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A NOTE ON THE EXTENSION OF A FAMILY OF BIORTHOGONAL COIFMAN WAVELET SYSTEMS

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Abstract

Wavelet systems with a maximum number of balanced vanishing moments are known to be extremely useful in a variety of applications such as image and video compression. Tian and Wells recently created a family of such wavelet systems, called the biorthogonal Coifman wavelets, which have proved valuable in both mathematics and applications. The purpose of this work is to establish along with direct proofs a very neat extension of Tian and Wells' family of biorthogonal Coifman wavelets by recovering other "missing" members of the biorthogonal Coifman wavelet systems.

1. Introduction

Wavelet systems and wavelet-based application techniques are constantly being refined, see, for example, [1, 2, 11]. Wavelet-based applications are known to hinge on a number of desirable wavelet properties such as the number of consecutive vanishing moments. There are already a variety of wavelets available including both orthogonal and biorthogonal wavelets [2, 3], many of which can in fact be described entirely on the basis of linear algebras [8, 9]. An evaluation of some of the existing wavelets in terms of their applications can be found in [10]. A typical good pair of wavelet filters often possess a maximum number of balanced vanishing moments such as, for instance, the so-called biorthogonal Coifman wavelets recently proposed by Tian and Wells [12]. Such wavelet systems are in general very useful in image, sound or video related applications due to the high order of consecutive vanishing moments associated with these systems. The purpose of this paper is thus to show that Tian and Wells' family of biorthogonal Coifman wavelets is in fact much larger than previously

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thought. More precisely, we shall establish explicitly a very neat extension of Tian and Wells' family of biorthogonal Coifman wavelets [12] by recovering those "missing" additional biorthogonal Coifman wavelet systems with a minimum length in one of the main filters.

One of the pivotal applications of wavelet systems is the use of analysis and synthesis filters, particularly in the field of processing multimedia signals. Suppose two real-valued finite sequences $\{h_k\}$ and $\{\bar{h}_k\}$ satisfy the biorthogonality condition $\sum_{k\in\mathbb{Z}}\bar{h}_k h_{k+m\ell} = \delta_{\ell,0}$ for all $\ell \in \mathbb{Z}$, where $\delta_{\ell,m}$ is the Kronecker delta symbol and $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$ is the set of all integers. Then for any integer $N > 0$ and any periodic sequence $\{x_k\}_{k\in\mathbb{Z}}$ with period $2N$, that is, $x_k = x_{k'}$ whenever $k \equiv k' \pmod{2N}$, the sequences $\{c_\ell\}$ and $\{d_\ell\}$ defined by the *analysis filters*

$$c_\ell = \sum_{k\in\mathbb{Z}} h_{k-2\ell}x_k \quad \text{and} \quad d_\ell = \sum_{k\in\mathbb{Z}}(-1)^k\bar{h}_{1-k+2\ell}x_k, \tag{1.1}$$

for $\ell \in \mathbb{Z}$ are all periodic with period N, and are sufficient to perfectly reconstruct the original sequence through the *synthesis filter*

$$x_k = \sum_{\ell\in\mathbb{Z}} \bar{h}_{k-2\ell}c_\ell + \sum_{\ell\in\mathbb{Z}}(-1)^k h_{1-k+2\ell}d_\ell, \quad k \in \mathbb{Z}. \tag{1.2}$$

This result is well known [2, 8, 9], and can be directly verified too. We note that the periodic sequence $\{x_k\}_{k\in\mathbb{Z}}$ can be represented by $\{x_k\}_{0\le k<2N}$, and $\{c_\ell\}_{\ell\in\mathbb{Z}}$ and $\{d_\ell\}_{\ell\in\mathbb{Z}}$ can be likewise represented by $\{c_\ell\}_{0\le\ell<N}$ and $\{d_\ell\}_{0\le\ell<N}$ respectively. It is also easy to see that for any sequence $\{x_k\}_{k\in\mathbb{Z}}$ of finite length, the sequences $\{c_\ell\}$ and $\{d_\ell\}$ defined by (1.1) are also of finite length, and are sufficient to perfectly reconstruct the original $\{x_k\}$ through the use of a synthesis filter (1.2).

