SOME DIRECT AND INVERSE THEOREMS IN APPROXIMATION OF FUNCTIONS

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Abstract

The paper is concerned with the determination of the degree of convergence of a sequence of linear operators connected with the Fourier series of a function of class \( L_p \) \((p > 1)\) to that function and some inverse results in relating the convergence to the classes of functions. In certain cases one can obtain the saturation results too. In all cases \( L_p \) norm is used.


Let \( f(x) \) be a periodic, Lebesgue integrable function with period \( 2\pi \). Let the Fourier series for \( f(x) \) be given by

\[
\frac{1}{2}a_0 + \sum_{k=1}^{\infty} \left( a_k \cos kx + b_k \sin kx \right) \equiv \sum_{k=0}^{\infty} A_k(x).
\]

Let \( S_n(f; x) \) be the \( n \)th partial sum of the series (1). The conjugate series of the series (1) is

\[
\sum_{n=1}^{\infty} B_n(x) = \sum_{k=1}^{\infty} \left( b_k \cos kx - a_k \sin kx \right).
\]

The conjugate function \( \tilde{f} \) of \( f \) is given by

\[
\tilde{f}(x) = (2\pi)^{-1} \int_0^{2\pi} \{ f(x + t) - f(x - t) \} \cot \frac{t}{2} \, dt
\]

the integral being interpreted as a Cauchy integral. It is known that \( \tilde{f} \) exists almost everywhere whenever \( f \) is integrable.
The space $L_p[-\pi, \pi]$ when $p = \infty$ will be replaced by the space $c_{2\pi}$ of all continuous functions defined over $[-\pi, \pi]$. Throughout the paper, norms will be taken with respect to the variable $x$ and $\| \cdot \|_p$ will denote the usual $L_p$ norm for $1 < p < \infty$, and the supremum norm when $p = \infty$. For $f \in L_p[-\pi, \pi]$ ($1 < p < \infty$), the modulus of continuity and the modulus of smoothness $w^{(p)}(\delta, f)$ and $w_2^{(p)}(\delta; f)$ are defined respectively by

\[
w^{(p)}(\delta; f) = \sup_{|h| \leq \delta} \|f(x + h) - f(x)\|_p, \quad \text{and} \quad w_2^{(p)}(\delta; f) = \sup_{|h| \leq \delta} \|f(x + h) + f(x - h) - 2f(x)\|_p.
\]

The classes Lip $\alpha$, Lip($\alpha$, $p$) ($p \geq 1$) will be as usual (see [5], page 612; also see [18], pages 42, 45). The class Lip($\alpha$, $p$) with $p = \infty$ will be taken as Lip $\alpha$.

Two functions $f$ and $g$ are said to be equivalent if $f(x) = g(x)$ almost everywhere.

Let $\{c_n\}, \{d_n\}$ be two non-zero sequences with $c_n, d_n > 0$. Suppose $C_n = \sum_{k=0}^{n} c_k$ and $D_n = \sum_{k=0}^{n} d_k$. Let $R_n = c_0 d_n + c_1 d_{n-1} + \cdots + c_n d_0 (n = 0, 1, \ldots)$.

Given $f$, let us associate with it the operator $t_n(f; x)$ defined by

\[
t_n(f; x) = (R_n)^{-1} \sum_{k=1}^{n} c_{n-k} d_k S_k(x).
\]

It should be remarked that $t_n(f; x)$ is the $(N, c, d)$ transform of $\{S_k(x)\}$ (see [2]).

We shall write $t_n(f; x) = N_n(f; x)$ or $\tilde{N}_n(f; x)$ according as $d_n = 1$ for all $n$ or $c_n = 1$ for all $n$.

