1 \leq n \leq 10^8$. Moreover, equality is attained for 733133 values of n within this interval, these being the square-free numbers $n = p_1p_2p_3$ satisfying $n^{1/4} < \min(p_1, p_2, p_3)$. So for small n it

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is certainly the case that $C = 8$ is the best possible constant, with evidence suggesting that this trend should continue as $n \to \infty$. Our aim is to investigate whether $C \leq 8$ is admissible for all n sufficiently large, as well as whether the sum can be made sharper.

We show that (1.2) indeed holds for $C = 8$. In addition we improve on the exponent of $\tau(d)$ in the sum, which (1.1) suggests should be much smaller than 8, at least for non-square-free n. Our main result to reach this goal is the following theorem.

Theorem 1.1. *Let $n \geq 1$. Then there exists $d \leq n^{1/4}$ with $d|n$ such that $\tau(n) \leq 8\tau(d)^7$.*

We shall also show that the constant C in (1.2) must satisfy $C \geq 8$.

Theorem 1.2. *We have*

$$\tau(n) \leq 8 \sum_{\substack{d|n \\ d \leq n^{1/4}}} \tau(d)^7 \quad \text{for } n \geq 1,$$

the constant 8 being best possible for all n.

The consideration of (1.2) by Friedlander and Iwaniec in [2] led to their study of sieving inequalities for Gaussian sequences. We shall see in Section 6 how Theorem 1.2 may be used to modestly improve one of their results [1].

2. A lower bound

Our first result describes a natural lower bound for the constant C in (1.2). This bound arises from the consideration of a particular set of square-free numbers. In fact, the result extends to the general case (1.1).

Proposition 2.1. *Fix an integer $k \geq 2$. For any multiplicative function $f : \mathbb{N} \to \mathbb{R}$,*

$$\limsup_{n \to \infty} \tau(n) \Big(\sum_{\substack{d|n \\ d \leq n^{1/k}}} f(d) \Big)^{-1} \geq 2^{k-1}.$$

Proof. Take a prime $p_1 > 2^{(k-1)(k-2)/2}$ and choose, using Bertrand's postulate, primes $p_2 < p_3 < \cdots < p_{k-1}$ such that $p_1 < p_2 < 2p_1$ and $p_i < 2^{i-1} p_1$ for $3 \leq i \leq k-1$. Then

$$p_1^{k-1} > 2^{(k-1)(k-2)/2} \times p_1^{k-2} = \prod_{i=2}^{k-1} 2^{i-1} p_1 > p_2 p_3 \cdots p_{k-1}.$$

Consider now $n = p_1 p_2 \cdots p_{k-1}$. We see that $p_1 > n^{1/k}$, whence there are no nontrivial divisors d of n with $d \leq n^{1/k}$. So for such an n we have $\tau(n) = 2^{k-1}$ and

$$\sum_{\substack{d|n \\ d \leq n^{1/k}}} f(d) = f(1) = 1. \qquad \square$$

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3. Some upper bounds

We now turn our attention to proving Theorem 1.1. The aim is to choose for any n a divisor $d \le n^{1/4}$ for which $\tau(d)$ is as large as possible. In this section we demonstrate this procedure for n with certain prime factorisations.

We shall make use of the following elementary inequalities. We write $[x]$ for the integer part of x.

Lemma 3.1. *For all integers $t \ge 4$, we have $7[t/4] \ge t$ and $([t/4]+1)^4 \ge 2(t+1)$.*

Proof. Let $i \ge 1$ be the unique integer such that $4i \le t \le 4i+3$. For the first inequality, we simply see that $7[t/4] = 7i \ge 4i+3 \ge t$. For the second, we have $([t/4]+1)^4 = (i+1)^4 \ge 8(i+1) = 2(4i+3)+2 \ge 2t+2$. □

We consider the various cases pertaining to how prime powers appear in the prime factorisation of n. Our first lemma deals with the case when all exponents are at least 4.

