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THE MEAN VALUE THEOREM AND ANALYTIC FUNCTIONS

by ELGIN H. JOHNSTON

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It is well known that the mean value theorem (MVT) does not, in general, hold for analytic functions. The most familiar example to this effect is $f(z)=e^z$ since $e^{2\pi i}-e^0 \neq 2\pi i e^{z_0}$ for any $z_0 \in \mathbb{C}$. On the other hand, it is easy to show that the MVT holds in $\mathbb{C}$ if $f(z)$ is a polynomial of degree at most 2. Thus it is natural to ask what conditions on a function $f(z)$ analytic in a domain D are necessary and sufficient for $f(z)$ to satisfy the MVT in D. This is one of the questions answered in this paper.

Many authors have devised "substitutes" for the MVT that do apply to all analytic functions. For example Samuelsson [**5**] (see also Robertson [**3**] and Novinger [**2**]) has proved the following local version of the mean value theorem.

Theorem A. *Let $f(z)$ be analytic in a domain containing z_0. Then there is a neighbourhood N of z_0 such that if z_1 is any point in N, then there exists a point z with $\left|z-\tfrac{1}{2}(z_0+z_1)\right|<\tfrac{1}{2}\left|z_0-z_1\right|$ and such that*

$$f(z_1)-f(z_0)=f'(z)(z_1-z_0).$$

Notice that the point z does not necessarily lie on the segment $[z_0, z_1]$. McLeod [**1**] has proved a version of the MVT that involves a convex combination of derivatives on $[z_0, z_1]$.

Theorem B. *Let $f(z)$ be analytic in a domain D. If $z_0, z_1 \in D$ and the segment $[z_0, z_1] \subseteq D$, then there are points $w_0, w_1 \in (z_0, z_1)$ and there is a λ $(0 \leqq \lambda \leqq 1)$ with*

$$f(z_1)-f(z_0)=(z_1-z_0)[\lambda f'(w_1)+(1-\lambda)f'(w_0)].$$

In this paper we look in a direction different from those in Theorems A and B and instead ask for what analytic functions does the classical MVT hold. We actually ask for more ... in particular when does a pair of analytic functions satisfy the generalised MVT? This is made more precise in the following definition.

Definition 1. Let $f(z)$ and $g(z)$ be functions analytic in a domain $D \subseteq \mathbb{C}$, and suppose that $g(z)$ is one-to-one in D. Then $f(z)$ and $g(z)$ satisfy the *generalised mean value property* (*GMVP*) on D if, whenever the line segment $[z_1, z_2] \subseteq D$ $(z_1 \neq z_2)$, there is a point $c \in (z_1, z_2)$ such that

$$\frac{f(z_2)-f(z_1)}{g(z_2)-g(z_1)}=\frac{f'(c)}{g'(c)}.$$

In the above definition, the classical MVP arises if we take $g(z)$ to be linear. Since the GMVP is clearly satisfied if $f(z)\equiv$constant, we eliminate this case from our considerations.

Theorem 2. *Let $f(z)$ ($\not\equiv$ constant) be analytic in a domain D and let $g(z)$ be analytic and one-to-one in D. Then the following are equivalent.*

(i) *$f(z)$ and $g(z)$ satisfy the GMVP on D.*

(ii) $\dfrac{f'''(z)}{f'(z)}=\dfrac{g'''(z)}{g'(z)}$ *and* $\dfrac{f^{(5)}(z)}{f'(z)}=\dfrac{g^{(5)}(z)}{g'(z)}$

as meromorphic functions in D.

(iii) *One of the following statements holds; in (a) and (c) it is assumed that g is univalent in D:*

(a) *$f(z)$ and $g(z)$ are nonconstant polynomials of degree at most* 2,

(b) *$f(z)=Ag(z)+B$ for some complex constants A and B* $(A\neq 0)$,

(c) *$f(z)=A\cos\alpha z+B\sin\alpha z+C$ and $g(z)=D\cos\alpha z+E\sin\alpha z+F$ where A, B, C, D, E, F are complex constants and* $(|A|+|B|)(|D|+|E|)\alpha\neq 0$.

Proof. (iii)⇒(i). This is easily checked.
(i)⇒(ii). Select $a\in D$ with $f'(a)\neq 0$. Since neither the addition of constants to $f(z)$ and $g(z)$ nor the translation of the variable affects the GMVP, we may assume $a=0=f(0)=g(0)$. Since $g'(0)\neq 0$, $g^{-1}(w)$ is analytic in a neighbourhood of $w=0$, with $g^{-1}(0)=0$. Thus $f\circ g^{-1}(w)$ is analytic in a neighbourhood of 0 and we can write

$$f\circ g^{-1}(w)=A_1 w+A_2 w^2+\cdots$$

for $|w|<\varepsilon$, where ε is a suitable positive number. Letting $w=g(z)$, we find that

$$f(z)=A_1 g(z)+A_2(g(z))^2+\cdots \tag{1}$$

for $|z|<\delta$, where δ is a sufficiently small positive number.

