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SYNTHETIC LIE THEORY

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Traditionally, an infinitesimal neighbourhood of the identity element of a Lie group is studied indirectly by using an appropriately chosen algebraic structure, such as a Lie algebra, to represent it. In this thesis, we use the theory of synthetic differential geometry [5] to work directly with this infinitesimal neighbourhood and reformulate Lie theory in terms of infinitesimals. We show how to carry out this reformulation for the established generalisation of Lie theory involving Lie groupoids and Lie algebroids [6] and make a further generalisation by replacing groupoids with categories. Our main result is a proof of Lie's second theorem in this context. Intuitively speaking, Lie's second theorem tells us that all relationships that can be expressed in terms of global data can also be expressed in terms of local data. (For a precise statement of the classical result please see [4, Section 3.8].) Finally, we show how our new constructions and definitions relate to the classical ones.

Synthetic differential geometry is a subject that makes precise arguments involving infinitesimals which can be found, for instance, in the work of Newton and Leibniz and which are often useful when thinking heuristically about modern differential geometry. The methods involved can be seen as an extension of the ideas of Grothendieck, which provide an algebraic treatment of nilpotent infinitesimals. Every well-adapted model of synthetic differential geometry (see for instance [3]) contains the category of smooth paracompact Hausdorff manifolds as a full subcategory, which means that the theory of smooth manifolds is preserved and subsumed in a well-adapted model.

Using these infinitesimals, we construct a correspondence between two different types of groupoid which generalises the classical Lie correspondence. The first type of groupoid (called jet groupoids) are those for which every arrow is infinitesimally close to an identity arrow. The second type of groupoid (called integral complete groupoids)

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are those which have solutions to a specific kind of differential equation. In [1], we show that the classical Lie groupoids are instances of integral complete groupoids and that the classical connectedness conditions interact with this generalisation in the expected manner.

This new correspondence is useful in several ways. Firstly, it provides a new infinitesimal approximation of a Lie groupoid that is analogous to the formal group law construction for Lie groups. Secondly, it applies to groupoids whose underlying space is nonclassical in nature. This second extension is significant because it avoids the problems existing in the literature concerning the nonintegrability of certain Lie algebroids. Until its recent solution in [2], the integrability of Lie algebroids was a major and long-standing unsolved problem in classical (multi-object) Lie theory. Any Lie algebroid integrates to a topological groupoid (its Weinstein groupoid), but there can be obstructions to putting a smooth structure on it. This suggests that, instead of working in the category of smooth manifolds, it would be more convenient to work in a category in which one can simultaneously make sense of tangent vectors and Weinstein groupoids. In their paper [7], Tseng and Zhu show that the category of differentiable stacks is suitable for this purpose and, in this thesis, we show that any well-adapted model of synthetic differential geometry is also.

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