Lemma 3.2. *Suppose $n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$ with $a_i \ge 4$ for all $1 \le i \le t$. Then there exists $d \le n^{1/4}$ with $d|n$ such that $\tau(n) \le 2^{-t}\tau(d)^4$.*

Proof. We let $d = \prod_{i=1}^t p_i^{[a_i/4]}$. Then $d \le n^{1/4}$ and, by Lemma 3.1,

$$\tau(d)^4 = \prod_{i=1}^t \left(\left[\frac{a_i}{4}\right]+1\right)^4 \ge 2^t \prod_{i=1}^t (a_i+1) = 2^t\tau(n). \qquad \square$$

We now consider the cases when all prime powers appearing in the prime factorisation of n occur with exponent k for $k \in \{1,2,3\}$.

Lemma 3.3. *Suppose $n = p_1 p_2 \cdots p_t$ with $p_1 < p_2 < \cdots < p_t$. Then there exists $d \le n^{1/4}$ with $d|n$ such that*

$$\tau(n) \le \begin{cases} 2^t\tau(d) & \text{if } t \in \{1,2,3\},\\ \tau(d)^7 & \text{if } t \ge 4.\end{cases}$$

Proof. Firstly, let $t \in \{1,2,3\}$ be fixed. In each of these cases we let $d = 1$. Then $2^t\tau(d) = \tau(n)$.

On the other hand, if $t \ge 4$, we take $d = p_1p_2\ldots p_{[t/4]}$. Then $d \le n^{1/4}$ and, by Lemma 3.1, $\tau(d)^7 = 2^{7\times[t/4]} \ge 2^t = \tau(n)$. □

Lemma 3.4. *Suppose $n = p_1^2p_2^2\ldots p_t^2$ with $p_1 < p_2 < \cdots < p_t$. Then there exists $d \le n^{1/4}$ with $d|n$ such that*

$$\tau(n) \le \begin{cases} 3\tau(d) & \text{if } t = 1,\\ 2^{-2}\tau(d)^7 & \text{if } t \in \{2,3\},\\ \tau(d)^7 & \text{if } t \ge 4.\end{cases}$$

Proof. If $t = 1$, we let $d = 1$. Then $3\tau(d) = \tau(p_1^2) = \tau(n)$. Next suppose $t \in \{2,3\}$. In these cases take $d = p_1$, whence $\tau(d)^7 = 2^7 > 2^2 \times 3^3 \ge 2^2\tau(n)$. Finally, suppose $t \ge 4$. Take $d = p_1^2p_2^2\cdots p_{[t/4]}^2$. Then $\tau(d)^7 = 3^{7\times[t/4]} \ge 3^t = \tau(n)$. □

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Lemma 3.5. *Suppose $n = p_1^3 p_2^3 \cdots p_t^3$ with $p_1 < p_2 < \cdots < p_t$. Then there exists $d \le n^{1/4}$ with $d|n$ such that*

$$\tau(n) \le \begin{cases} 4\tau(d) & \text{if } t = 1, \\ 2^{-3}\tau(d)^7 & \text{if } t = 2, \\ 2^{-5}\tau(d)^7 & \text{if } t = 3, \\ \tau(d)^7 & \text{if } t \ge 4. \end{cases}$$

Proof. As before, if $t = 1$, let $d = 1$, whence $4\tau(d) = \tau(n)$. If $t = 2$, we take $d = p_1$, giving $\tau(d)^7 = 2^7 = 2^3\tau(n)$. If $t = 3$, let $d = p_1^2$, so that $\tau(d)^7 = 3^7 > 2^5 \times 4^3 = 2^5\tau(n)$. Finally, for $t \ge 4$, take $d = p_1^3 p_2^3 \cdots p_{[t/4]}^3$, whence $\tau(d)^7 = 4^{7\times[t/4]} \ge 4^t = \tau(n)$. □

We are now ready to combine these estimates to prove Theorem 1.1.