In terms of a typical application in image compression, for instance, an image $\mathscr{X}$ is composed of a sequence of pixel values $\{x_k\}$. These values will first be transformed, or filtered, to $\{c_\ell\}$ and $\{d_\ell\}$ via a relation such as (1.1) so as to decorrelate the image data. The transformed data will then be quantised, that is, specifically "rounded", before being entropy coded to produce a compressed image. Although no wavelet filter properties can guarantee superior performance in prospective applications, a larger number of vanishing moments and higher regularity of wavelet functions are preferred before other accompanying side-effects become an issue. For a given image and a given bitrate or Peak Noise Signal Ratio [1], it is possible to select dynamically an optimal wavelet filter from a prescribed family of, say, the orthogonal wavelet filters of a given length. The search is extremely computationally intensive. For orthogonal wavelets, however, it has been shown [10] that the performance improvement is negligible with an optimal wavelet when dealing with highly active images. Moreover one observes that the parameters for the optimal orthogonal wavelets are in general not too far away from the same length wavelet filters of the maximum vanishing moments. Hence it is always worthwhile to search for *individual* wavelet filters that possess excellent overall properties such as balanced vanishing moments. This is indeed the main purpose of our current work.

In the rest of this paper we shall first introduce briefly the basic concept and terminology of vanishing moments and biorthogonal Coifman wavelet systems. The bulk of the paper is then dedicated to the derivation and the proof of the main theorem, Theorem 2.1, which gives a neat extension of Tian and Wells' family of biorthogonal Coifman wavelet systems. It is perhaps also pertinent to note that the new biorthogonal Coifman wavelet systems given by Theorem 2.1 will typically outperform the industrial standard JPEG and some of the classical wavelets in the applications of image compression. However we hasten to add that no knowledge on any form of image processing is required or assumed in this work.

2. Biorthogonal Coifman wavelet systems

A fundamental question in the construction of wavelet systems is how one may choose wavelets so that they will possess relevant useful features and result in good application performance in such applications as image and video compressions. It turns out that an important aspect of wavelet filters is characterised by the moments or the discrete moments, and in particular by the number and order of the vanishing moments. By *discrete moments* we here mean those defined by

$$\begin{aligned} \mu_r^{(0)} &= \sum_{k\in\mathbb{Z}} k^r h_k, & \mu_r^{(1)} &= \sum_{k\in\mathbb{Z}} k^r g_k, \\ \bar{\mu}_r^{(0)} &= \sum_{k\in\mathbb{Z}} k^r \bar{h}_k, & \bar{\mu}_r^{(1)} &= \sum_{k\in\mathbb{Z}} k^r \bar{g}_k, \end{aligned} \tag{2.1}$$

where $r \geq 0$ is an integer, $g_k = (-1)^k \bar{h}_{1-k}$ and $\bar{g}_k = (-1)^k h_{1-k}$. The fundamental importance of the vanishing moments lies in the fact that, if $\mu_r^{(1)} = 0$ for $r = 0, \ldots, N$ and $\mu_r^{(0)} = 0$ for $r = 1, \ldots, N$, then for any input signals $\{x_k\}$ sampled from any N degree polynomial, the details d_ℓ produced by the analysis filter (1.1) are all zero and the corresponding averages c_ℓ are also polynomial signals. This commonly known result says, roughly, that signals of any N degree polynomial sampled at an equal step are completely decorrelated by (1.1) when there are sufficiently many vanishing moments. It will thus need less storage space to encode the filtered signals in a compression application because all d_ℓ are 0. The other half of the vanishing moments, $\bar{\mu}_r^{(1)} = 0$ for $r = 0, \ldots, N$ and $\bar{\mu}_r^{(0)} = 0$ for $r = 1, \ldots, N$ on the other hand, ensure that the synthesis filter (1.2) will filter out quantisation noises of N degree polynomials. The impact of vanishing moments in terms of the regularity of the corresponding wavelet functions and their favourable effects upon the applications are also significant although we will not delve into the actual details in this regard.

