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PERCOLATION OF HARD DISKS

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Abstract

Random arrangements of points in the plane, interacting only through a simple hard-core exclusion, are considered. An intensity parameter controls the average density of arrangements, in analogy with the Poisson point process. It is proved that, at high intensity, an infinite connected cluster of excluded volume appears almost surely.

Keywords: Percolation; Poisson point process; Gibbs measure; grand canonical Gibbs distribution; statistical mechanics; hard sphere; hard disk; excluded volume; gas/liquid transition; phase transition

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1. Introduction

Consider a random arrangement of points in the plane. Suppose that each pair of points at a distance less than L from one another are joined by an edge, and let G be the resulting graph. An important question in percolation theory is: does G have an infinite connected component?

A key problem in answering this question is in defining what is meant by a random arrangement of points. A standard model is the *Poisson point process*, in which the probability that a (Borel) set A contains k points of the random arrangement is Poisson distributed with parameter $\lambda|A|$, where $|\cdot|$ is the Lebesgue measure and λ is the *intensity* of the process. Events in disjoint sets are independent; see [3]. Here λ is the (average) density of arrangements of points. It can be shown that if λ is greater than some critical value λ_c then G has an infinite connected component with probability 1; see [10]. (Of course, λ_c depends on the connection distance L.)

The Poisson point process is closely related to the (grand canonical) *Gibbs distribution* of statistical mechanics (with particle interaction set to 0 and momentum variables integrated out) in the sense that they give nearly identical probabilistic descriptions of arrangements of points in large finite subsets of the plane. The Gibbs distributions, however, also allow for interactions among the points. Suppose that the points interact through a simple exclusion of radius $2r > 0$. (That is, each pair of points is separated by a distance of at least $2r$.) Each arrangement of points can then be imagined as a collection of *hard-core* (i.e. nonoverlapping) disks of radius r.

There is a Gibbs distribution on arrangements of points with exclusion radius $2r$ in finite subsets of the plane which, like the Poisson process, gives equal probabilistic weight to every arrangement of the same density. Furthermore, a probability measure can be defined on such arrangements in the whole plane such that, in a certain sense, its restriction to finite subsets has the Gibbs distribution. This probability measure, called an (infinite volume) *Gibbs measure*, has been extensively studied (see, e.g. [5], [8], and [12]).

It is natural to ask whether G has an infinite connected component when the points in G are sampled from a Gibbs measure with an exclusion of radius $2r$. If $r \ll L$, one can argue

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that the exclusion is insignificant and that, by analogy with the Poisson process, there is some critical *activity*, z_c, such that G almost surely has an infinite connected component for $z > z_c$. (See Section 7 of [2] for a sketch of a proof in this direction.) Here the activity z is a parameter analogous to the intensity of the Poisson process.

If r and L are close, the qualitative relationship with the Poisson point process is less clear, at least as it pertains to percolation. In particular, let $L < 4r$. Then the percolation question is closely related to the *excluded volume*. (The excluded volume corresponding to an arrangement of points is the set of all points which, due to the exclusion radius, cannot be added to the arrangement.) If G has an infinite component for such L, then there is an infinite connected region of excluded volume. The latter event has been associated with the gas/liquid phase transition in equilibrium statistical mechanics; see [7] and [13]. Below it is proved that, given $L > 3r$, with points distributed under a Gibbs measure with an exclusion of radius $2r$, G has an infinite connected component almost surely whenever the activity z is sufficiently large.

Little is known about qualitative properties of typical samples from a Gibbs measure (with exclusion) when z is large; even simulations have been inconclusive, although a recent large-scale study [1] may settle some questions. It is expected (but not proven) that when z is large, typical arrangements exhibit long-range orientational order; see [1]. On the other hand, it has been shown that there can be no long-range positional order at any z (see [11]; this is an extension of the famous Mermin–Wagner theorem to the case of hard-core interactions). The absence of long-range positional order makes the percolation question even more pertinent.

2. Notation, probability measure, and sketch of proof

Fix $r > 0$, and define

$$\Omega = \{\omega \subset \mathbb{R}^2 \colon |x - y| \geq 2r \text{ for all } x \neq y \in \omega\} \subset \mathcal{P}(\mathbb{R}^2).$$

In particular, $\varnothing \in \Omega$. (Here $\mathcal{P}(\mathbb{R}^2)$ is the set of subsets of $\mathbb{R}^2$.) Let $\mathcal{T}$ be the topology on Ω generated by the subbasis of sets of the form $\{\omega \in \Omega \colon \#(\omega \cap U) = \#(\omega \cap K) = m\}$ for compact sets $K \subset \mathbb{R}^2$, open sets $U \subset K$, and positive integers m. Here $\#\zeta$ is the number of elements in the set ζ. Let $\mathcal{F}$ be the σ-algebra of Borel sets with respect to the topology $\mathcal{T}$; it is known that $\mathcal{F}$ is generated by sets of the form $\{\omega \in \Omega \colon \#(\omega \cap B) = m\}$ for bounded Borel sets $B \subset \mathbb{R}^2$ and nonnegative integers m; see [12]. Let $\Lambda_n = [-n, n]^2 \subset \mathbb{R}^2$, and, given $A \in \mathcal{F}$, define

$$A_{n,N} = \{(x_1, \ldots, x_N) \colon \{x_1, \ldots, x_N\} \in A,\ \{x_1, \ldots, x_N\} \subset \Lambda_n\} \subset (\mathbb{R}^2)^N,$$

$$L_{n,N}(A) = \frac{1}{N!} \int_{A_{n,N}} \mathrm{d}x_1 \cdots \mathrm{d}x_N, \qquad L_{n,z}(A) = \sum_{N=1}^{\infty} z^N L_{n,N}(A).$$