4. Proof of Theorem 1.1

Let $n \ge 1$ and consider the unique prime factorisation of n. We group the prime powers according to their exponents: for each $i \in \{1, 2, 3\}$, let m_i be the product of those occurring with exponent i and let l be the product of those with exponent at least 4. The relations $m_i = 1$ and $l = 1$ will be understood to mean that no primes of the corresponding form divide n.

Write $n = m_1 m_2 m_3 l$. First observe by Lemma 3.2 that there exists a divisor d_l of l with $d_l \le l^{1/4}$ for which

$$\tau(n) = \tau(m_1 m_2 m_3)\tau(l) \le \tau(m_1 m_2 m_3)\tau(d_l)^7. \tag{4.1}$$

Thus to prove our theorem it suffices to consider those n whose prime factorisations consist solely of prime powers with exponents strictly less than 4. That is, if for each such $n = m_1 m_2 m_3$ we can find a divisor $d \le n^{1/4}$ with $\tau(n) \le 8\tau(d)^7$, then the assertion in the theorem follows from (4.1).

In each of the following cases the numbers d_1, d_2, d_3 are chosen according to Lemmas 3.3, 3.4 and 3.5. Note that these satisfy $d_i | m_i$ and $d_i \le m_i^{1/4}$. Moreover, if $m_i = 1$, we may choose $d_i = 1$.

(I) Let $m_1 \ge 1$.

- (i) If $\omega(m_2) \in \{2, 3\}$, then $\tau(n) \le 8\tau(d_1)^7 \times 2^{-2}\tau(d_2)^7 \times 4\tau(d_3)^7 \le 8\tau(d_1 d_2 d_3)^7$.
- (ii) If $\omega(m_3) \in \{2, 3\}$, then $\tau(n) \le 8\tau(d_1)^7 \times 3\tau(d_2)^7 \times 2^{-3}\tau(d_3)^7 \le 3\tau(d_1 d_2 d_3)^7$.

Henceforth we only consider the cases $m_2, m_3 = 1$ and $\omega(m_2), \omega(m_3) \in \mathbb{N} \setminus \{2, 3\}$.

(II) Suppose $m_1 = 1$ or $\omega(m_1) \ge 4$.

- (i) If at least one of $\omega(m_2) \ge 4$ or $\omega(m_3) \ge 4$ holds, then $\tau(n) \le \tau(d_1)^7 \times 4\tau(d_2)^7 \times \tau(d_3)^7 = 4\tau(d_1 d_2 d_3)^7$.
- (ii) On the other hand, suppose $\omega(m_2) = \omega(m_3) = 1$. Write $n = m_1 p_1^2 p_2^3$. Let $d' = \min(p_1, p_2) \le (p_1^2 p_2^3)^{1/4}$. Then $\tau(d')^7 = 2^7 > \tau(p_1^2 p_2^3)$ and so $\tau(n) < \tau(d_1)^7 \times \tau(d')^7 \le \tau(d_1 d')^7$.

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(III) Suppose $\omega(m_1) = 1$.

(i) If at least one of $\omega(m_2) \geq 4$ or $\omega(m_3) \geq 4$ holds, then $\tau(n) \leq 2\tau(d_1)^7 \times 4\tau(d_2)^7 \times \tau(d_3)^7 \leq 8\tau(d_1d_2d_3)^7$.

(ii) On the other hand, suppose $\omega(m_2) = \omega(m_3) = 1$. Write $n = m_1p_1^2p_2^3$. Let $d' = \min(p_1, p_2) \leq (p_1^2p_2^3)^{1/4}$. Then $\tau(d')^7 > \tau(p_1^2p_2^3)$ and so $\tau(n) < 2\tau(d_1) \times \tau(d')^7 \leq 2\tau(d_1d')^7$.