A biorthogonal wavelet system with compact support is called a *biorthogonal Coifman wavelet system* of order N if [12]

(i) $\int_{\mathbb{R}} \bar{\phi}(x)dx = \int_{\mathbb{R}} \phi(x)\,dx = 1$;
(ii) $\int_{\mathbb{R}} x^i \bar{\psi}(x)\,dx = \int_{\mathbb{R}} x^i \psi(x)\,dx = 0,\ i = 0, \dots, N$;
(iii) $\int_{\mathbb{R}} x^j \bar{\phi}(x)\,dx = \int_{\mathbb{R}} x^j \phi(x)\,dx = 0,\ j = 1, \dots, N$,

where the scaling functions $\phi(x)$ and $\bar{\phi}(x)$ are determined by

$$\phi(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} h_k \phi(2x-k), \quad \bar{\phi}(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} \bar{h}_k \bar{\phi}(2x-k),$$

and the wavelet functions $\psi(x)$ and $\bar{\psi}(x)$ are then defined by the dilation equations

$$\psi(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} g_k \phi(2x-k), \quad \bar{\psi}(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} \bar{g}_k \bar{\phi}(2x-k).$$

We recall [2, 8] that with the existential conditions

$$\sum_{k\in\mathbb{Z}} h_k = \sum_{k\in\mathbb{Z}} \bar{h}_k = \sqrt{2},$$

the scaling and wavelet functions can be constructed from the filter coefficients $\{h_k\}$ and $\{\bar{h}_k\}$. Due to the essential equivalence [2, 8] of the vanishing discrete moments (2.1) and the vanishing continuum moments defined by

$$M_r^{(0)} = \int_{\mathbb{R}} x^r \phi(x)\,dx, \qquad M_r^{(1)} = \int_{\mathbb{R}} x^r \psi(x)\,dx,$$
$$\bar{M}_r^{(0)} = \int_{\mathbb{R}} x^r \bar{\phi}(x)\,dx, \qquad \bar{M}_r^{(1)} = \int_{\mathbb{R}} x^r \bar{\psi}(x)\,dx,$$

the conditions (i)–(iii) for the biorthogonal Coifman wavelets of order N are simply equivalent to

$$\mu_r^{(0)} = \bar{\mu}_r^{(0)} = \sqrt{2}\delta_r^0, \quad \mu_r^{(1)} = \bar{\mu}_r^{(1)} = 0 \tag{2.2}$$

for $r = 0, 1, \dots, N$. It is thus equally valid to treat conditions (2.2) as the very definition of a biorthogonal Coifman wavelet system in terms of $\{h_k, g_k, \bar{h}_k, \bar{g}_k\}$ without having to get involved with the scaling and wavelet *functions* at all. We are now ready to present below our main results in the form of a theorem, which gives an explicit construction of a family of biorthogonal Coifman wavelet systems of order N.

THEOREM 2.1. *For any integer $N \geq 0$ and $\alpha \in \mathbb{Z}$, if the nonzero elements of the synthesis scaling coefficients $\bar{h}_i$ are given by*

$$\bar{h}_0 = 1/\sqrt{2}, \quad \bar{h}_{1-2\alpha+2k} = \frac{\prod_{\ell=0,\ell\neq k}^{N}[2(\ell-\alpha)+1]}{2^N\sqrt{2}\prod_{\ell=0,\ell\neq k}^{N}(\ell-k)}, \quad k = 0, \dots, N, \tag{2.3}$$

and the analysis scaling coefficients h_i are calculated in closed form recursively from

$$h_{2\ell} = \sqrt{2}\left[\delta_{\ell,0} - \sum_{n\in\mathbb{Z}} \bar{h}_{1+2n}\bar{h}_{1+2(n-\ell)}\right], \quad h_{2\ell+1} = \bar{h}_{2\ell+1}, \quad \ell \in \mathbb{Z}, \tag{2.4}$$