For $\zeta \in \Omega$ and $n \in \mathbb{N}$, define

$$\Omega_{n,\zeta} = \{\omega \in \Omega \colon \omega \subset \Lambda_n,\ \omega \cup (\zeta \setminus \Lambda_n) \in \Omega\}.$$

It is easily seen that $\Omega_{n,\zeta} \in \mathcal{F}$. For $\zeta \in \Omega$, $z \in \mathbb{R}$, and $n \in \mathbb{N}$, define the *grand canonical Gibbs distribution $G_{n,z,\zeta}$ with boundary condition ζ on Λ_n* by

$$G_{n,z,\zeta}(A) = \frac{L_{n,z}(A \cap \Omega_{n,\zeta})}{L_{n,z}(\Omega_{n,\zeta})} \quad \text{for } A \in \mathcal{F}.$$

The Gibbs distribution $G_{n,z,\zeta}$ is a probability measure on $(\Omega, \mathcal{F})$ with support in $\Omega_{n,\zeta}$. A measure μ_z on $(\Omega, \mathcal{F})$ is called a *Gibbs measure* if $\mu_z(\Omega) = 1$ and, for all $n \in \mathbb{N}$ and all measurable functions $f: \Omega \to [0, \infty)$,

$$\int_{\Omega} f(\omega)\mu_z(\mathrm{d}\omega) = \int_{\Omega} \mu_z(\mathrm{d}\zeta) \int_{\Omega_{n,\zeta}} G_{n,z,\zeta}(\mathrm{d}\omega) f(\omega \cup (\zeta \setminus \Lambda_n)).$$

It is well known that μ_z exists for every z. (For a proof of existence, see [12].) However, μ_z may be nonunique. When μ_z is referred to below, it is assumed that μ_z is an arbitrary Gibbs measure, unless otherwise specified.

For $s > 0$, $P, Q \subset \mathbb{R}^2$, and $x \in \mathbb{R}^2$, define

$$\begin{aligned} B_s(x) &= \{y \in \mathbb{R}^2 : |x - y| \le s\}, \\ d(P, Q) &= \inf\{|p - q| : p \in P, q \in Q\}, \\ P - x &= \{p - x : p \in P\}, \end{aligned}$$

and call P *infinite* if, for every n, P is not a subset of Λ_n.

Let $L > 3r$. The main result of this paper, Theorem 3, states that, for sufficiently large z, $\bigcup_{x\in\omega} B_{L/2}(x)$ has an infinite connected component μ_z-almost surely for all Gibbs measures μ_z. As a preliminary step the following is shown in Theorem 2: let A_{inf} be the event that $\bigcup_{x\in\omega} B_{L/2}(x)$ has an infinite connected component, W, such that $d(0, W) \le L/2$. Then $\lim_{z\to\infty} \mu_z(A_{\mathrm{inf}}) = 1$ uniformly in all Gibbs measures μ_z.

Here an outline of the proof of Theorem 2 is sketched. Write $R = \delta + 3r/2$ with $\delta > 0$, with R chosen to be slightly smaller than $L/2$. Let $\Psi: \mathbb{R}^2 \to (\varepsilon\mathbb{Z})^2$ be a discretization of space, with ε much smaller than r and δ. Let $\omega \in \Omega$, and suppose that $\bigcup_{x\in\omega} B_R(\Psi(x))$ has a finite connected component W. The boundary of W is comprised of a number of closed curves; let γ be the one which encloses a region W_γ containing all the others, and assume that γ is comprised of exactly K arcs. Let A_γ be the set of all $\omega \in \Omega$ for which the curve γ arises as above. It can be shown that there is a vector $u_0 \in \mathbb{R}^2$ of magnitude approximately r and a map $\phi: A_\gamma \to \Omega$ defined by $\phi(\omega) = ((\omega \cap W_\gamma) - u_0) \cup (\omega \setminus W_\gamma)$ with the following properties: $L_{n,z}(\phi(A)) = L_{n,z}(A)$ for all measurable $A \subset A_\gamma$, and there exist $x_1, x_2, \dots, x_M \in \mathbb{R}^2$, with $M = \lceil cK \rceil$ and c a positive constant (depending only on δ and r, and not on γ), such that, for all $\omega \in A_\gamma$ and $i \ne j \in \{1, 2, \dots, M\}$,

$$d(x_i, \phi(\omega)) \ge \frac{\delta}{2} + 2r \quad \text{and} \quad |x_i - x_j| \ge \delta + 2r.$$

Then, with $A_\gamma^\phi = \{\phi(\omega) \cup \{y_1, y_2, \dots, y_M\} : \omega \in A_\gamma,\ y_i \in B_{\delta/2}(x_i)\}$,

$$G_{n,z,\zeta}(A_\gamma) \le \frac{G_{n,z,\zeta}(A_\gamma)}{G_{n,z,\zeta}(A_\gamma^\phi)} = \left(\frac{\pi\delta^2 z}{4}\right)^{-M}$$

provided n is large enough. It follows that $\mu_z(A_\gamma) \le (\pi\delta^2 z/4)^{-M}$.