If there exists a positive non-increasing function $\phi(n)$ and a normed linear space $K$ of functions such that

\[
\|f(x) - t_n(f; x)\| = o(\phi(n)) \Rightarrow f \text{ is a constant a.e.,}
\]
\[
\|f(x) - t_n(f; x)\| = O(\phi(n)) \Rightarrow f \in K, \quad \text{and}
\]
\[
f \in K \Rightarrow \|f(x) - t_n(f; x)\| = O(\phi(n)),
\]

then we say that the operator $t_n(f)$ or the corresponding method $(N, c, d)$ is saturated with order $\phi(n)$ and class $K$.
summability methods. Sunouchi and Watari [15], [16] have obtained the saturation order and class for Cesàro, Abel and the Riesz method \( (R, n^\xi, 1) \) \( (\xi = 1, 2, \ldots) \). Mohapatra and Sahney [11] have obtained results on saturation for a general class of summability methods in the supremum norm. Sunouchi [14] has studied the local saturation properties of the convolution operator (also see [13], [17]).

Concerning the saturation property of the Nörlund method, Goel, Holland, Nasim and Sahney [4] have proved the following theorem:

**Theorem A** ([4], compare [9]). Let \( f \in c_{2^m} \) and \( C_n > 0 \) (all \( n \)). Then the following hold:

\[
\| f - N_n(f) \|_\infty = o \left( \frac{c_n}{C_n} \right) \Rightarrow f \text{ is a constant a.e.} \tag{9}
\]

\[
\| f - N_n(f) \|_\infty = O \left( \frac{c_n}{C_n} \right) \Rightarrow f \in \{ f | \tilde{f} \in \text{Lip 1} \} \tag{10}
\]

whenever

\[
\lim_{n \to \infty} \frac{c_n - k}{c_n} = 1 \quad (k = 0, 1, \ldots; c_n > 0 \text{ for all } n). \tag{11}
\]

\[
f \in \{ f | \tilde{f} \in \text{Lip 1} \} \Rightarrow \| f - N_n(f) \|_\infty = O \left( \frac{c_n}{C_n} \right) \tag{12}
\]

whenever

\[
\sum_{k=0}^{n} |c_k - c_{k-1}| = O(c_n) \quad (c_{-1} = 0). \tag{13}
\]

In Section 3 we obtain the order and class of saturation of the method \( (N, c, d) \) or the operator \( t_n(f) \) in the \( L_p \) \( (1 < p \leq \infty) \) norm. Special cases of this result extend Theorem A and yield a saturation result for a type of Riesz method.

The other object of this paper is to obtain the degree of convergence of \( t_n(f) \) to \( f \in L_p \) in terms of the integral modulus of continuity and integral modulus of smoothness with a view to generalizing the following results:

**Theorem B** ([12]). If \( f \in \text{Lip}(\alpha, p) \) \( (0 < \alpha \leq 1, p > 1, p^{-1} + p'^{-1} = 1) \) and if \( C_n \to \infty \) and

\[
\left( \int_1^n \frac{C(y)}{y^{p'a + 2-p'^{-1}}} \, dy \right)^{1/p'} = O \left( \frac{C_n}{n^{a-p^{-1}}} \right) \tag{14}
\]

(where \( C(y) = C_{\lfloor y \rfloor} \)) then

\[
\| f - N_n(f) \|_p = O(n^{-a+p^{-1}}). 
\]
THEOREM C ([10]). Let $C_n \to \infty$ as $n \to \infty$, and $R(y)/y^\alpha$ be nondecreasing where $R(y) = R_{\{y\}}$. Then $f \in \text{Lip}(\alpha, p)$ ($0 < \alpha < 1, p > 1$) implies

$$
\|f - t_n(f)\|_p = O(n^{-\alpha+p-1}).
$$

THEOREM D ([8]). If $w(t)$ is the modulus of continuity of $f \in C[-\pi, \pi]$ and $c_n > 0$, $c_n/C_n = O(n^{-1})$,

$$
\|f - N_n(f)\|_\infty = O\left(\frac{1}{C_n} \sum_{k=1}^n \frac{w(1/k)}{k} C_k\right).
$$

In Section 4 we shall generalize these results and obtain some other special cases.