(IV) Suppose $\omega(m_1) = 2$.

(i) If $\omega(m_2) \geq 4$ and $\omega(m_3) \geq 4$, then $\tau(n) \leq 4\tau(d_1) \times \tau(d_2)^7 \times \tau(d_3)^7 \leq 4\tau(d_1d_2d_3)^7$.

(ii) If $\omega(m_2) = 1$ and $\omega(m_3) \geq 4$, write $n = p_1p_2p_3^2m_3$. Let $d' = \min(p_1, p_2, p_3) \leq (p_1p_2p_3^2)^{1/4}$. Then $\tau(d')^7 > \tau(p_1p_2p_3^2)$ and so $\tau(n) < \tau(d')^7 \times \tau(d_3)^7 = \tau(d'd_3)^7$.

(iii) If $\omega(m_2) \geq 4$ and $\omega(m_3) = 1$, write $n = p_1p_2p_3^3m_2$. Let $d' = \min(p_1, p_2, p_3) \leq (p_1p_2p_3^3)^{1/4}$. Then $\tau(d')^7 > \tau(p_1p_2p_3^3)$ and so $\tau(n) < \tau(d')^7 \times \tau(d_2)^7 = \tau(d'd_2)^7$.

(iv) Suppose $\omega(m_2) = \omega(m_3) = 1$. Write $n = m_1p_1^2p_2^3$. Let $d' = \min(p_1, p_2) \leq (p_1^2p_2^3)^{1/4}$. Then $\tau(d')^7 > \tau(p_1^2p_2^3)$ and so $\tau(n) < 4\tau(d_1) \times \tau(d')^7 \leq 4\tau(d_1d')^7$.

(V) Suppose $\omega(m_1) = 3$.

(i) If $\omega(m_2) \geq 4$ and $\omega(m_3) \geq 4$, then $\tau(n) \leq 8\tau(d_1) \times \tau(d_2)^7 \times \tau(d_3)^7 \leq 8\tau(d_1d_2d_3)^7$.

(ii) If $\omega(m_2) = 1$ and $\omega(m_3) \geq 4$, write $n = p_1p_2p_3p_4^2m_3$. Let $d' = \min(\{p_i\}) \leq (p_1p_2p_3p_4^2)^{1/4}$. Then $\tau(d')^7 > \tau(p_1p_2p_3p_4^2)$ and so $\tau(n) < \tau(d')^7 \times \tau(d_3)^7 = \tau(d'd_3)^7$.

(iii) If $\omega(m_2) \geq 4$ and $\omega(m_3) = 1$, write $n = p_1p_2p_3p_4^3m_2$. Let $d' = \min(\{p_i\}) \leq (p_1p_2p_3p_4^3)^{1/4}$. Then $\tau(d')^7 > \tau(p_1p_2p_3p_4^3)$ and so $\tau(n) < \tau(d')^7 \times \tau(d_2)^7 = \tau(d'd_2)^7$.

(iv) If $\omega(m_2) = \omega(m_3) = 1$, write $n = m_1p_1^2p_2^3$. Let $d' = \min(p_1, p_2) \leq (p_1^2p_2^3)^{1/4}$. Then $\tau(d')^7 > \tau(p_1^2p_2^3)$ and so $\tau(n) < 8\tau(d_1) \times \tau(d')^7 \leq 8\tau(d_1d')^7$.

5. Further speculation

Returning to (1.1), one may consider for any $k \geq 2$ and $\eta \geq 1$ the generalised inequality

$$\tau(n) \leq C_{k,\eta} \sum_{\substack{d|n \\ d \leq n^{1/k}}} \tau(d)^\eta. \tag{5.1}$$

Clearly if (5.1) holds then it must also be true for any $\eta' > \eta$, in which case we may choose $C_{k,\eta'} = C_{k,\eta}$. Thus for fixed k and $C_k = C_{k,\eta}$ we would like to know the smallest η for which (5.1) holds.