Let A_{inf}^{Ψ} be the event that $\bigcup_{x\in\omega} B_R(\Psi(x))$ has an infinite connected component W such that $d(0, W) \le r/2$. Consider only those *finite* connected components W of $\bigcup_{x\in\omega} B_R(\Psi(x))$ such that $d(0, W) \le r/2$. A counting argument shows that the number of curves γ with K arcs corresponding to such W is bounded above by

$$\left(\frac{(K+1)H}{\varepsilon}\right)^2 \left(\frac{H}{\varepsilon}\right)^{2(K-1)},$$

where H depends only on δ and r. So the μ_z-probability that there is a finite connected component W of $\bigcup_{x\in\omega} B_R(\Psi(x))$ such that $d(0, W) \le r/2$ is less than

$$\sum_{K=1}^{\infty}\left(\frac{(K+1)H}{\varepsilon}\right)^2\left(\frac{H}{\varepsilon}\right)^{2(K-1)}\left(\frac{\pi\delta^2 z}{4}\right)^{-\lceil cK\rceil}.$$

This summation approaches 0 as $z \to \infty$. A simpler version of the above arguments shows that the μ_z-probability that $d(0, W) > r/2$ for *all* connected components W of $\bigcup_{x\in\omega} B_R(\Psi(x))$ also approaches 0 as $z \to \infty$. It follows that $\lim_{z\to\infty}\mu_z(A_{\text{inf}}^{\Psi}) = 1$. The continuous space corollary is the statement $\lim_{z\to\infty}\mu_z(A_{\text{inf}}) = 1$, which is deduced by an appropriate choice of R; since all of the above estimates apply to arbitrary Gibbs measures μ_z, the convergence is uniform in μ_z.

3. Discretization and contours

Throughout R, δ, and ε are fixed with $R = \delta + 3r/2$, $\delta \in (0, r/2)$, and $\varepsilon \in (0, \delta/2)$. Define $\Psi\colon \mathbb{R}^2 \to (\varepsilon\mathbb{Z})^2$ as follows. For $n, m \in \mathbb{Z}$, if

$$(x, y) \in \left[\varepsilon m - \frac{\varepsilon}{2}, \varepsilon m + \frac{\varepsilon}{2}\right) \times \left[\varepsilon n - \frac{\varepsilon}{2}, \varepsilon n + \frac{\varepsilon}{2}\right)$$

then set

$$\Psi(x, y) = (\varepsilon m, \varepsilon n).$$

Note that $|\Psi(x) - x| < \varepsilon$ for all $x \in \mathbb{R}^2$. Furthermore, Ψ is Borel measurable in the sense that $\Psi^{-1}(P)$ is a Borel set for any $P \subset (\varepsilon\mathbb{Z})^2$. (The dependence of Ψ on ε will be suppressed.)

Let $\omega \in \Omega$. The connected components of $\bigcup_{x\in\omega} B_R(\Psi(x))$ naturally partition ω into subsets $\omega' \subset \omega$; each ω' consists exactly of all the points $x \in \omega$ such that $\Psi(x)$ belongs to a given connected component of $\bigcup_{x\in\omega} B_R(\Psi(x))$. The subsets ω' will be called *components* of ω. A component ω' of ω is said to be *finite* if $\omega' \subset \Lambda_n$ for some n. For each finite component ω' of $\omega \in \Omega$, consider the set $W_{\omega,\omega'} = \bigcup_{x\in\omega'} B_{\delta+2r}(\Psi(x))$. Since $\delta + 2r \ge R$, $W_{\omega,\omega'}$ is connected. (It will also be assumed throughout that $r, \delta \in \mathbb{Q}$ and that ε is transcendental. This assumption implies that if two disks in $W_{\omega,\omega'}$ intersect then they overlap.) Consider now the boundary $\partial W_{\omega,\omega'}$ of $W_{\omega,\omega'}$. By the above, $\partial W_{\omega,\omega'}$ is a union of (images of) simple closed curves, one of which encloses a region containing all the others. Define $\gamma = \gamma_{\omega,\omega'} \subset \mathbb{R}^2$ to be the latter curve; γ will be called a *contour* of ω. A contour γ is (the image of) a simple closed curve comprised of circle arcs. The total number of circle arcs in γ is called the *size of the contour*; see Figure 1. *The region enclosed by* γ will be denoted by W_γ. It is emphasized that a contour $\gamma = \gamma_{\omega,\omega'}$ is defined only when ω' is a finite component of some $\omega \in \Omega$.

Lemma 1. *There exists $c > 0$ such that the following holds. Let γ be any contour of size $K > 0$, and let A_γ be the (nonempty) set of all $\omega \in \Omega$ such that $\gamma = \gamma_{\omega,\omega'}$ for some finite component ω' of ω. Then $A_\gamma \in \mathcal{F}$. Choose n such that $\gamma \subset \Lambda_n$. There is a map $\phi\colon A_\gamma \to \Omega$ and $x_1, x_2, \dots, x_M \in \mathbb{R}^2$, with $M = \lceil cK\rceil$, such that*

(i) *$L_{n,z}(A) = L_{n,z}(\phi(A))$ for all z and $\mathcal{F}$-measurable $A \subset A_\gamma$;*

(ii) *$|x_i - x_j| \ge \delta + 2r$ for all $i \ne j \in \{1, 2, \dots, M\}$;*

(iii) *$d(x_i, \phi(\omega)) \ge \delta/2 + 2r$ for all $i \in \{1, 2, \dots, M\}$ and all $\omega \in A_\gamma$.*

Proof. To see that $A_\gamma \in \mathcal{F}$, note that A_γ can be written as a finite intersection of sets of the form $\{\omega \in \Omega\colon \#(\omega \cap \Psi^{-1}(\{x\})) = \ell\}$, where $x \in (\varepsilon\mathbb{Z})^2$ and $\ell \in \{0, 1\}$.