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Following the method of Sunouchi and Watari [16] we can obtain

THEOREM 1. Let $1 \leq p \leq \infty$. The following hold:

$$
\|f - t_n(f)\|_p = o\left(\frac{c_n}{R_n}\right) \Rightarrow f \text{ is equivalent to a constant.}
$$

When $c_n - k/c_n \to 1$ as $n \to \infty, k$ fixed, we have

$$
\|f - t_n(f)\|_p = O\left(\frac{c_n}{R_n}\right) \Rightarrow \left\|\sum_{k=1}^n D_{k-1}\left(1 - \frac{k}{N+1}\right) A_k(x)\right\|_p = O(1).
$$

Thus $\|f - t_n(f)\|_p = O(c_n/R_n)$ implies

$$
\sum_{k=1}^\infty D_{k-1} A_k(x) \text{ is the Fourier series of a bounded function, when } p = \infty;
$$

$$
\sum_{k=1}^\infty D_{k-1} A_k(x) \text{ is the Fourier series of a function of class } L_p,
$$

when $1 < p < \infty$;

$$
\sum_{k=1}^\infty D_{k-1} A_k(x) \text{ is the Fourier-Stieltjes series of a function of bounded variation, when } p = 1.
$$

Throughout the paper, we write for $1 \leq p < \infty$, $K_p = \{f \in L_p | \tilde{f} \in \text{Lip}(1, p)\}$, and $K_{\infty} = \{f \in c_2^{\infty} | f \in \text{Lip}(1)\}$. 

If \( d_n = 1 \) for all \( n \), then we have, from Theorem 1:

**COROLLARY 1.** Let \( C_n > 0 \) (all \( n \)). Then

\[
\| f - N_n(f) \|_p = o\left(\frac{c_n}{C_n}\right) \Rightarrow f \text{ is equivalent to a constant,}
\]

and if (11) holds then

\[
\| f - N_n(f) \|_p = O\left(\frac{c_n}{C_n}\right) \Rightarrow f \in K_p \quad (1 \leq p \leq \infty).
\]

**PROOF.** It is enough to deduce (24). When (11) holds we observe that the conclusion in (19) shows that the \((C, 1)\) mean of \( \sum_k k A_k(x) \) is uniformly bounded in the \( L_p \) norm \((1 \leq p \leq \infty)\). Since \( -\sum_k k A_k(x) = \sum_k B_k(x) \) where \( \sum B_k(x) \) is the conjugate series of the Fourier series of \( f(x) \), we have \( \| \sigma_N \|_p = O(1) \) where \( \sigma_N(x) \) is the first Cesàro mean of \( \sum B_k \). This is known to be equivalent to \( f \in K_p \).

**REMARKS.** 1. If \( p > 1 \), then the conclusion \( f \in K_p \) in Corollary 1 can be replaced by \( f \in \text{Lip}(1, p) \) (see [6], Lemma 13, page 621).

2. (20), (21) and (22) refer to the Fourier series \( \sum_{k=1}^\infty D_{k-1} A_k(x) \). Since we do not know much about the behaviour of that series the saturation problem for \((\bar{N}, d)\) turns out to be difficult. However when \( p = 2 \) we get the following as an easy consequence of Parseval’s identity:

**COROLLARY 2.** Let \( f \in L_2 \). Corresponding to the order of saturation \( 1/D_n \) the saturation class of the method \((\bar{N}, d)\) or of the operator \( \bar{N}_n(f) \) is the class of all functions \( f \in L_2 \) with Fourier series \( \sum_{k=1}^\infty D_{k-1} A_k(x) \).