It is clear that the pair of functions $f(z)$ and $g(z)$ satisfy the GMVP on D if and only if the pair $f(z)-kg(z)$ and $g(z)$ do (k=constant). So taking $F(z)=f(z)-A_1 g(z)$, we may assume that $F(z)$ and $g(z)$ satisfy the GMVP on D.

If $F(z)$ is a constant function, then $f(z)=A_1 g(z)+B$, and $A_1\neq 0$ since $f(z)$ is not identically constant. In this case it is easy to check that (ii) holds.

Suppose $F(z)$ is not constant. Since $F(0)=F'(0)=0$ it follows that $F(z)$ takes the value 0 at $z=0$ with multiplicity $n\geqq 2$. Thus [4, p. 216] there is a disc $\Delta(0,\eta)\subseteq\Delta(0,\delta)$ such that $F'(z)\neq 0$ in $\Delta'(0,\eta)$ and there is a neighbourhood N of 0 with $N\subseteq\Delta(0,\eta)$ such that $F(z)$ is an n-to-one mapping on N. In fact each $w\in F(N)-\{0\}$ is taken on at n distinct points of

$N-\{0\}$. If $n \geqq 3$, then we can find distinct points $z_1, z_2, z_3 \in N-\{0\}$ with $F(z_1)=F(z_2)=F(z_3)$. By the GMVP, each of the segments (z_1, z_2), (z_1, z_3), (z_2, z_3) must contain a zero of $F'(z)$. However these segments are all contained in $\Delta(0,\eta)$ and 0 is the only zero of $F'(z)$ in $\Delta(0,\eta)$. Thus $0 \in (z_1, z_2) \cap (z_2, z_3) \cap (z_1, z_3)$. This is impossible for distinct points z_1, z_2, z_3. It follows that $n=2$. Thus $F(z)$ is a two-to-one mapping on N and, if z_1, z_2 are distinct points of $N-\{0\}$, then

$$F(z_1)=F(z_2) \quad \text{implies that} \quad 0 \in (z_1, z_2), \quad \text{i.e.} \quad z_1/z_2<0. \tag{2}$$

Let l be any line through 0 and let H_1, H_2 be the two open half planes determined by l. Let $N_i=H_i \cap N$ $(i=1,2)$. It follows from (2) that $F(z)$ is univalent on each of N_1, N_2. Furthermore, for $i=1,2$, $F(N_i)=F(N)-\gamma$ where $\gamma=F(l \cap N)$ is a simple analytic arc with one endpoint 0. We can then define analytic functions $h_i\colon F(N)-\gamma \to N_i$ $(i=1,2)$ with $(h_i \circ F)(z)=z$ $(z \in N_i)$. Since $0 \notin N_i$ $(i=1,2)$, neither h_1 nor h_2 takes the values 0. Thus $h_1(w)/h_2(w)$ is analytic on $F(N)-\gamma$ and by (2), $h_1(w)/h_2(w)<0$ on $F(N)-\gamma$. Thus $h_1(w)/h_2(w) \equiv k$ (k some negative constant). Now let $\tilde{l}$ be a line through 0 and assume the acute angle formed by l and $\tilde{l}$ is less than $\pi/10$. Let $\tilde{H}_1$ and $\tilde{H}_2$ be the half planes determined by $\tilde{l}$, and labeled so that $\tilde{H}_1 \cap H_1$ is a sector of angle measure greater than $9\pi/10$. Analagous to the previous development, define $\tilde{N}_i=\tilde{H}_i \cap N$ $(i=1,2)$, $\tilde{\gamma}=F(N \cap \tilde{l})$ and $\tilde{h}_i\colon F(N)-\tilde{\gamma} \to \tilde{N}_i$ $(i=1,2)$ with $(\tilde{h}_i \circ F)(z)=z$ $(z \in \tilde{N}_i)$. Then

$$\tilde{h}_i\big|_{F(N_i \cap \tilde{N}_i)}=h_i\big|_{F(N_i \cap \tilde{N}_i)}$$

$(i=1,2)$, and it follows that $\tilde{h}_1(w)/\tilde{h}_2(w) \equiv k$ on $F(N)-\tilde{\gamma}$. Now let $z_0 \in N_1 \cap \tilde{N}_2$. Then there is a point $z_0' \in N_2 \cap \tilde{N}_1$ with $F(z_0)=F(z_0')$. We then have

$$k=\frac{z_0}{z_0'}=\frac{h_1(F(z_0))}{h_2(F(z_0))}=\frac{\tilde{h}_2(F(z_0))}{\tilde{h}_1(F(z_0))}=\frac{z_0'}{z_0}=\frac{1}{k}.$$

Hence $k=-1$.

