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ON THE COMMUTATIVITY OF TORSION FREE RINGS

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If R is $(n(n-1)/2)$-torsion free ring with 1 and satisfies the identity $(xy)^n = x^n y^n$, then R is commutative provided that $n = 4k$.

A theorem of Bell [2] states that if a ring R with identity 1 is n-torsion free and satisfies the two identities $(xy)^n = x^n y^n$ and $(xy)^{n+1} = x^{n+1}y^{n+1}$, then R is commutative. In [1] Abu-Khuzam proved that if R is $n(n-1)$-torsion free ring with 1 and satisfies the identity $(xy)^n = x^n y^n$, then R is commutative. Recently Kobayashi [4] has stated the following conjecture: if R is $(n(n-1)/2)$-torsion free ring with 1 and satisfies the identity $(xy)^n = x^n y^n$, then R is commutative provided that n is even. Considering the ring

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix} : a, b, c \in \mathrm{GF}(4) \right\}$$

we see that, with $n = 6$, R is $(n(n-1)/2)$-torsion free ring with 1 and satisfies the identity $(xy)^n = x^n y^n$. Note that R is not commutative. Therefore, Kobayashi's conjecture is not true in general.

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However, we prove that if $n = 4k$, then the above conjecture is true. Namely, we prove the following

THEOREM. *Let* n *be a fixed positive integer. If* R *is* $n(2n-1)$*-torsion free ring with* 1 *and satisfies the identity* $(xy)^{2n} = x^{2n}y^{2n}$ *, then* R *is commutative provided that* n *is even.*

Throughout this note R will be an associative ring with 1 , $Z(R)$ the center, $N(R)$ the set of all nilpotent elements and $C(R)$ the commutator ideal of R . As usual we write $[x, y] = xy - yx$.

We shall use freely the following well known results.

(I) *If* $[x, [x, y]] = 0$ *, then* $[x^m, y] = mx^{m-1}[x, y]$ *for any positive integer* m *.*

(II) *If* $x^m[x, y] = 0$ *for some positive integer* m *, then* $[x, y] = 0$.

We now proceed to prove our theorem.

Proof of the theorem. In hypothesis $(xy)^{2n} = x^{2n}y^{2n}$ replace x by $u^{-1}x$ and y by u , where u is an invertible element of R . Then

$$u^{-2n}x^{2n}u^{2n} = \left(u^{-1}xu\right)^{2n} = u^{-1}x^{2n}u$$

which implies

(1) $[u^{2n-1}, x^{2n}] = 0$, for all x in R and all invertible elements u of R .

Let $a \in N(R)$; then there exists a positive integer p such that

(2) $$[a^k, x^{2n}] = 0 \text{ , for all } k \geq p \text{ , } p \text{ minimal.}$$

Suppose $p > 1$; then $1 + a^{p-1}$ is invertible, and by (1) and (2) we obtain

$$0 = [(1+a^{p-1})^{2n-1}, x^{2n}] = (2n-1)[a^{p-1}, x^{2n}] .$$

Since R is $(2n-1)$-torsion free, we conclude that $[a^{p-1}, x^{2n}] = 0$ which contradicts the minimality of p . Thus $p = 1$ and (2) implies

(3) $$[a, x^{2n}] = 0 \text{ , for all } x \in R \text{ and } a \in N(R) .$$

Consider the subring $S = \langle x^{2n} : x \in R \rangle$ of R generated by all $2n$th powers of elements of R. Then (3) implies $N(S) \subseteq Z(S)$, and by Herstein's theorem [3],

(4) $$C(S) \subseteq N(S) \subseteq Z(S) .$$

Since $(xy)^{2n}x = x(yx)^{2n}$ for all x, y in S, we have

$$x^{2n}y^{2n}x = xy^{2n}x^{2n} ,$$

$$x[x^{2n-1}, y^{2n}]x = 0 .$$

In view of (I) and (4) the last identity implies

$$(2n-1)x^{2n}[x, y^{2n}] = 0 \text{ , for all } x, y \text{ in } S .$$

But S is $(2n-1)$-torsion free, thus $x^{2n}[x, y^{2n}] = 0$ which, in view of (II), implies

$$[x, y^{2n}] = 0 \text{ , for all } x, y \text{ in } S .$$

Applying (I), in view of (4), to the last identity we obtain

$$2ny^{2n-1}[x, y] = 0 \text{ , for all } x, y \text{ in } S .$$

Since S is n-torsion free and n is even, we conclude that $y^{2n-1}[x, y] = 0$ which together with (II) implies

$$[x, y] = 0 \text{ , for all } x, y \text{ in } S .$$

Therefore

$$x^{2n}y^{2n} = y^{2n}x^{2n} \text{ , for all } x, y \text{ in } R .$$

Then

$$x^{2n+1}y^{2n+1} = x(x^{2n}y^{2n})y = x(y^{2n}x^{2n})y = x(yx)^{2n}y = (xy)^{2n}xy = (xy)^{2n+1} ;$$

that is,

(5) $$(xy)^{2n+1} = x^{2n+1}y^{2n+1} \text{ , for all } x, y \text{ in } R .$$

Furthermore,

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$$(xy)^{2n} = x^{2n}y^{2n} = y^{2n}x^{2n} = (yx)^{2n} .$$

Since $(xy)^{2n}x = x(yx)^{2n} = x(xy)^{2n}$, we have

$$0 = [x, (xy)^{2n}] = [x, x^{2n}y^{2n}] = x^{2n}[x, y^{2n}] .$$

Combining (II) and the last identity we obtain

(6) $$[x, y^{2n}] = 0 \text{ , for all } x, y \text{ in } R .$$

Let u be an invertible element of R. In (5) replace x by $u^{-1}x$ and y by u to get

$$u^{-1}x^{2n+1}u = \left(u^{-1}xu\right)^{2n+1} = u^{-2n-1}x^{2n+1}u^{2n+1} ;$$

that is,

(7) $$[u^{2n}, x^{2n+1}] = 0 .$$

Now the same argument we used in (1) to obtain (3) works also in (7), since R is n-torsion free and n is even. Thus we can obtain

(8) $$[a, x^{2n+1}] = 0 \text{ , for all } x \text{ in } R \text{ and } a \text{ in } N(R) .$$

Combining (3) and (8) we see that

$$a \in Z(R) \text{ , for all } a \text{ in } N(R) ;$$

that is,

$$N(R) \subseteq Z(R) .$$

Hence Herstein's theorem [3] implies

(9) $$C(R) \subseteq Z(R) .$$

Finally, combining (6), (9) and (I) we obtain

$$0 = [x, y^{2n}] = 2ny^{2n-1}[x, y] \text{ , for all } x, y \text{ in } R .$$

Since R is n-torsion free and n even, the last identity implies $y^{2n-1}[x, y] = 0$ which together with (II) yields $[x, y] = 0$, for all x, y in R.

This completes the proof of the theorem.

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