Our next result gives an estimate for the error in approximating a function \( f \in K_p \) by \( t_n(f) \). Precisely, we prove

**THEOREM 2.** Let \( 1 < p \leq \infty \) and \( \{c_n\} \) and \( \{d_n\} \) satisfy

\[
\sum_{k=0}^n |c_{n-k} d_k - c_{n-k-1} d_{k+1}| = O(c_n).
\]

Then, for \( f \in K_p \),

\[
\| f - t_n(f) \|_p = O\left(\frac{c_n}{R_n}\right).
\]

We shall need the following lemmas for the proof of our theorem:

**LEMMA 1 ([5], Theorem 24(i), page 599).** If \( f \) belongs to \( \text{Lip}(1, p) \) \((1 < p \leq \infty)\) then \( f \) is equivalent to the indefinite integral of a function belonging to \( L_p \). If \( f \in \text{Lip} 1 \) then \( f \) is the indefinite integral of a bounded function.
Lemma 2 ([6], Theorem 5, page 627). Suppose \( f \in \text{Lip}(\alpha, p) \) where \( p \geq 1, 0 < \alpha \leq 1 \).

(i) If \( \alpha p \leq 1 \) and \( p < q < p/(1 - \alpha p) \), then \( f \in \text{Lip}(\alpha - 1/p + 1/q, q) \).
(ii) If \( \alpha p > 1 \) then \( f \in \text{Lip}(\alpha - 1/p + 1/q, q) \) for all \( q > p \), and \( f \) is equivalent to a function of \( \text{Lip}(\alpha - 1/p) \).

Lemma 3. Let

\[
K_n(t) = (R_n)^{-1} \sum_{k=1}^{n} c_{n-k} d_k \frac{\cos(k+1/2)t}{\sin t/2},
\]

and

\[
L_n(t) = \int_{t}^{\pi} K_n(u) \, du.
\]

Then

\[
f \in K_p \ (1 < p \leq \infty) \implies \|f - t_n(f)\|_p = O(c_n/R_n) \quad \text{if} \quad \int_{0}^{\pi} |L_n(t)| \, dt = O(c_n/R_n).
\]

Proof. Let \( \hat{S}_n(f; x) \) denote the partial sums of the conjugate series associated with \( \hat{f}(x) \). We have, from the definition,

\[
t_n(\hat{S}_n(f; x)) = (2\pi R_n)^{-1} \sum_{k=0}^{n} c_{n-k} d_k \int_{0}^{\pi} \left[ \hat{f}(x + t) - \hat{f}(x - t) \right] \cot \frac{t}{2} \, dt
- (2\pi R_n)^{-1} \sum_{k=0}^{n} c_{n-k} d_k \int_{0}^{\pi} \left[ \hat{f}(x + t) - \hat{f}(x - t) \right] \cos \left( k + \frac{1}{2} \right) t \csc \frac{t}{2} \, dt.
\]

By M. Riesz's theorem (Zygmund [18], Theorem (2.4), page 253) \( f \in L_p \ (1 < p < \infty) \Rightarrow \hat{f} \in L_p = \hat{f} \in L_p \) and \( \hat{S}(\hat{f}) = S(\hat{f}) \). If \( p = \infty \), \( \hat{f} \in \text{Lip} 1 \) (by hypothesis) and then \( -\hat{f} + \frac{1}{2}a_0 \) is identical to \( \hat{f} \). Thus from (30) and (27),

\[
f(x) - t_n(f; x) = (2\pi)^{-1} \int_{0}^{\pi} \left[ \hat{f}(x + t) - \hat{f}(x - t) \right] K_n(t) \, dt
\]

almost everywhere.