A natural question to consider is whether Theorem 1.1 can be improved to show that for all $n \geq 1$ there exists a divisor $d \leq n^{1/4}$ such that $\tau(n) \leq 8\tau(d)^6$. It appears, however, that the purely elementary methods presented in this paper cannot achieve this in any practical sense. To see why, consider a number $n = p_1^2 p_2^2 \cdots p_{t_1}^2 q_1^3 q_2^3 \cdots q_{t_2}^3$ with $t_1 \geq 4$ and $t_2 \geq 4$. Suppose $p_1 < p_2 < \cdots < p_{t_1}$ and $q_1 < q_2 < \cdots < q_{t_2}$. Without additional assumptions on n the best choice of divisor $d \leq n^{1/4}$ for which $\tau(d)$ is as large as possible is $d = p_1^2 p_2^2 \cdots p_{[t_1/4]}^2 q_1^3 q_2^3 \cdots q_{[t_2/4]}^3$. But then (cf. Lemma 3.1)

$$\tau(d)^6 = 3^{6\times[t_1/4]} \times 4^{6\times[t_2/4]} \geq 3^{t_1-1} \times 4^{t_2-1} = 12^{-1}\tau(n).$$

Thus the best estimate we can produce unconditionally is $\tau(n) \leq 12\tau(d)^6$. One may enumerate each of the various cases in regard to the relative sizes of the p_i, q_j to produce a divisor d with $\tau(d)$ large enough; this seems a formidable task in general.

In any case it remains an open problem to determine the smallest $\eta > 0$ such that

$$\tau(n) \ll_\eta \sum_{\substack{d|n \\ d \leq n^{1/4}}} \tau(d)^\eta. \tag{5.2}$$

At least in the square-free case this problem has been solved. Iwaniec and Munshi [3] showed that (5.2) holds for square-free n with any $\eta > 3\log 3/\log 2 - 4 = 0.75488\ldots$, this lower bound being best possible.

6. An application to Gaussian sequences

Of significant interest in sieve theory is the detection of primes in Gaussian sequences, namely sequences supported on integers which can be expressed as the sum of two squares.

Here we consider a generalised Gaussian sequence $\mathcal{A} = (a_n)$ defined by

$$a_n = \sum_{\substack{l^2+m^2=n \\ (l,m)=1}} \gamma_l, \tag{6.1}$$

where l, m run over positive integers and γ_l are any complex numbers with $|\gamma_l| \leq 1$. We further suppose that the γ_l are supported on rth powers, that is, $\gamma_l = 0$ if $l \neq k^r$.

In the process of sieving $\mathcal{A}$ one requires good estimates for

$$A_d(x) = \sum_{\substack{n \leq x \\ d|n}} a_n. \tag{6.2}$$

It can be shown (see [1, equations (6) and (7)]) that

$$\sum_{n \leq x} a_n = \sum_{l < \sqrt{x}} \gamma_l \frac{\varphi(l)}{l} \sqrt{x - l^2} + O(x^{1/2r} \log x),$$

so for d not too large we expect $A_d(x)$ to be uniformly well approximated by

$$M_d(x) = \frac{\rho(d)}{d} \sum_{\substack{l < \sqrt{x} \\ (l,d)=1}} \gamma_l \frac{\varphi(l)}{l} \sqrt{x - l^2},$$

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where $\rho(d)$ is the number of solutions to the congruence $\nu^2 + 1 \equiv 0 \mod d$.

To estimate (6.2), we may consider instead the smoothed sum

$$A_d(f) = \sum_{n\equiv 0 \bmod d} a_n f(n),$$

where $f \in C^\infty([0,\infty))$ is such that $f(t) = 1$ if $0 \le t \le (1-\kappa)x$ and $f(t) = 0$ if $t \ge x$. Here $x^{-1/4r} \le \kappa \le 1$ is some parameter to be optimised later.