From the above argument we may conclude that if z_1, z_2 are distinct points in N with $F(z_1)=F(z_2)$, then $z_1=-z_2$. Conversely, if $z_1=-z_2$, we must have $F(z_1)=F(-z_2)=F(z_2)$, showing that $F(z)$ is an even function. Thus

$$f'''(0)-A_1 g'''(0)=F'''(0)=0$$

and

$$f^{(5)}(0)-A_1 g^{(5)}(0)=F^{(5)}(0)=0.$$

Since $A_1=f'(0)/g'(0)$ and $a=0$ was chosen without loss of generality, we see that (ii) holds for all $z \in D$ at which $f'(z) \neq 0$. Since $g'(z)$ is never 0 on D, it follows that $f'''(z)/f'(z)$ and $f^{(5)}(z)/f'(z)$ have only removable singularities on D. Thus (ii) holds in D.

(ii)$\Rightarrow$(iii). We consider three cases

Case a. If $f'''(z)/f'(z) \equiv g'''(z)/g'(z) \equiv 0$ on D, then it follows that f and g are polynomials of degree one or two. Observe that $f^{(5)}(z)/f'(x) = g^{(5)}(z)/g'(z)$ is also satisfied by such f and g.

Case b. If $f'''(z)/f'(z) = g'''(z)/g'(z) \equiv k \neq 0$ on D, then it follows immediately that $f(z) = A\cos\alpha z + B\sin\alpha z + C$ and $g(z) = D\cos\alpha z + E\sin\alpha z + F$ where $|A| + |B| \neq 0$, $|D| + |E| \neq 0$ and $\alpha^2 = -k$. Note that $f^{(5)}(z)/f'(z) = g^{(5)}(z)/g'(z)$ is also true for such f and g.

Case c. If $f'''(z)/f'(z) = g'''(z)/g'(z) = u(z)$ where $u(z)$ is not constant on D, then since $g'(z) = 0$ we see that $u(z)$ is analytic on D. Thus

$$g'''(z) = g'(z)u(z)$$

and differentiation gives

$$g^{(5)}(z) = g'''(z)u(z) + 2g''(z)u'(z) + g'(z)u''(z).$$

Dividing by $g'(z)$ we have

$$\frac{g^{(5)}(z)}{g'(z)} = (u(z))^2 + \frac{2g''(z)}{g'(z)}u'(z) + u''(z).$$

Similarly,

$$\frac{f^{(5)}(z)}{f'(z)} = (u(z))^2 + \frac{2f''(z)}{f'(z)}u'(z) + u''(z).$$

Since $u'(z) \not\equiv 0$ in D it follows that

$$\frac{f''(z)}{f'(z)} = \frac{g''(z)}{g'(z)}$$

for all $z \in D$. Hence

$$\frac{d}{dz}\frac{f'(z)}{g'(z)} \equiv 0$$

and so,

$$\frac{f'(z)}{g'(z)} \equiv A,$$

Thus $f'(z) = Ag'(z)$ and $f(z) = Ag(z) + B$. □

As an immediate corollary we can characterise those analytic functions that satisfy the MVT on D.

Corollary 3. *Let $f(z)$ be analytic on a domain D. Then $f(z)$ satisfies the MVT on D (i.e. the GMVP with $g(z)=z$) if and only if $f(z)$ is a polynomial of degree at most* 2.

Remark 4. Since the arguments in the proof of Theorem 2 were all local, it is clear that the univalence hypothesis on g can be dropped. If we instead require only that $g(z)$ is not constant on D. then the proof of Theorem 2 shows that (ii) holds at all points where $f'(z)g'(z)\neq 0$. It follows that the equalities in (ii) actually hold in all of D as equalities between meromorphic functions. The equation

$$\frac{f(z_1)-f(z_2)}{g(z_1)-g(z_2)}=\frac{f'(c)}{g'(c)}$$

when $g(z_1)=g(z_2)$ $(z_1\neq z_2)$ can then be interpreted as saying that there is a point $c\in(z_1,z_2)$ such that the order of the pole of $(f(z_1)-f(z))/(g(z_1)-g(z))$ at z_2 is the same as the order of the pole of $f'(z)/g'(z)$ at c. Indeed this is easily checked for those functions $f(z)$ and $g(z)$ listed in (iii).

Remark 5. Although the condition $f^{(5)}(z)/f'(z)=g^{(5)}(z)/g'(z)$ was used in only one case of the proof of (ii)$\Rightarrow$(iii), the condition cannot be dropped. For example, it is easy to check that if $f(z)=z^4$ and $g(z)=z^{-1}$ are analytic on a domain D, then $f'''(z)/f'(z)=g'''(z)/g'(z)$. However $f(z)$ and $g(z)$ will not satisfy the GMVP on D. (Note that $f^{(5)}(z)/f'(z)\neq g^{(5)}(z)/g'(z)$ on D).