Since \( f \in K_p \), by Lemma 1, we can take \( \hat{f}(u) \) equivalent to the indefinite integral of a function, say \( \hat{f}'(u) \in L_p \ (p > 1) \). By integration by parts, we have from (31)

\[
f(x) - t_n(f; x) = (2\pi)^{-1} \int_{0}^{\pi} \left[ \hat{f}'(x + t) + \hat{f}'(x - t) \right] L_n(t) \, dt.
\]
By using the generalized Minkowski’s inequality ([7], page 148, 6.13.9)

\[ \|f - t_n(f)\|_p \leq (2\pi)^{-1} \int_0^\pi \|\tilde{f}'(x + t) + \tilde{f}'(x - t)\|_p |L_n(t)| \, dt \]

\[ = O\left( \int_0^\pi |L_n(t)| \, dt \right) = O(c_n/R_n). \]

**Lemma 4 ([4]).**

(32) \[ \left| \int_0^\pi \frac{\sin(k + 1)u}{u^2} \, du \right| \leq \begin{cases} 
2(k + 1)\log \frac{1}{(k + 1)t} & \text{for } 0 < (k + 1)t < 1/e; \\
2/(k + 1)t^2 & \text{for any } k \geq 0, t > 0.
\end{cases} \]

The lemma can be proved easily.

**Proof of Theorem 2.** In view of Lemma 3, it is enough to prove (29). By Abel’s transformation

(33) \[ -K_n(t) = \left(2R_n \sin^2 \frac{t}{2}\right)^{-1} \sum_{k=0}^n (c_{n-k}d_k - c_{n-k-1}d_{k+1}) \sin(k + 1)t. \]

Since

\[ \left(2\sin^2 \frac{t}{2}\right)^{-1} = \frac{2}{t^2} + O(1), \]

we get, from (33) and (25), that

\[ -K_n(t) = \left(2/R_n t^2\right) \sum_{k=0}^n (c_{n-k}d_k - c_{n-k-1}d_{k+1}) \sin(k + 1)t + O(c_n/R_n). \]

From (33), we observe that (29) holds if

(34) \[ \sum_{k=0}^n |(c_{n-k}d_k - c_{n-k-1}d_{k+1})| \int_0^\pi \left| \int_0^\pi \frac{\sin(k + 1)u}{u^2} \, du \right| dt = O(1). \]

In view of (25), (34) is true whenever

(35) \[ \int_0^\pi \left| \int_0^\pi \frac{\sin(k + 1)u}{u^2} \, du \right| dt = O(1) \]

uniformly in \( k \).
By Lemma 4, the integral on the left of (35) does not exceed
\[ \int_0^{1/e(k+1)} \left| \int_t^\pi \frac{\sin(k+1)u}{u^2} \, du \right| \, dt + \int_0^{1/e(k+1)} \left| \int_t^\pi \frac{\sin(k+1)u}{u^2} \, du \right| \, dt \leq \int_0^{1/e(k+1)} 2 \log(1/(k+1)t) \, dt + \int_0^{1/e(k+1)} 2(k+1)^{-1}t^{-2} \, dt. \]
Since each integral is bounded the result follows.

**COROLLARY 3.** Let \( \{d_n\} \in \text{bv}, d_n \geq 0, D_n > 0. \) If \( f \in K_p \) \((1 < p \leq \infty)\), then
\[ \|f - \overline{N}_n(f)\|_p = O(D_n^{-1}). \]

**COROLLARY 4.** Let \( \{c_n\} \) satisfy \( C_n \geq 0, C_n > 0, \) and
\[ \sum_{k=0}^n |c_k - c_{k-1}| = O(c_n) \quad (c_{-1} = 0). \]
Then \( f \in K_p \) implies \( \|f - N_n(f)\|_p = O(c_n/C_n) \) \((1 < p \leq \infty)\).

The case \( p = \infty \) is given in [4, Lemma 2.3].
Combining Corollary 1 and Corollary 4, we get the following:

**THEOREM 3.** Let \( \{c_n\} \) satisfy (11) and (36). Then the Nörlund method \( (N, c_n) \) is saturated with order \( c_n/C_n \) and class \( K_p \).

**REMARK.** Lemma 3 shows that (29) is a sufficient condition for \( \|f - t_n(f)\|_p = O(c_n/R_n) \) whenever \( f \in K_p \) \((1 < p \leq \infty)\). We do not know if (29) is also necessary.