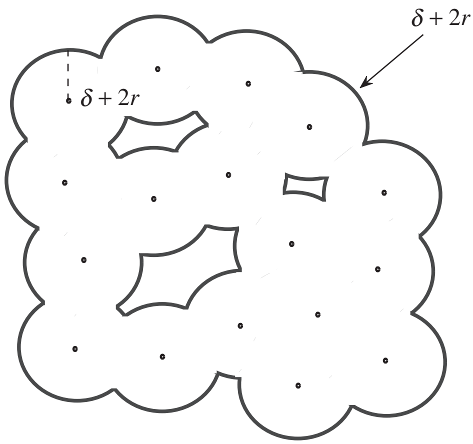

FIGURE 1: The outer curve is a contour $\gamma = \gamma_{\omega,\omega'}$ of size 13. All the points pictured belong to $\Psi(\omega')$.

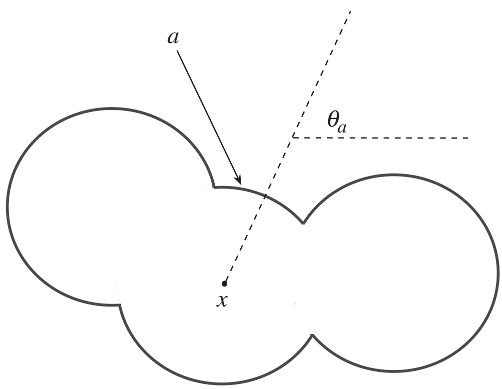

FIGURE 2: A contour $\gamma_{\omega,\omega'}$ with the arc a. Here θ_a is the outward normal angle with respect to the midpoint of a and $x \in \Psi(\omega')$.

For each circle arc a of γ, let $\theta_a \in [0, 2\pi)$ be the outward normal angle with respect to the midpoint of the arc (see Figure 2). Choose $0 < \alpha < \delta/(\delta + 2r)$ so that $\alpha = 2\pi/n$ for some $n \in \mathbb{N}$. By the pigeonhole principle, there is a subinterval $I = [v, v + \alpha) \subset [0, 2\pi)$ such that $\lceil (2\pi)^{-1}\alpha K \rceil$ of the angles θ_a belong to I. Fix $\theta_0 \in I$, and let

$$u_0 = \left(\left(\frac{\delta}{2} + r\right)\cos\theta_0, \left(\frac{\delta}{2} + r\right)\sin\theta_0\right)$$

be the vector in the direction of θ_0 with magnitude $\delta/2 + r$. Define $\phi\colon \mathcal{P}(\mathbb{R}^2) \to \mathcal{P}(\mathbb{R}^2)$ by

$$\phi(X) = ((X \cap W_\gamma) - u_0) \cup (X \setminus W_\gamma).$$

It will be shown below that $\phi(A_\gamma) \subset \Omega$.

Let $\omega \in A_\gamma$ be arbitrary, and let ω' be the unique component of ω such that $\gamma = \gamma_{\omega,\omega'}$. Assume that $x \in \omega \setminus W_\gamma$. Then $d(\Psi(x), \Psi(\omega')) > 2\delta + 3r$, and so

$$d\left(\Psi(x), \bigcup_{y \in \omega'} B_{\delta+2r}(\Psi(y))\right) > \delta + r.$$

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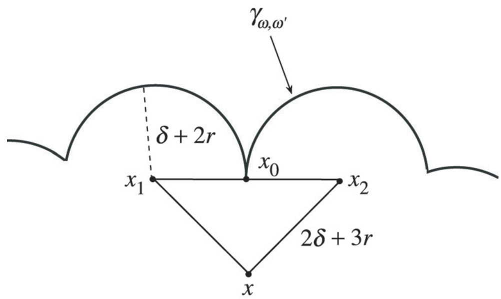

FIGURE 3: Pictured are $x_1, x_2 \in \Psi(\omega') \subset W_\gamma$ and $x \in \Psi(\omega \cap W_\gamma)$, but $x \notin \Psi(\omega')$. For such x, $d(x, \gamma) > \sqrt{5r^2 + 8r\delta + 3\delta^2}$. This can be seen in the above picture, in which the distance from x to γ is minimized by placing x_1 and x_2 as far apart as possible.

It follows that $d(\Psi(x), \gamma) > \delta + r$, and so $d(x, \gamma) > \delta/2 + r$. Now assume that $x \in \omega \cap W_\gamma$. If $x \in \omega'$ then $d(\Psi(x), \gamma) \geq \delta + 2r$, and so $d(x, \gamma) > \delta/2 + 2r$. If $x \notin \omega'$ then

$$\Psi(x) \notin \bigcup_{y \in \omega'} B_{2\delta+3r}(\Psi(y))$$

and a simple computation shows that $d(\Psi(x), \gamma) > \sqrt{5r^2 + 8r\delta + 3\delta^2} > \delta + 2r$, and so $d(x, \gamma) > \delta/2 + 2r$. (See Figure 3.)

Now let $A \subset A_\gamma$ with $A \in \mathcal{F}$, and define

$$A^{\text{in}} = \{\omega \cap W_\gamma : \omega \in A\}, \qquad A^{\text{out}} = \{\omega \setminus W_\gamma : \omega \in A\}.$$

Let $\omega^{\text{in}} \in A^{\text{in}}$ and $\omega^{\text{out}} \in A^{\text{out}}$. By the preceding paragraph,

$$d(\omega^{\text{out}}, \gamma) > \frac{\delta}{2} + r, \qquad d(\omega^{\text{in}}, \gamma) > \frac{\delta}{2} + 2r.$$

Let $x \in \omega^{\text{in}}$ and $y \in \omega^{\text{out}}$, and let z be any point on the intersection of γ with the line segment $\overline{xy}$. Then

$$|x - y| = |x - z| + |y - z| > \frac{\delta}{2} + 2r + \frac{\delta}{2} + r = \delta + 3r.$$

