ON THE NUMBER OF ASSOCIATIVE TRIPLES IN AN ALGEBRA OF n ELEMENTS

P. J. BROCKWELL

1. Introduction. Consider a set of n elements $\alpha_1, \ldots, \alpha_n$ (denoted by S) and the set of all possible multiplication tables on these elements. The total number of such tables is clearly n^{n^2} and each table can be represented by a square matrix $[\mu_{ij}]$ where μ_{ij} is the product $\alpha_i \alpha_j$ ($\mu_{ij} \in S$, $i = 1, \ldots, n$; $j = 1, \ldots, n$). The triple $(\alpha_i, \alpha_j, \alpha_k)$ is said to be associative if the following equation is satisfied:

$$(\alpha_i \alpha_j)\alpha_k = \alpha_i(\alpha_j \alpha_k). \tag{1.1}$$

The purpose of this paper is to examine the function $\phi_n(m)$, defined as the number of $n \times n$ tables in which exactly m triples are associative

$$(m = 0, 1, \ldots, n^3).$$

It has already been shown by Climescu **(1)** (and independently by Straus and Wilf **(2)** that $\phi_n(m)$ has the property

$$\phi_n(m) > 0 \qquad (m = 0, 1, \ldots, n^3; n \geqslant 3). \tag{1.2}$$

In the present paper a method will be developed for determining the moments

$$M_n(k) = \sum_{m=0}^{n^3} m^k \phi_n(m) \tag{1.3}$$

and explicit solutions will be given for $M_n(1)$ and $M_n(2)$.

It is convenient to introduce a random variable X_n, defined as the number of associative triples in a multiplication table selected at random from the set of all $n \times n$ tables. Then if $p_n(m)$ denotes the probability that X_n is equal to m, we have

$$p_n(m) = n^{-n^2}\phi_n(m) \qquad (m = 0, 1, 2, \ldots, n^3). \tag{1.4}$$

The moments of the random variable X_n are therefore given by

$$\mathscr{E} X_n^k = n^{-n^2} M_n(k). \tag{1.5}$$

It has been conjectured by Straus and Wilf **(2)**, on the basis of a computer study programmed by Mrs. Nancy Clark at Argonne National Laboratory (to appear), that the mean and standard deviation, μ_n and σ_n, of X_n are asymptotically n^2 and n respectively and that the distribution of the random variable $\sigma_n^{-1}(X_n - \mu_n)$ is asymptotically normal.

Received May 20, 1966. Work performed in part under the auspices of the USAEC.

https://doi.org/10.4153/CJM-1967-079-3 Published online by Cambridge University Press

We shall show that the conjectures of Straus and Wilf regarding the mean and standard deviation of X_n are in fact true. The asymptotic normality remains an open question; however, the conjecture is supported by the asymptotic vanishing of the third moment of $\sigma_n^{-1}(X_n - \mu_n)$.

2. The expected number of associative triples. We select a multiplication table at random from the set of n^{n^2} tables and introduce the n^3 random variables defined by

$$\epsilon_{i_1 i_2 i_3} = \begin{cases} 1 & \text{if } (\alpha_{i_1} \alpha_{i_2})\alpha_{i_3} = \alpha_{i_1}(\alpha_{i_2} \alpha_{i_3}), \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Then

$$X_n = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \sum_{i_3=1}^{n} \epsilon_{i_1 i_2 i_3},$$

and the expected value of X_n is given by

$$\mathscr{E}X_n = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \sum_{i_3=1}^{n} P(i_1, i_2, i_3) \tag{2.2}$$

where $P(i_1, i_2, i_3)$ is the probability that the triple $(\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3})$ is associative, i.e. that $\epsilon_{i_1 i_2 i_3} = 1$.