Let us write \( R(y) = R_{\lfloor y \rfloor} \). With a view to generalizing Theorem B and Theorem C, and extending Theorem D, we prove the following:

**THEOREM 4.** Let \( \{c_n\}, \{d_n\} \) be non-negative, non-increasing sequences and \( R_n > 0 \). Let \( f \in L_p[-\pi, \pi] \) \((1 < p < \infty)\) or \( f \in c_{2\pi} \) \((p = \infty)\). Then
\[ \|f - t_n(f)\|_p = O \left( \frac{1}{R_n} \sum_{k=1}^n \frac{w_2^{(p)}(1/k)}{k} R_k \right) + O \left( w_2^{(p)}(\frac{1}{n}) \right). \]
Remark. If in addition to the hypotheses assumed on the sequences \( \{c_n\} \) and \( \{d_n\} \), we assume that there exists \( l > 0 \) such that

\[
(R_n)^{-1} \sum_{k=1}^{n} (R_k/k) \geq l \quad (n = 1, 2, \ldots)
\]

then

\[
w_2^{(p)} \left( \frac{1}{n} \right) \leq (lR_n)^{-1} \sum_{k=1}^{n} \left\{ \frac{R_k w_2^{(p)}(1/k)}{k} \right\}.
\]

Hence we can get from (42) that

\[
\|f - t_n(f)\|_p = O \left( \frac{1}{R_n} \sum_{k=1}^{n} \left( \frac{R_k w_2^{(p)}(1/k)}{k} \right) \right).
\]

We shall need the following lemma for the proof of our theorem.

**Lemma 5 ([10]).** \( \{c_n\} \) and \( \{d_n\} \) are non-negative, non-increasing sequences and \( \tau = [1/\ell] \) then for \( 0 < a < b < n \) (any \( n \)), and \( 0 < |\ell| \leq \tau \), we have

\[
\left| \sum_{k=1}^{b} c_{n-k} d_k \sin k\ell \right| = O(R(\tau)) \text{ as } \ell \to 0.
\]

**Proof of Theorem 4.** We easily get

\[
f(x) - t_n(f; x) = \int_{0}^{\tau} \{ f(x + \ell) + f(x - \ell) - 2f(x) \} M_n(\ell) \, d\ell
\]

where

\[
M_n(\ell) = (2\pi R_n)^{-1} \sum_{k=0}^{n} c_{n-k} d_k \frac{\sin (k + \ell/2)}{\sin \ell/2}.
\]

Hence, by generalized Minkowski's inequality

\[
\|f(x) - t_n(f; x)\| \leq I_1 + I_2,
\]

where

\[
I_1 = \int_{0}^{\tau/n} w_2^{(p)}(\ell; f) \, |M_n(\ell)| \, d\ell, \quad \text{and} \quad I_2 = \int_{\tau/n}^{\tau} w_2^{(p)}(\ell; f) \, |M_n(\ell)| \, d\ell.
\]

Since \( 0 < \sin((k + \ell/2)) < (2k + 1) \sin \ell/2 \) for \( 0 < k \leq n, 0 < \ell < \pi/n \), we have

\[
I_1 = O\left( (2n + 1) \int_{0}^{\tau/n} w_2^{(p)}(\ell; f) \, d\ell \right) = O\left( w_2^{(p)} \left( \frac{\tau}{n}; f \right) \right).
\]
By Lemma 5,
\[
I_2 = O\left(\frac{1}{R_n} \frac{R(1/t)}{t} w_2^p(t; f) dt\right)
\]
\[
= O\left(\frac{1}{R_n} \sum_{k=1}^{n-1} \int_{k/\pi}^{(k+1)/\pi} \frac{w_2^p(1/t, f)}{t} dt\right)
\]
\[
= O\left(\frac{1}{R_n} \sum_{k=1}^{n} \left(\frac{R_k w_2^p(1/k)}{k}\right)\right).
\]

On collecting the estimates, the theorem follows.