Proposition 6.1. *Suppose* $\sqrt{x} \le D \le x^{(r+1)/2r}$. *Then*

$$\sum_{d\le D} |A_d(x) - A_d(f)| \ll \kappa x^{(r+1)2r} (\log x)^{128}.$$

Proof. A rearrangement of the sum gives

$$\begin{aligned}
\sum_{d\le D} |A_d(x) - A_d(f)| &= \sum_{d\le D} \Bigg| \sum_{\substack{(1-\kappa)x<n\le x \\ d\mid n}} (1 - f(n)) \sum_{\substack{l^2+m^2=n \\ (l,m)=1}} \gamma_l \Bigg| \\
&\ll \sum_{\substack{(1-\kappa)x<l^2+m^2\le x \\ (l,m)=1}} |\gamma_l| \sum_{d\mid (l^2+m^2)} 1 \\
&\ll \sideset{}{'}\sum_{\substack{(1-\kappa)x<l^2+m^2\le x \\ (l,m)=1}} |\gamma_l| \tau(l^2+m^2) + \sqrt{x}\log x,
\end{aligned}$$

where $\sum'$ means that the terms with a value of l which is nearest to $\sqrt{x}$ are omitted. We deduce from Theorem 1.2 that

$$\sideset{}{'}\sum_{\substack{(1-\kappa)x<l^2+m^2\le x \\ (l,m)=1}} |\gamma_l| \tau(l^2+m^2) \ll \sideset{}{'}\sum_{l<\sqrt{x}} |\gamma_l| \sum_{\substack{d\le x^{1/4} \\ (d,l)=1}} \tau(d)^7 \sum_{\substack{(1-\kappa)x<l^2+m^2\le x \\ l^2+m^2\equiv 0 \bmod d}} 1.$$

Now split the range of m into residue classes $m \equiv \nu l \mod d$, where $\nu^2 + 1 \equiv 0 \mod d$. This, combined with the observation that m runs over an interval of length $O(\kappa x/\sqrt{x - l^2})$, allows us to estimate the above by

$$\begin{aligned}
&\ll \kappa x \Big(\sum_{d\le x^{1/4}} \tau(d)^7 \frac{\rho(d)}{d} \Big) \Big(\sideset{}{'}\sum_{l<\sqrt{x}} \frac{|\gamma_l|}{\sqrt{x - l^2}} \Big) + x^{1/4+1/2r} (\log x)^{128} \\
&\ll \kappa x \times (\log x)^{128} \times x^{1-r/2r} + x^{1/4+1/2r} (\log x)^{128} \\
&\ll \kappa x^{(r+1)/2r} (\log x)^{128}.
\end{aligned}$$

□

We can now use Proposition 6.1 to improve the error term in the main theorem of [1] by a factor of $O((\log x)^{64.75})$.

Theorem 6.2. *Let* a_n *and* $A_d(x)$ *be as in (6.1) and (6.2), respectively. Suppose* $\sqrt{x} \le D \le x^{(r+1)/2r}$. *Then*

$$\sum_{d\le D} |A_d(x) - M_d(x)| \ll D^{1/4} x^{3(r+1)/8r} (\log x)^{65.25}.$$

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PROOF. We combine equations (19) and (35) from [1] with Proposition 6.1 above to obtain the estimate

$$\sum_{d\le D}|A_d(x)-M_d(x)| \ll \sum_{d\le D}|A_d(x)-A_d(f)| + \kappa^{-1}D^{1/2}x^{r+1/4r}(\log x)^{5/2} + \kappa x^{r+1/2r}\log x$$
$$\ll \kappa x^{r+1/2r}(\log x)^{128} + \kappa^{-1}D^{1/2}x^{r+1/4r}(\log x)^{5/2}.$$

Choosing

$$\kappa = D^{1/4}x^{-r+1/8r}(\log x)^{5/4-64}$$

yields the desired result. □

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