Since $|u_0| = \delta/2 + r$, it follows that

$$|\phi(x) - \phi(y)| = |(x - u_0) - y| > \frac{\delta}{2} + 2r.$$

By the preceding statements,

$$d(\omega^{\text{in}}, \omega^{\text{out}}) > \delta + 3r \geq 2r, \qquad d(\phi(\omega^{\text{in}}), \phi(\omega^{\text{out}})) > \frac{\delta}{2} + 2r \geq 2r.$$

In particular, this shows that $\phi(A) \subset \Omega$, and so $\phi(A_\gamma) \subset \Omega$. Also, note that $d(\omega^{\text{in}}, \gamma) > \delta/2 + 2r$ and $\gamma \subset \Lambda_n$ together imply that $\phi(\omega^{\text{in}}) = \omega^{\text{in}} - u_0 \subset \Lambda_n$. Combining the above statements gives

$$\begin{aligned} L_{n,N}(A) &= L_{n,N}(A^{\text{in}}) L_{n,N}(A^{\text{out}}) \\ &= L_{n,N}(A^{\text{in}} - u_0) L_{n,N}(A^{\text{out}}) \\ &= L_{n,N}(\phi(A^{\text{in}})) L_{n,N}(\phi(A^{\text{out}})) \\ &= L_{n,N}(\phi(A)). \end{aligned}$$

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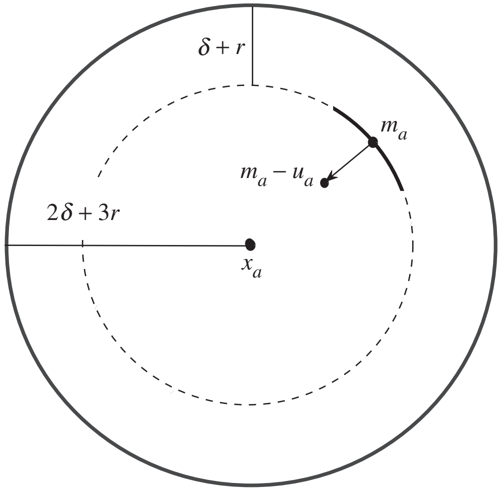

FIGURE 4: The midpoint m_a of the arc a with corresponding normal vector u_a. Here $x_a \in \Psi(\omega')$. No points in $\Psi(\omega \setminus W_\gamma)$ can be inside the large circle. The magnitude of u_a is $r + \delta/2$, and so the d-distance between $m_a - u_a$ and the large circle is $3\delta/2 + 2r$.

Since $\#(\omega \cap \Lambda_n) = \#(\phi(\omega) \cap \Lambda_n)$ for each $\omega \in A_\gamma$, it follows that $L_{n,z}(A) = L_{n,z}(\phi(A))$. This proves (i).

Consider now (ii) and (iii). Again, let $\omega \in A_\gamma$, and let ω' be the unique component of ω such that $\gamma = \gamma_{\omega,\omega'}$. Let a be an arc of γ such that $\theta_a \in I$. Let m_a be the midpoint of the arc, let x_a be the center of the circle (of radius $\delta + 2r$) which forms the arc, and let u_a be the vector in the direction of θ_a with magnitude $\delta/2 + r$.

Since $x_a \in \Psi(\omega')$, no points of $\Psi(\omega \setminus W_\gamma)$ are in $B_{2\delta+3r}(x_a)$. Since $|u_a| = \delta/2 + r$, it follows that, for any $x \in \omega \setminus W_\gamma$, $|\Psi(x) - (m_a - u_a)| > 3\delta/2 + 2r$. (See Figure 4.) So, for each $x \in \omega \setminus W_\gamma$,

$$|\Psi(x) - (m_a - u_0)| \geq |\Psi(x) - (m_a - u_a)| - |u_a - u_0| > \frac{3\delta}{2} + 2r - \left(\frac{\delta}{2} + r\right)\alpha > \delta + 2r,$$

where the last inequality follows by the choice of α. Therefore, if $x \in \omega \setminus W_\gamma$ then

$$|\phi(x) - (m_a - u_0)| = |x - (m_a - u_0)| > \frac{\delta}{2} + 2r.$$

On the other hand, if $x \in \omega \cap W_\gamma$ then $d(\Psi(x), \gamma) \geq \delta + 2r$, and so

$$|\phi(x) - (m_a - u_0)| = |x - m_a| > \frac{\delta}{2} + 2r.$$

Combining the above statements, if $x \in \omega$ then $|\phi(x) - (m_a - u_0)| > \delta/2 + 2r$.

Now note that, for any $x \in \Psi(\omega')$, a disk $B_{2r+\delta}(x)$ contributes to no more than six distinct circle arcs in γ. In turn, each circle arc corresponds to a unique $x \in \Psi(\omega')$ which is the center of the circle forming the arc. If two arc midpoints in γ are at a distance less than $\delta + 2r$ from one another, then the corresponding $x, y \in \Psi(\omega')$ are at a distance less than $3\delta + 6r$, so that the (unique) points in ω' which Ψ maps to x and y are at a distance less than $4\delta + 6r < 8r$ from each other. By a simple area comparison, the number of points $x \in \omega$ contained in a disk

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of radius $8r$ is bounded above by $(9r)^2/r^2 = 81$. The preceding shows that, given any arc midpoint m_a in γ, the number of arc midpoints $m_{\tilde{a}} \neq m_a$ in γ such that $|m_a - m_{\tilde{a}}| < \delta + 2r$ is bounded above by $J = 6 \cdot 81 = 486$. So, with $c = (2\pi(J+1))^{-1}\alpha$, there exists a subcollection

$$\{m_1, m_2, \ldots, m_M\} \subset \{m_a : \theta_a \in I\}, \qquad M = \lceil cK \rceil,$$

of arc midpoints such that $d(m_i, m_j) \geq \delta + 2r$ for all $i \neq j \in \{1, 2, \ldots, M\}$. By taking $x_i = m_i - u_0$ for $i \in \{1, 2, \ldots, M\}$, the proof is completed.