then the corresponding wavelet system, along with the wavelet coefficients g_ℓ and $\bar{g}_\ell$ given by

$$g_\ell = (-1)^\ell \bar{h}_{1-\ell}, \quad \bar{g}_\ell = (-1)^\ell h_{1-\ell}, \quad \ell \in \mathbb{Z}, \tag{2.5}$$

is a biorthogonal Coifman wavelet system of order N. If $1 \le \alpha \le N$, then the sequences $\{\bar{h}_i\}_{i\in\mathbb{Z}}$ given by (2.3) have the minimum length $2N+1$ among all the biorthogonal Coifman wavelet systems of order N.

We note that the special case of $\alpha = \lceil N/2 \rceil$ in Theorem 2.1 will reproduce Tian and Wells' whole list [12] of biorthogonal Coifman wavelet systems. However Theorem 2.1 does contain new biorthogonal Coifman wavelet systems, as can be easily seen from the relative position of $\bar{h}_0$ among the other nonzero elements of $\{\bar{h}_i\}$. In the case of $N = 3$ and $\alpha = 3$, for instance, the filter parameters $\sqrt{2}\bar{h}_i$ for $-3 \le i \le 3$ are given by $5/16, 1, 15/16, 0, -5/16, 0, 1/16$ respectively, and the other filter parameters $\sqrt{2}h_i$ for $-6 \le i \le 6$ are given by

$$-\frac{5}{256}, 0, \frac{5}{128}, 0, \frac{5}{256}, \frac{5}{16}, \frac{59}{64}, \frac{15}{16}, \frac{5}{256}, -\frac{5}{16}, \frac{5}{128}, \frac{1}{16}, -\frac{5}{256}$$

respectively.

PROOF OF THEOREM 2.1. The proof is of a constructive nature and is based on the direct verification of biorthogonality (2.6) and the vanishing moments (2.2) through the use of (2.3)–(2.5). First we observe that under the biorthogonality condition

$$\sum_{k\in\mathbb{Z}} \bar{h}_k h_{k+2\ell} = \delta_{\ell,0}, \quad \ell \in \mathbb{Z}, \tag{2.6}$$

the following additional equations

$$\sum_{\ell\in\mathbb{Z}} (k+2\ell)^i \bar{h}_{k+2\ell} = \sum_{\ell\in\mathbb{Z}} (k+2\ell)^i h_{k+2\ell} = \frac{\delta_{i,0}}{\sqrt{2}} \tag{2.7}$$

for $k \in \mathbb{Z}$ and $i = 0, \ldots, N$ ensure that $\mu_p^{(0)} = \bar{\mu}_p^{(0)} = 0$ will hold for $p = 1, \ldots, N$. This is because (2.7) implies for $p, j = 1, \ldots, N$ and $p \ge j$ that

$$\sum_{k,\ell\in\mathbb{Z}} k^{p-j}(k+2\ell)^j h_k \bar{h}_{k+2\ell} = \sum_{k\in\mathbb{Z}} k^{p-j} h_k \left[\sum_{\ell\in\mathbb{Z}} (k+2\ell)^j \bar{h}_{k+2\ell}\right] = 0,$$

and hence for $p = 1, \dots, N$ that

$$\sum_{k\in\mathbb{Z}} k^p h_k \overset{(2.7)}{=} \sqrt{2} \sum_{k,\ell\in\mathbb{Z}} k^p h_k \bar{h}_{k+2\ell}$$
$$= \sqrt{2}\left[\sum_{k,\ell\in\mathbb{Z}} k^p h_k \bar{h}_{k+2\ell} + \sum_{j=1}^{p}\sum_{k,\ell\in\mathbb{Z}} (-1)^j k^{p-j}(k+2\ell)^j h_k \bar{h}_{k+2\ell}\binom{p}{j}\right]$$
$$= \sqrt{2}\sum_{k,\ell\in\mathbb{Z}} (-2\ell)^p h_k \bar{h}_{k+2\ell} \overset{(2.6)}{=} \sqrt{2}\sum_{\ell\in\mathbb{Z}}(-2\ell)^p \delta_{\ell,0} = 0.$$