REMARK. If \(0 < \alpha \leq 1, \ p > 1, \ \alpha p > 1\) then, by Lemma 2(ii), \(f \in \text{Lip}(\alpha, p)\) implies \(w_2^p(\delta; f) = O(\delta^{a-1/p})\). In this case
\[
w_2^p(1/n) = O(n^{-\alpha+1/p})
\]
and
\[
\frac{1}{R_n} \sum_{k=1}^{n} \frac{R_k w_2^p(1/k)}{k} = O\left(\frac{1}{R_n} \sum_{k=1}^{n} \frac{1}{k^{1+1-1/p}}\right).
\]

Let \(\delta > 0\) and \(A_n^\delta\) be given by \(\sum_{n=0}^{\infty} A_n^\delta \cdot n^n = (1 - x)^{-\delta - 1}(\ |x| < 1)\). Let \(N_n(f)\) be written as \(\sigma_n^\delta(f)\) or \(H_n^\delta(f)\) according as \(c_n = A_n^\delta\) or \(c_n = (n + 1)^{-1}\) for all \(n\).

By putting \(d_n = 1\) for all \(n\), we get the following results:

**Corollary 5.** Let \(f \in \text{Lip}(\alpha, p), \ 1 < p \leq \infty\). Then
\[
\|f - \sigma_n^\delta(f)\|_p = \begin{cases} 
O(n^{-\delta+1/p}) & (0 < \delta < \alpha \leq 1); \\
O\left(\frac{\log n}{n^{\delta-1/p}}\right) & (0 < \delta \leq \alpha \leq 1).
\end{cases}
\]

REMARK. The case \(p = \infty\) of Corollary 5 was proved by Alexits [1].

**Corollary 6.** If \(\{c_n\}\) is a positive non-increasing sequence and \(f \in L_p[-\pi, \pi]\) (\(1 < p < \infty\)) or \(f \in c_{2\pi}(p = \infty)\), then
\[
\|f - N_n(f)\|_p = O\left(\frac{1}{C_n} \sum_{k=1}^{n} \frac{C_k}{k} w_2^p(1/k; f)\right).
\]

REMARK. The case \(p = \infty\) of this Corollary is Theorem D.
COROLLARY 7. If \( f \in \text{Lip}(\alpha, p) \), \( \alpha p > 1 \), \( 0 < \alpha \leq 1 \), \( p > 1 \), then
\[
\| f - H_n(f) \|_p = O\left(\left(\log n\right)^{-1}\right).
\]
In what follows, we shall write \( \overline{H}_n(f) \) for \( \overline{N}_n(f) \) when \( d_n = 1/(n + 1) \).

COROLLARY 8. Let \( \{d_n\} \) be a non-negative, non-increasing sequence. Then for \( f \in L_p[-\pi, \pi] \) (\( 1 < p < \infty \)), or \( f \in c_2 \pi \) (\( p = \infty \)),
\[
\| f - \overline{N}_n(f) \|_p = O\left(w^{(p)}(1/n)\right) + O\left(\frac{1}{D_n} \sum_{k=1}^{n} \frac{D_kw^{(p)}(1/k)}{k}\right).
\]

COROLLARY 9. If \( f \in \text{Lip}(\alpha, p) \), \( \alpha p > 1 \), \( 0 < \alpha \leq 1 \), \( p > 1 \), then
\[
\| f - \overline{H}_n(f) \|_p = O\left(\left(\log n\right)^{-1}\right).
\]

REMARKS. (i) Since \( w^{(p)}(\delta, f) \leq 2w^{(p)}(\delta, f) \), Corollary 6 and Corollary 8 are stated with estimates using modulus of continuity in place of integral modulus of smoothness.

(ii) It can be observed that our corollaries contain assumptions on \( \{c_n\} \) and \( \{d_n\} \) but we do not use conditions of the type (14) (see Theorem B and Theorem C).

Finally we are grateful to the referee for his valuable comments.

References