4. Estimates

Using Lemma 1, the μ_z-probability of seeing a given contour γ is shown to be exponentially small in the size, K, of the contour.

Lemma 2. *There exists $c > 0$ such that the following holds. Let γ be any contour of size K, and let A_γ be the set of all $\omega \in \Omega$ such that $\gamma = \gamma_{\omega,\omega'}$ for some finite component ω' of ω. Then, for every Gibbs measure μ_z,*

$$\mu_z(A_\gamma) \leq \left(\frac{\pi\delta^2 z}{4}\right)^{-\lceil cK \rceil}.$$

Proof. Choose $c > 0$, ϕ, and $x_1, x_2, \ldots, x_M$ satisfying the conclusion of Lemma 1. Choose $\hat{n}$ so that $\gamma \subset \Lambda_{\hat{n}}$, and let $\zeta \in \Omega$ be arbitrary. For each $A \subset A_\gamma$ such that $A \in \mathcal{F}$, define

$$A^\phi = \{\omega^\phi \subset \mathbb{R}^2 : \omega^\phi = \phi(\omega) \cup \{y_1, y_2, \ldots, y_M\},\ \omega \in A,\ y_i \in B_{\delta/2}(x_i)\}.$$

(See Figure 5.) By conditions (ii) and (iii) of Lemma 1, $A_\gamma^\phi \subset \Omega$, and, since $A_\gamma \in \mathcal{F}$, it is easy to see that $A_\gamma^\phi \in \mathcal{F}$.

By the definition of ϕ and choice of $\hat{n}$, if $\omega \in A_\gamma$ and $\omega^\phi = \phi(\omega) \cup \{y_1, y_2, \ldots, y_M\}$ with $y_i \in B_{\delta/2}(x_i)$, then $\omega \setminus \Lambda_{\hat{n}+l} = \omega^\phi \setminus \Lambda_{\hat{n}+l}$, where $l = \lceil \delta + r \rceil$. Now let $n = \hat{n} + l + \lceil 2r \rceil$. If $\omega \in A_\gamma$ and $\omega^\phi = \phi(\omega) \cup \{y_1, y_2, \ldots, y_M\}$ with $y_i \in B_{\delta/2}(x_i)$, then $\omega \in \Omega_{n,\zeta}$ if and only if $\omega^\phi \in \Omega_{n,\zeta}$. Let $A_{\gamma,n,\zeta} = A_\gamma \cap \Omega_{n,\zeta}$. The preceding shows that $A_{\gamma,n,\zeta}^\phi = A_\gamma^\phi \cap \Omega_{n,\zeta}$.

Now, since each disk $B_{\delta/2}(x_i)$ has (Lebesgue) area $\pi\delta^2/4$, Lemma 1 implies that

$$L_{n,z}(A_{\gamma,n,\zeta}^\phi) = \left(\frac{\pi\delta z}{4}\right)^M L_{n,z}(\phi(A_{\gamma,n,\zeta})) = \left(\frac{\pi\delta z}{4}\right)^M L_{n,z}(A_{\gamma,n,\zeta}).$$

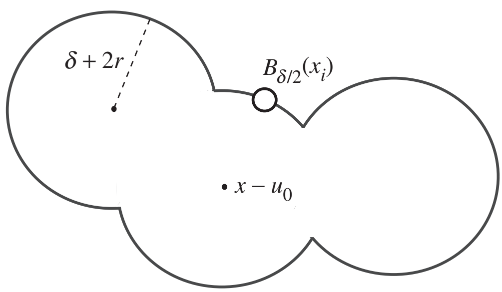

FIGURE 5: A disk $B_{\delta/2}(x_i)$ centered at a midpoint of an arc of $\gamma_{\omega,\omega'} - u_0$, with $x \in \Psi(\omega')$.

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From the definitions, it is easy to see that $G_{n,z,\zeta}(A_\gamma)$ and $G_{n,z,\zeta}(A_\gamma^\phi)$ are positive. Thus,

$$G_{n,z,\zeta}(A_\gamma) \le \frac{G_{n,z,\zeta}(A_\gamma)}{G_{n,z,\zeta}(A_\gamma^\phi)} = \frac{L_{n,z}(A_\gamma \cap \Omega_{n,\zeta})}{L_{n,z}(A_\gamma^\phi \cap \Omega_{n,\zeta})} = \frac{L_{n,z}(A_{\gamma,n,\zeta})}{L_{n,z}(A_{\gamma,n,\zeta}^\phi)} = \left(\frac{\pi\delta^2 z}{4}\right)^{-M}.$$