To determine the probabilities $P(i_1, i_2, i_3)$ we proceed as follows. Let $\pi(i_1, i_2, i_3; a, b)$ be the probability that in a randomly selected multiplication table

$$\alpha_{i_1} \alpha_{i_2} = \alpha_a \quad \text{and} \quad \alpha_{i_2} \alpha_{i_3} = \alpha_b. \tag{2.3}$$

If $P(i_1, i_2, i_3 | a, b)$ denotes the conditional probability that $(\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3})$ is associative given that Conditions (2.3) are satisfied, then

$$P(i_1, i_2, i_3) = \sum_{a=1}^{n} \sum_{b=1}^{n} P(i_1, i_2, i_3 | a, b) \pi(i_1, i_2, i_3; a, b). \tag{2.4}$$

For any particular triple $(\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3})$ the probabilities $\pi(i_1, i_2, i_3; a, b)$ and the conditional probabilities $P(i_1, i_2, i_3 | a, b)$ take a very simple form and the associativity probabilities $P(i_1, i_2, i_3)$ can be found from (2.4) by direct summation.

Consider first the case where i_1, i_2, i_3 are all different, say $i_1 = i$, $i_2 = j$, and $i_3 = k$. Then

$$\pi(i, j, k; a, b) = n^{-2} \qquad \text{for all } a, b$$

and

$$\begin{aligned} P(i, j, k | a, b) &= \text{Prob}[(\alpha_i \alpha_j)\alpha_k = \alpha_i(\alpha_j \alpha_k) | \alpha_i \alpha_j = \alpha_a, \alpha_j \alpha_k = \alpha_b] \\ &= \text{Prob}[\alpha_a \alpha_k = \alpha_i \alpha_b | \alpha_i \alpha_j = \alpha_a, \alpha_j \alpha_k = \alpha_b] \\ &= \begin{cases} 1 & \text{if } (a, b) = (i, k), \\ 1 & \text{if } (a, b) = (j, j), \\ n^{-1} & \text{otherwise.} \end{cases} \end{aligned}$$

https://doi.org/10.4153/CJM-1967-079-3 Published online by Cambridge University Press

Substituting in (2.4) and summing, it is found that

$$P(i, j, k) = n^{-1} + 2n^{-2} - 2n^{-3} \tag{2.5}$$

and there are $n(n-1)(n-2)$ probabilities of this form appearing in the sum (2.2).

An argument analogous to that of the preceding paragraph shows that

$$P(i, i, k) = P(i, k, i) = P(k, i, i) = n^{-1} + 2n^{-2} - 2n^{-3}, \tag{2.6}$$

and terms of each of these three types appear $n(n-1)$ times in the sum (2.2).

In the case when all three indices are the same, say $i_1 = i_2 = i_3 = i$, it is found that

$$\pi(i, i, i; a, b) = \begin{cases} n^{-1} & \text{if } a = b, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$P(i, i, i|\, a, b) = \begin{cases} 1 & \text{if } a = b = i, \\ n^{-1} & \text{if } a = b \neq i. \end{cases}$$

Hence from (2.4)

$$P(i, i, i) = 2n^{-1} - n^{-2} \tag{2.7}$$

and there are n such terms in the sum (2.2).

From equations (2.2), (2.5), (2.6), and (2.7) we find by direct summation that

$$\mathscr{E} X_n = n^2 + 2n - 1 - 3n^{-1} + 2n^{-2}. \tag{2.8}$$

The corresponding expression for the first moment $M_n(1)$ follows immediately from (1.5).

3. The variance of the number of associative triples. Using the notation of §2, the variance of the number of associative triples in a randomly selected multiplication table is given by

$$\operatorname{Var} X_n = \mathscr{E} X_n^2 - (\mathscr{E} X_n)^2 \tag{3.1}$$

where

$$\mathscr{E} X_n^2 = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \ldots \sum_{i_6=1}^{n} P(i_1, i_2, \ldots, i_6) \tag{3.2}$$

and $P(i_1, \ldots, i_6)$ is the probability that in a randomly selected multiplication table the triples $(\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3})$ and $(\alpha_{i_4}, \alpha_{i_5}, \alpha_{i_6})$ are both associative, i.e. the probability that $\epsilon_{i_1 i_2 i_3} \epsilon_{i_4 i_5 i_6} = 1$.