On the other hand, since (2.7) for $k = 0$ and $k = 1$ implies

$$\sum_{s\in\mathbb{Z}} (-1)^s s^i \bar{h}_s = \sum_{s\in\mathbb{Z}} (-1)^s s^i h_s = 0$$

for $i = 0, \dots, N$, we see (2.5) gives for $p = 0, \dots, N$

$$\mu_p^{(1)} = \sum_{\ell\in\mathbb{Z}} \ell^p g_\ell = \sum_{\ell\in\mathbb{Z}} (-1)^\ell \ell^p \bar{h}_{1-\ell} = \sum_{s\in\mathbb{Z}} (-1)^{1-s}(1-s)^p \bar{h}_s$$
$$= \sum_{j=0}^{p} \binom{p}{j} (-1)^{1+j} \sum_{s\in\mathbb{Z}} (-1)^s s^j \bar{h}_s = 0$$

and likewise $\bar{\mu}_p^{(1)} = 0$. In other words (2.7) and (2.5) also imply $\mu_p^{(1)} = \bar{\mu}_p^{(1)} = 0$ for $p = 0, \dots, N$. Hence the solutions of (2.6) and (2.7) will result in biorthogonal Coifman wavelet systems of order N. Incidentally we observe that Equations (2.7) for all $k \in \mathbb{Z}$ are in fact equivalent to those for just $k = 0$ and 1.

We now show that the filters given by (2.3)–(2.5) indeed satisfy (2.6) and (2.7). We do this by solving explicitly part of the equations in (2.7) and then verifying the remaining conditions. Let $\beta_j = j - 2\alpha_j$ for $j = 0, 1$ with $\alpha_j \in \mathbb{Z}$, $\alpha_1 \equiv \alpha$ and $0 \le \alpha_0 \le N$. For the construction of (2.3) and (2.4), we assume that the nonzero elements from $\{\bar{h}_{\beta_j+2\ell}\}_{\ell\in\mathbb{Z}}$ can only come from $\bar{h}_{\beta_j}, \bar{h}_{\beta_j+2}, \dots, \bar{h}_{\beta_j+2N}$. Hence the linear system

$$\sum_{\ell\in\mathbb{Z}} (j+2\ell)^k \bar{h}_{j+2\ell} = \delta_{k,0}/\sqrt{2}, \tag{2.8}$$

for $k = 0, \dots, N$ and $j = 0, 1$, as a part of (2.7), becomes

$$\sum_{\ell=0}^{N} (\beta_j + 2\ell)^k \bar{h}_{\beta_j+2\ell} = \delta_{k,0}/\sqrt{2},$$

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or equivalently

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \beta_j & \beta_j+2 & \cdots & \beta_j+2N \\ \vdots & \vdots & & \vdots \\ \beta_j^N & (\beta_j+2)^N & \cdots & (\beta_j+2N)^N \end{bmatrix} \begin{bmatrix} \bar{h}_{\beta_j} \\ \bar{h}_{\beta_j+2} \\ \vdots \\ \bar{h}_{\beta_j+2N} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.9}$$

with an $N+1$ by $N+1$ Vandermonde matrix. If we replace the k-th column of the Vandermonde matrix by $(1/\sqrt{2}, 0, \dots, 0)^T$ and denote by Δ_k the determinant of the resulting matrix, then for $k = 0, \dots, N$

$$\Delta_k = \frac{(-1)^k}{\sqrt{2}} \left[\prod_{\ell=0,\ell\neq k}^{N} (\beta_j + 2\ell) \right] \prod_{\ell>i,\ell\neq k,i\neq k} (2\ell - 2i),$$

and the determinant Δ of the Vandermonde coefficient matrix in (2.9) reads

$$\Delta = \prod_{\ell,k=0,\ell>k}^{N} (2\ell - 2k) = \left[\prod_{\ell>i} (2\ell - 2k) \right] \left[\prod_{i>k} (2i - 2k) \right].$$