Remark 6. When discussing the MVT it seems appropriate to mention Rolle's Theorem. We will say that a non constant function $f(z)$ analytic on D satisfies the Rolle Property (RP) on D if $[z_1,z_2]\subseteq D$ $(z_1\neq z_2)$ and $f(z_1)=f(z_2)$ imply the existence of a point $c\in(z_1,z_2)$ such that $f'(c)=0$. It is clear that if $f'(z_0)\neq 0$ $(z_0\in D)$ then $f(z)$ satisfies RP in a neighbourhood of z_0. This illustrates a major difference between RP and the MVP. A function may satisfy RP locally on a domain D without satisfying RP on the whole domain D, while if functions $f(z)$ and $g(z)$ satisfy the GMVP on any open subset of D, then they satisfy the GMVP on D. However, since only local arguments were used in the proof of Theorem 2, the same arguments can be used to prove the following result.

Theorem 7. *If $f(z)$ is analytic in a domain D, $f'(z_0)=0$ and $f(z)$ satisfies the RP on D, then $f(z+z_0)$ is an even function in a neighbourhood of* 0. *Furthermore, if f is not identically constant, then $f''(z_0)\neq 0$.*

A somewhat longer argument can be used to prove the following more informative result.

Theorem 8. *If $f(z)$ is analytic, nonconstant and satisfies RP on a convex domain D, then $f(z)$ takes no value with multiplicity greater than two in D.*

Proof. Suppose $f(z)$ takes the value a with multiplicity at least 3. Then three cases might arise:

Case 1. There is a point $z_0 \in D$ with $f(z_0)=a$ and $f'(z_0)=f''(z_0)=0$.

Case 2. There are distinct points $z_0, z_1 \in D$ with $f(z_0)=f(z_1)=a$ and $f'(z_0)=0$.

Case 3. There are distinct points $z_0, z_1, z_2 \in D$ with $f(z_0)=f(z_1)=f(z_2)=a$.

In Case 1 we arrive at a contradiction by applying Theorem 7. In Cases 2 and 3 it can be shown that $f'(z) \equiv 0$ on D, contradicting the fact that f is not identically constant.

If Case 2 holds, then by the Open Mapping Theorem and Theorem 7 we may find a neighbourhood N of z_0 such that $f(z+z_0)$ is even in $N-z_0=\{z-z_0: z \in N\}$ and such that $f(z)$ is a two-to-one mapping on N, with $f'(z) \neq 0$ on $N-\{z_0\}$. Depending on whether $f'(z_1) \neq 0$ or $f'(z_1)=0$ we may find a neighbourhood N' of z_1 on which $f(z)$ is a one-to-one mapping or a two-to-one mapping. In either case both $f(N)$ and $f(N')$ are neighbourhoods of a. Let l be the line determined by z_0 and z_1. Let H_1 and H_2 be the open half planes determined by l and let $N_i=H_i \cap N$, $N'_i=H_i \cap N'$ $(i=1,2)$. Since $f(N_1)$, $f(N'_1)$ are both open, since $\overline{f(N_1)}$ contains a neighbourhood of a and since $a \in \overline{f(N'_1)}$, it follows that $f(N_1) \cap f(N'_1)$ is a non empty open set with $a \in \overline{f(N_1) \cap f(N'_1)}$. Thus we can find sequences $\{\xi_k\} \subseteq N_1$ and $\{\eta_k\} \subseteq N'_1$ with the following properties:

(i) $\lim_{k \to \infty} \xi_k = z_0$ and $\lim_{k \to \infty} \eta_k = z_1$

(ii) $f(\xi_k)=f(\eta_k)\,(k=1,2,\ldots)$.

Since N_1 and N_2 are open, we may assume, by a subsequence argument that the segments $\{[\xi_k, \eta_k]\}$ are pairwise disjoint. By (ii) and RP, there is a point $\rho_k \in (\xi_k, \eta_k)$ with $f'(\rho_k)=0$ $(k=1,2,\ldots)$. Since the segments $\{[\xi_k, \eta_k]\}$ are disjoint, the ρ_k's produced are distinct. By another subsequence argument we may assume $\lim_{k \to \infty} \rho_k = \rho$ exists where $\rho \in [z_0, z_1] \subseteq D$. But then the Identity Theorem [4, p. 209] implies $f'(z) \equiv 0$ on D, contradicting the assumption that f is not identically constant. Thus Case 2 cannot occur.

The argument that Case 3 cannot occur is similar and is left to the reader.

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IOWA STATE UNIVERSITY
AMES, IOWA 50010