Also, by the choice of n, if $\omega \in \Omega_{n,\zeta}$ then $\mathbf{1}_{A_\gamma}(\omega) = \mathbf{1}_{A_\gamma}(\omega \cup (\zeta \setminus \Lambda_n))$, where $\mathbf{1}_{A_\gamma} : \Omega \to [0, \infty)$ is the (measurable) function $\mathbf{1}_{A_\gamma}(\omega) = 1$ if $\omega \in A_\gamma$, and $\mathbf{1}_{A_\gamma}(\omega) = 0$ otherwise. Since ζ was arbitrary,

$$\begin{aligned}
\mu_z(A_\gamma) &= \int_\Omega \mu(\mathrm{d}\zeta) \int_{\Omega_{n,\zeta}} G_{n,z,\zeta}(\mathrm{d}\omega)\, \mathbf{1}_{A_\gamma}(\omega \cup (\zeta \setminus \Lambda_n)) \\
&= \int_\Omega \mu(\mathrm{d}\zeta) \int_{\Omega_{n,\zeta}} G_{n,z,\zeta}(\mathrm{d}\omega)\, \mathbf{1}_{A_\gamma}(\omega) \\
&= \int_\Omega G_{n,z,\zeta}(A_\gamma)\mu(\mathrm{d}\zeta) \\
&\le \int_\Omega \left(\frac{\pi\delta^2 z}{4}\right)^{-M} \mu(\mathrm{d}\zeta) \\
&= \left(\frac{\pi\delta^2 z}{4}\right)^{-M}.
\end{aligned}$$

As μ_z was an arbitrary Gibbs measure, the proof is complete.

Next an upper bound for the number of contours enclosing the origin is obtained.

Lemma 3. *Let Γ_K be the set of all contours γ of size K such that $0 \in W_\gamma$. Then*

$$\#\Gamma_K \le \left(\frac{(K+1)H}{\varepsilon}\right)^2 \left(\frac{H}{\varepsilon}\right)^{2(K-1)},$$

where H is a constant depending only on r.

Proof. Note that each contour γ is completely determined by its set of arcs, with each arc naturally corresponding to a unique point in $(\varepsilon\mathbb{Z})^2$, namely, the center of the circle of which the arc is part. Let $\gamma \in \Gamma_K$. Since γ is the (image of a) simple closed curve comprised of circle arcs, there is a sequence of circle arcs $a_1, a_2, \dots, a_K$ such that a_i and a_{i+1} are adjacent for $i = 1, 2, \dots, K-1$. Choose the corresponding sequence $x_1, x_2, \dots, x_K$ of points in $(\varepsilon\mathbb{Z})^2$. Then $|x_{i+1} - x_i| < 2\delta + 4r < 5r$ for $i = 1, 2, \dots, K-1$.

By a simple area comparison, the number of points in $(\varepsilon\mathbb{Z})^2$ inside any disk $B_s(x)$ is bounded above by

$$\frac{\pi(s+\varepsilon)^2}{\varepsilon^2} < \frac{2\pi s^2}{\varepsilon^2}$$

if $s > 3\varepsilon$. As γ encloses the origin, x_1 must be contained in a disk of radius $(K+1)5r$ around 0. Therefore, there are at most $2\pi[(K+1)5r]^2/\varepsilon^2$ possibilities for x_1. For $i = 1, 2, \dots, K-1$, x_{i+1} must be contained in a disk of radius $5r$ around x_i, so, given x_i, there are no more than $2\pi(5r)^2/\varepsilon^2$ possibilities for x_{i+1}. Taking $H = 5\sqrt{2\pi}r$, the result follows.

5. Main results

Let $\omega \in \Omega$. If the origin is not close to an infinite component of ω, then it is either close to a finite component of ω, or it is not close to any component of ω. The probability of the former event can be handled by combining Lemma 2 with Lemma 3, while it is easy to control the probability of the latter event. This leads to the following result.

Theorem 1. *Let A_{inf}^{Ψ} be the set of all $\omega \in \Omega$ such that $d(0, \Psi(\omega')) \le \delta + 2r$ for some infinite component ω' of ω. Then $A_{\mathrm{inf}}^{\Psi} \in \mathcal{F}$ and $\lim_{z\to\infty} \mu_z(A_{\mathrm{inf}}^{\Psi}) = 1$ uniformly in all Gibbs measures μ_z.*

Proof. We define

$$\begin{aligned} A_{\mathrm{orig}} &= \{\omega \in \Omega\colon d(0, \Psi(\omega')) > \delta + 2r \text{ for all components } \omega' \text{ of } \omega\}, \\ A_{\mathrm{fin}} &= \{\omega \in \Omega\colon d(0, \Psi(\omega')) \le \delta + 2r \text{ for some finite component } \omega' \text{ of } \omega\}, \\ A_{\mathrm{cont}} &= \{\omega \in \Omega\colon 0 \in W_\gamma \text{ for some contour } \gamma = \gamma_{\omega,\omega'}\}. \end{aligned}$$

Note that A_{orig}, A_{fin}, and A_{cont} can each be written as a countable union of finite intersections of sets of the form $\{\omega \in \Omega\colon \#(\omega \cap \Psi^{-1}(\{x\})) = \ell\}$, where $x \in (\varepsilon\mathbb{Z})^2$ and $\ell \in \{0, 1\}$. Thus, $A_{\mathrm{orig}}, A_{\mathrm{fin}}, A_{\mathrm{cont}} \in \mathcal{F}$.

Let A_n be the set of all $\omega \in \Omega$ with the following property: there exist a positive integer k and $x_1, x_2, \dots, x_k \in \Psi(\omega)$ such that $|x_1| \le \delta + 2r$, $|x_i - x_{i+1}| \le 2R$ for $i = 1, 2, \dots, k-1$, and $x_k \notin \Lambda_n$. Note that A_n can be written as a finite union of finite intersections of sets of the form $\{\omega \in \Omega\colon \#(\omega \cap \Psi^{-1}(\{x\})) = 1\}$, where $x \in (\varepsilon\mathbb{Z})^2$. Hence, $A_n \in \mathcal{F}$. Since $A_{\mathrm{inf}}^{\Psi} = \bigcap_{n=1}^{\infty} A_n$ it follows that $A_{\mathrm{inf}}^{\Psi} \in \mathcal{F}$.