The probabilities $P(i_1, \ldots, i_6)$ are determined using a method analogous to that of §2. Thus we define $\pi(i_1, \ldots, i_6; a, b, c, d)$ to be the probability that in a randomly selected multiplication table

$$\alpha_{i_1} \alpha_{i_2} = \alpha_a, \qquad \alpha_{i_2} \alpha_{i_3} = \alpha_b, \qquad \alpha_{i_4} \alpha_{i_5} = \alpha_c, \qquad \alpha_{i_5} \alpha_{i_6} = \alpha_d. \tag{3.3}$$

If $P(i, \ldots, i_6 | \, a, b, c, d)$ denotes the conditional probability that

$$\epsilon_{i_1 i_2 i_3} \epsilon_{i_4 i_5 i_6} = 1$$

given that Conditions (3.3) are satisfied, then

$$(3.4) \qquad P(i_1, \ldots, i_6) = \sum_{a=1}^{n} \sum_{b=1}^{n} \sum_{c=1}^{n} \sum_{d=1}^{n} P(i_1, \ldots, i_6 | a, b, c, d) \times \pi(i_1, \ldots, i_6; a, b, c, d).$$

For any particular pair of triples $(\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3})$ and $(\alpha_{i_4}, \alpha_{i_5}, \alpha_{i_6})$ the probabilities $\pi(i_1, \ldots, i_6; a, b, c, d)$ and the conditional probabilities

$$P(i_1, \ldots, i_6 | \, a, b, c, d)$$

again take a very simple form and the unconditional probability $P(i_1, \ldots, i_6)$ can be found for each choice of $i_1, \ldots, i_6$ from (3.4).

It is convenient from an algebraic point of view to subdivide the probabilities $P(i_1, \ldots, i_6)$ into classes as follows. Class 1 consists of the

$$n(n-1)(n-2)(n-3)(n-4)(n-5)$$

probabilities in which $i_1, \ldots, i_6$ are all different, e.g. $P(i, j, k, l, m, p)$. Class 2 contains those in which only five of $i_1, \ldots, i_6$ are distinct, e.g. $P(i, i, k, l, m, p)$, and there are $15n(n-1)(n-2)(n-3)(n-4)$ of these. Similarly class 3 consists of those with four distinct indices, e.g. $P(i, i, i, l, m, p)$, $P(i, i, k, k, m, p)$, and there are $65n(n-1)(n-2)(n-3)$ of these. Class 4 contains the $90n(n-1)(n-2)$ probabilities having three distinct indices, class 5 contains the $31n(n-1)$ probabilities with two distinct indices, and class 6 contains the n probabilities in which all indices are the same.

In the Appendix the probabilities $P(i_1, \ldots, i_6)$ have been evaluated for all possible distinct choices of the indices $i_1, \ldots, i_6$ and are set out in classes together with the frequency with which each of the probabilities appears in the sum (3.2). This sum can therefore be evaluated directly from the table to give $\mathscr{E} X_n^2$.

To show how the table was constructed we take as an example the case where $i_1, \ldots, i_6$ are all different, say $i_1 = i$, $i_2 = j$, $i_3 = k$, $i_4 = l$, $i_5 = m$, and $i_6 = p$. In this case, by simple enumeration,

$$(3.5) \qquad \pi(i, j, k, l, m, p; a, b, c, d) = n^{-4} \qquad (a, b, c, d = 1, \ldots, n).$$

Consider now the set of all multiplication tables in which

$$(3.6) \qquad \alpha_i \alpha_j = \alpha_a, \qquad \alpha_j \alpha_k = \alpha_b, \qquad \alpha_l \alpha_m = \alpha_c, \qquad \text{and } \alpha_m \alpha_p = \alpha_d.$$