Hence, due to $\bar{h}_{\beta_j+2k} = \Delta_k/\Delta$, we have

$$\bar{h}_{\beta_j+2k} = \frac{\prod_{\ell=0,\ell\neq k}^{N} (\beta_j + 2\ell)}{2^N\sqrt{2}\prod_{\ell=0,\ell\neq k}^{N} (\ell - k)}$$

for $k = 0, \dots, N$ and $j = 0, 1$. These equations are then simplified to the first expression in (2.3). Since $0 \le \alpha_0 \le N$ implies $\beta_0 \ge -2N$ and

$$\bar{h}_{2(k-\alpha_0)} = \bar{h}_{\beta_0+2k} = \frac{\prod_{\ell=0,\ell\neq k}^{N} (\ell - \alpha_0)}{\sqrt{2}\prod_{\ell=0,\ell\neq k}^{N} (\ell - k)} = \frac{1}{\sqrt{2}}\delta_{\alpha_0,k} = \frac{1}{\sqrt{2}}\delta_{\beta_0+2k,0}$$

for $k = 0, \dots, N$, that is, $\bar{h}_{2k} = \delta_{k,0}/\sqrt{2}$, we obtain the second expression in (2.3). It is now easy to verify that the orthogonality condition (2.6) is assured by (2.3) and (2.4) because

$$\begin{aligned} \sum_{k\in\mathbf{Z}} \bar{h}_k h_{k+2\ell} &= \sum_{s=0}^{1} \sum_{n\in\mathbf{Z}} \bar{h}_{s+2n} h_{s+2n+2\ell} \\ &= \sum_{n\in\mathbf{Z}} \bar{h}_{\beta_1+2n} h_{\beta_1+2n+2\ell} + \sum_{n\in\mathbf{Z}} \bar{h}_{\beta_0+2n} h_{\beta_0+2n+2\ell} \\ &\overset{(2.3)}{=} \left[\sum_{n\in\mathbf{Z}} \bar{h}_{\beta_1+2n-2\ell} \tilde{h}_{\beta_1+2n} \right] + \frac{1}{\sqrt{2}} h_{2\ell} \overset{(2.4)}{=} \delta_{\ell,0}. \end{aligned}$$

To verify (2.7) we observe that the first half, that is, (2.8), is automatically satisfied by the construction of the $\bar{h}_i$. The remaining half can be further rewritten as

$$\sum_{k\in\mathbb{Z}}(2k+1)^p h_{2k+1} = \frac{1}{\sqrt{2}}\delta_{p,0}, \quad \sum_{k\in\mathbb{Z}}(2k)^p h_{2k} = \frac{1}{\sqrt{2}}\delta_{p,0} \tag{2.10}$$

for $p = 0, \dots, N$. Since (2.8) and $h_{2k+1} = \bar{h}_{2k+1}$ in (2.4) already imply the first identity in (2.10), the second identity in (2.10) can be established algebraically on the basis of (2.4) because