Note that $\Omega \setminus A_{\mathrm{inf}}^{\Psi} \subset A_{\mathrm{orig}} \cup A_{\mathrm{fin}}$ and $A_{\mathrm{fin}} \subset A_{\mathrm{cont}}$, so

$$\mu_z(\Omega \setminus A_{\mathrm{inf}}^{\Psi}) \le \mu_z(A_{\mathrm{orig}}) + \mu_z(A_{\mathrm{fin}}) \le \mu_z(A_{\mathrm{orig}}) + \mu_z(A_{\mathrm{cont}}).$$

Choose $c > 0$ such that the conclusion of Lemma 2 holds, and choose H such that the conclusion of Lemma 3 holds. Then, for any Gibbs measure μ_z,

$$\mu_z(A_{\mathrm{cont}}) \le \sum_{K=1}^{\infty} \#\Gamma_K \left(\frac{\pi\delta^2 z}{4}\right)^{-\lceil cK\rceil} \le \sum_{K=1}^{\infty} \left(\frac{(K+1)H}{\varepsilon}\right)^2 \left(\frac{H}{\varepsilon}\right)^{2(K-1)} \left(\frac{\pi\delta^2 z}{4}\right)^{-\lceil cK\rceil}.$$

This shows that $\mu_z(A_{\mathrm{cont}}) \to 0$ as $z \to \infty$ uniformly in μ_z.

Now, for any $\omega \in A_{\mathrm{orig}}$, $d(0, \Psi(\omega)) > \delta + 2r$, and so $d(0, \omega) > \delta/2 + 2r$. It follows that, for any $\omega \in A_{\mathrm{orig}}$ and any $x \in B_{\delta/2}(0)$, $\omega \cup x \in \Omega$. A simplified version of the proof of Lemma 2 then implies that $\mu_z(A_{\mathrm{orig}}) \le (\pi\delta^2 z/4)^{-1}$ for any Gibbs measure μ_z. Thus, $\mu_z(A_{\mathrm{orig}}) \to 0$ as $z \to \infty$ uniformly in μ_z, and the result follows.

Below Theorem 1 is extended to continuous space.

Theorem 2. *Let $L > 3r$. Let A_{inf} be the set of all $\omega \in \Omega$ such that $\bigcup_{x\in\omega} B_{L/2}(x)$ has an infinite connected component, W, with $d(0, W) \le L/2$. Then $A_{\mathrm{inf}} \in \mathcal{F}$ and $\lim_{z\to\infty} \mu_z(A_{\mathrm{inf}}) = 1$ uniformly in all Gibbs measures μ_z.*

Proof. It is standard to show that $A_{\mathrm{inf}} \in \mathcal{F}$, so this part of the proof is omitted. To see that $\lim_{z\to\infty} \mu_z(A_{\mathrm{inf}}) = 1$, choose $\delta \in (0, r/2)$ and $\varepsilon \in (0, \delta/2)$ such that $3r + 2\delta + 2\varepsilon < L$, and define A_{inf}^{Ψ} as in Theorem 1. Then $A_{\mathrm{inf}}^{\Psi} \subset A_{\mathrm{inf}}$, and so $\mu_z(A_{\mathrm{inf}}) \ge \mu_z(A_{\mathrm{inf}}^{\Psi})$. The result now follows from Theorem 1.

The main result can now be proved.

Theorem 3. *Let $L > 3r$. Let A be the set of all $\omega \in \Omega$ such that $\bigcup_{x\in\omega} B_{L/2}(x)$ has an infinite connected component. Then $A \in \mathcal{F}$, and for sufficiently large z, $\mu_z(A) = 1$ for all Gibbs measures μ_z.*

Proof. The proof of measurability is again omitted. It is clear that A is in the tail sub-σ-algebra of $\mathcal{F}$, so $\mu_z(A) = 0$ or 1 for all extremal Gibbs measures μ_z (see Theorem 7.7 of [4, Chapter 7]). Let A_{inf} be defined as in Theorem 2. Since $A_{\text{inf}} \subset A$, Theorem 2 implies that $\lim_{z\to\infty} \mu_z(A) = 1$ uniformly in all Gibbs measures μ_z. So, for sufficiently large z, $\mu_z(A) = 1$ for all extremal Gibbs measures μ_z. The result now follows from extremal decomposition of Gibbs measures (see Theorem 7.26 of [4, Chapter 7]).

6. Conclusion

Percolation of excluded volume has been proved for points in the plane distributed according to Gibbs measures with a pure hard-core interaction. This model, commonly called the *hard disk model*, is among the simplest continuum models of particles with pair interactions. The proof, which generalizes to 3 dimensions, relies on a Peierls-type argument; see [6]. (The generalization requires a slightly more complicated argument for choosing u_0 and estimating the number of contours of a given size.) A similar result is expected in a hard disk model with an added attraction which extends beyond the hard core, though this generalization is not pursued here. The hard disk model with attraction is believed to exhibit a gas/liquid phase transition, which has been heuristically connected to percolation of excluded volume; see [7], [13]. (There is no proof in the literature of a gas/liquid transition in a continuum model with pair interactions; see, however, [9].) To this author's knowledge, there is no previous proof of percolation of excluded volume for hard disks (or spheres) in the literature. (See [2] for a proof in a model with a complicated exclusion.) In general, very little is known (or proved) about the qualitative properties of the hard disk model at large activity. The result of this paper is of particular interest because of the known absence of long-range translational order in the model. It remains an open question whether percolation occurs for an arbitrarily small connection radius, that is, for a connection radius extending just beyond the exclusion radius; see [2].

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