The conditional probability $P(i, j, k, l, m, p | a, b, c, d)$ is the proportion of these tables in which the triples $(\alpha_i, \alpha_j, \alpha_k)$ and $(\alpha_l, \alpha_m, \alpha_p)$ are both associative, i.e. in which the relations

$$(3.7) \qquad \alpha_a \alpha_k = \alpha_i \alpha_b,$$

$$(3.8) \qquad \alpha_c \alpha_p = \alpha_l \alpha_d,$$

https://doi.org/10.4153/CJM-1967-079-3 Published online by Cambridge University Press

are both satisfied. In evaluating this probability for each choice of the quadruple (a, b, c, d) there are several possibilities to distinguish:

(a) The associativity conditions (3.7) and (3.8) may be contradictory or impossible to satisfy for certain quadruples (a, b, c, d),

$$\text{e.g. } (a, b, c, d) = (l, j, m, k);$$

in such cases $P(i, j, k, l, m, p| a, b, c, d) = 0$.

(b) Both conditions may be satisfied identically,

$$\text{e.g. } (a, b, c, d) = (i, k, l, p);$$

in this case $P(i, j, k, l, m, p| a, b, c, d) = 1$.

(c) The conditions may be equivalent, not impossible to satisfy, and not satisfied identically,

$$\text{e.g. } (a, b, c, d) = (l, p, i, k);$$

in this case $P(i, j, k, l, m, p| a, b, c, d) = n^{-1}$.

(d) One of the conditions may be satisfied identically while the other is neither contradictory nor satisfied identically,

$$\text{e.g. } (a, b, c, d) = (i, k, l, m);$$

in this case again $P(i, j, k, l, m, p| a, b, c, d) = n^{-1}$.

(e) In all cases except those already mentioned

$$P(i, j, k, l, m, p|a, b, c, d) = n^{-2}.$$

Thus, by examining Conditions (3.7) and (3.8) for each possible choice of the quadruple (a, b, c, d), the conditional probabilities $P(i, j, k, l, m, p| a, b, c, d)$ can be determined and the unconditional probability $P(i, j, k, l, m, p)$ is then found from (3.4) and (3.5).

The process is repeated for all possible choices of the indices $(i_1, \ldots, i_6)$ and the results are as tabulated in the Appendix. Using the table to form the sum (3.2) we find that

$$\text{(3.9)} \qquad \mathscr{E} X_n^2 = n^4 + 4n^3 + 3n^2 + 15n - 44 - 175n^{-1} + 507n^{-2} - 190n^{-3} - 472n^{-4} + 352n^{-5}.$$

The corresponding moment $M_n(2)$ is found immediately from (1.5). From (3.1) the variance of X_n is given by

$$\text{(3.10)} \qquad \operatorname{Var} X_n = n^2 + 25n - 37 - 189n^{-1} + 502n^{-2} - 178n^{-3} - 476n^{-4} + 352n^{-5}.$$

4. The third moment of X_n. The third moment of X_n about its mean $\mu_n = \mathscr{E} X_n$ is given by

$$\text{(4.1)} \qquad \mathscr{E}(X_n - \mu_n)^3 = \mathscr{E} X_n^3 - 3\mu_n \mathscr{E} X_n^2 + 2\mu_n^3$$

https://doi.org/10.4153/CJM-1967-079-3 Published online by Cambridge University Press

where

$$\mathscr{E}X_n^{\,3} = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_9=1}^{n} P(i_1, i_2, \ldots, i_9) \tag{4.2}$$

and $P(i_1, \ldots, i_9)$ is the probability that in a randomly selected multiplication table the three triples $(\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3})$, $(\alpha_{i_4}, \alpha_{i_5}, \alpha_{i_6})$, and $(\alpha_{i_7}, \alpha_{i_8}, \alpha_{i_9})$ are all associative, i.e. that $\epsilon_{i_1 i_2 i_3}\, \epsilon_{i_4 i_5 i_6}\, \epsilon_{i_7 i_8 i_9} = 1$.