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}(2k)^p h_{2k} \\
&= \sum_{k\in\mathbb{Z}}(2k)^p \left\{\sqrt{2}\delta_{k,0} - \sqrt{2}\left[\sum_{n\in\mathbb{Z}} \bar{h}_{\beta_1+2n}\bar{h}_{\beta_1+2n-2k}\right]\right\} \\
&= \sqrt{2}\delta_{p,0} - \sum_{n,k\in\mathbb{Z}} \sqrt{2}\left[(2k)^p \bar{h}_{\beta_1+2n}\bar{h}_{\beta_1+2n-2k}\right] \\
&= \sqrt{2}\delta_{p,0} - \sqrt{2}\sum_{n,k\in\mathbb{Z}} \left\{[(\beta_1+2n)-(\beta_1+2n-2k)]^p\, \bar{h}_{\beta_1+2n}\bar{h}_{\beta_1+2n-2k}\right\} \\
&= \sqrt{2}\delta_{p,0} - \sqrt{2}\sum_{n,k\in\mathbb{Z}}\sum_{\ell=0}^{p}\binom{p}{\ell}(-1)^\ell(\beta_1+2n)^{p-\ell}(\beta_1+2n-2k)^\ell \bar{h}_{\beta_1+2n}\bar{h}_{\beta_1+2n-2k} \\
&= \sqrt{2}\delta_{p,0} - \sqrt{2}\sum_{\ell=0}^{p}\binom{p}{\ell}(-1)^\ell\left[\sum_{n\in\mathbb{Z}}(\beta_1+2n)^{p-\ell}\bar{h}_{\beta_1+2n}\right]\left[\sum_{k'\in\mathbb{Z}}(\beta_1+2k')^\ell \bar{h}_{\beta_1+2k'}\right] \\
&= \sqrt{2}\delta_{p,0} - \sqrt{2}\sum_{\ell=0}^{p}\binom{p}{\ell}(-1)^\ell \frac{\delta_{p-\ell,0}}{\sqrt{2}}\frac{\delta_{\ell,0}}{\sqrt{2}} \\
&= \sqrt{2}\left[\delta_{p,0} - \frac{(-1)^p\delta_{p,0}}{2}\right] = \sqrt{2}\delta_{p,0}\left[\frac{2-(-1)^p}{2}\right] = \frac{1}{\sqrt{2}}\delta_{p,0}.
\end{aligned}$$

Hence the condition of vanishing moments (2.7) is now verified, implying (2.2) are completely satisfied by (2.3)–(2.5) because (2.6) has already been verified earlier on. We have thus proved that (2.3)–(2.5) indeed give a family of biorthogonal wavelet systems of order N.

We now show the last part of the theorem, that is, that the constructed biorthogonal Coifman wavelet systems of order N given by (2.3)–(2.5) exhibit a minimum length in the sequence $\{\bar{h}_i\}$ when $1 \le \alpha \le N$. For this purpose, we first observe (2.6) and (2.7) are in fact also equivalent to (2.2). Then we observe that the coefficient matrix of the linear equations (2.8) for $\bar{h}_{1+2i}$, like (2.9), has rank $N+1$ and is again of the form of a Vandermonde matrix. Hence the most distant nonzero $\bar{h}_{1+2i}$ must be at least $2N+1$ index positions apart otherwise a solution for a linear system like (2.9) won't exist. In other words we have argued that the length of $\{\bar{h}_i\}$ is at least $2N + 1$. With the choice of (2.3)–(2.5) and $1 \leq \alpha \leq N$, we see immediately $0 \in [\beta_1, \beta_1 + 2N] \equiv [1 - 2\alpha, 1 - 2\alpha + 2N]$, that is, the index of $\bar{h}_0$ will not go outside the index interval of the nonzero $\bar{h}_{2k+1}$'s. Hence the $\{\bar{h}_i\}$ will have exactly the minimum length $2N + 1$.

We note that in the proof of the above theorem, the choice of $0 \leq \alpha_0 \leq N$ gives the simplest solution to (2.9) for $j = 0$. If this condition were not enforced, then we would get more than 1 nonzero $\bar{h}_{2i}$ and this could result in a wavelet system with the length of $\{\bar{h}_i\}$ larger than $2N + 1$.

To conclude this work, we point out that the making or the choice of a good wavelet, in terms of a given application perspective, is typically a balancing act on trading away minor useful features for certain more dominant desirable properties. For instance, we could also trade away the least significant vanishing moment for the minimisation of the norm of a synthesis filter so that the quantisation errors would be least magnified, see [6]. Biorthogonal Coifman wavelets however aim to provide universally well-behaved filters by achieving a symmetry in vanishing moments in regard to analysis and synthesis filters. Incidentally, symmetry is a well appreciated feature for wavelets just as in many other fields including dynamical systems [4, 5] and weighted finite automata [7]. We finally note that the explicit listing of the new wavelet filter coefficients dictated by Theorem 2.1 is straightforward and will thus not be tabulated here in any greater detail.

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