The n^9 probabilities $P(i_1, \ldots, i_9)$ can be determined just as in §§2, 3 from the relation

$$P(i_1, \ldots, i_9) = \sum_{a=1}^{n} \sum_{b=1}^{n} \sum_{c=1}^{n} \sum_{d=1}^{n} \sum_{e=1}^{n} \sum_{f=1}^{n} P(i_1, \ldots, i_9 |\, a, b, c, d, e, f) \times \pi(i_1, \ldots, i_9; a, b, c, d, e, f) \tag{4.3}$$

where $\pi(i_1, \ldots, i_9; a, b, c, d, e, f)$ is the probability that in a randomly selected multiplication table

$$\begin{cases} \alpha_{i_1}\alpha_{i_2} = \alpha_a, & \alpha_{i_2}\alpha_{i_3} = \alpha_b, \\ \alpha_{i_4}\alpha_{i_5} = \alpha_c, & \alpha_{i_5}\alpha_{i_6} = \alpha_d, \\ \alpha_{i_7}\alpha_{i_8} = \alpha_e, & \alpha_{i_8}\alpha_{i_9} = \alpha_f, \end{cases} \tag{4.4}$$

and $P(i_1, \ldots, i_9 |\, a, b, c, d, e, f)$ denotes the conditional probability that $\epsilon_{i_1 i_2 i_3}\, \epsilon_{i_4 i_5 i_6}\, \epsilon_{i_7 i_8 i_9} = 1$ given that Conditions (4.4) are satisfied. These conditional probabilities and the probabilities π again take a simple form and can be determined as in §§2 and 3.

Proceeding in this way, $\mathscr{E}X_n^3$ was determined to within terms of order n^2 and it was found that

$$\mathscr{E}X_n^3 = n^6 + 6n^5 + 12n^4 + 68n^3 + O(n^2). \tag{4.5}$$

Hence from (2.8), (3.9), (4.1), and (4.5)

$$\mathscr{E}(X_n - \mu_n)^3 = O(n^2). \tag{4.6}$$

This shows that the third moment of $\sigma_n^{-1}(X_n - \mu_n)$ converges to the third moment of a normal distribution (i.e. zero) as n tends to infinity, which supports the conjecture of Straus and Wilf regarding the asymptotic normality of the distribution.

Appendix. The following table shows the probabilities $P(i_1, i_2, \ldots, i_6)$ of §3 for each distinct choice of the indices $i_1, i_2, \ldots, i_6$. The indices i_1, i_2, i_3, and i_4, i_5, i_6 are tabulated together with the coefficients of $n^{-1}, n^{-2}, \ldots, n^{-6}$ in the corresponding probability $P(i_1, \ldots, i_6)$. Each entry in class k $(k = 1, \ldots, 6)$ appears $n(n-1)\ldots(n-6+k)$ times in the sum (3.2).

Indices		Coefficients of					Indices		Coefficients of				
	n^{-1}	n^{-2}	n^{-3}	n^{-4}	n^{-5}	n^{-6}		n^{-1}	n^{-2}	n^{-3}	n^{-4}	n^{-5}	n^{-6}
CLASS 1													
ijk lmp	0	1	4	0	−7	1							
CLASS 2													
iij klm	0	1	4	0	−6	0	*ijk lmj*	0	1	4	1	−10	4
iji klm	0	1	4	0	−7	1	*ijk klm*	0	1	4	0	−6	1
ijk ilm	0	1	4	4	−18	6	*ijk lkm*	0	1	4	1	−10	4
ijk lim	0	1	4	1	−10	4	*ijk lmk*	0	1	4	4	−18	6
ijk lmi	0	1	4	0	−6	1	*ijk llm*	0	1	4	0	−6	0
ijj klm	0	1	4	0	−6	0	*ijk lml*	0	1	4	0	−7	1
ijk jlm	0	1	4	1	−10	4	*ijk lmm*	0	1	4	0	−6	0
ijk ljm	0	1	4	0	−7	1							
CLASS 3													
iii jkl	0	2	3	−6	2	0	*jik lki*	0	1	4	2	−14	9
iij ikl	0	1	4	6	−24	12	*iij klk*	0	1	4	0	−6	0
iij kil	0	1	4	1	−9	3	*iij lkk*	0	1	4	0	−5	−2
iij kli	0	1	4	2	−10	2	*iji lkk*	0	1	4	0	−6	0
iji ikl	0	1	4	4	−17	6	*ikj ilk*	0	1	4	5	−21	12
iji kil	0	1	4	3	−17	10	*ikj lik*	0	1	6	−8	2	0
iji kli	0	1	4	4	−17	6	*ijk lik*	0	1	4	5	−21	12
jii ikl	0	1	4	2	−10	2	*ikj kli*	0	1	4	1	−9	4
jii kil	0	1	4	1	−9	3	*ikj lki*	0	1	4	0	−6	1
jii kli	0	1	4	6	−24	12	*ijk lki*	0	1	4	1	−9	4
jkl iii	0	2	3	−6	2	0	*iik kjl*	0	1	4	0	−5	0
ikl iij	0	1	4	6	−24	12	*iik jkl*	0	1	4	2	−13	6
kil iij	0	1	4	1	−9	3	*iik jlk*	0	1	4	5	−20	12
kli iij	0	1	4	2	−10	2	*iki kjl*	0	1	4	1	−10	4
ikl iji	0	1	4	4	−17	6	*iki jkl*	0	1	4	0	−7	1
kil iji	0	1	4	3	−17	10	*iki jlk*	0	1	4	1	−10	4
kli iji	0	1	4	4	−17	6	*kii kjl*	0	1	4	5	−20	12
ikl jii	0	1	4	2	−10	2	*kii jkl*	0	1	4	2	−13	6
kil jii	0	1	4	1	−9	3	*kii jlk*	0	1	4	0	−5	0
kli jii	0	1	4	6	−24	12	*klk iij*	0	1	4	0	−6	0
iij kkl	0	1	4	0	−5	−1	*lkk iij*	0	1	4	0	−5	−2
iji klk	0	1	4	0	−7	1	*lkk iji*	0	1	4	0	−6	0
jii lkk	0	1	4	0	−5	1	*ilk ikj*	0	1	4	5	−21	12
ikj ikl	0	1	7	−9	2	0	*lik ikj*	0	1	6	−8	2	0
ijk ilk	0	1	5	12	−49	30	*lik ijk*	0	1	4	5	−21	12
jik lik	0	1	7	−9	2	0	*kli ikj*	0	1	4	1	−9	4
ikj kil	0	1	4	2	−14	9	*lki ikj*	0	1	4	0	−6	1
ijk kli	0	1	4	0	−5	1	*lki ijk*	0	1	4	1	−9	4
kjl iik	0	1	4	0	−5	0	*jlk iki*	0	1	4	1	−10	4
jkl iik	0	1	4	2	−13	6	*kjl kii*	0	1	4	5	−20	12
jlk iik	0	1	4	5	−20	12	*jkl kii*	0	1	4	2	−13	6
kjl iki	0	1	4	1	−10	4	*jlk kii*	0	1	4	0	−5	0
jkl iki	0	1	4	0	−7	1							
CLASS 4													
iii ijk	0	2	4	−8	3	0	*jjk iii*	0	2	3	−6	2	0
iii jik	0	2	3	−6	2	0	*jkj iii*	0	2	3	−5	1	0
iii jki	0	2	4	−8	3	0	*kjj iii*	0	2	3	−6	2	0
iij iki	0	1	4	9	−34	24	*jik iij*	0	1	4	1	−8	3
iij kii	0	1	8	−12	4	0	*jki iij*	0	1	4	2	−8	2
iji kii	0	1	4	9	−34	24	*kji iij*	0	1	4	4	−18	12
ijk iii	0	2	4	−8	3	0	*kij iij*	0	1	7	−9	2	0
jik iii	0	2	3	−6	2	0	*ikj iij*	0	1	5	16	−64	60
jki iii	0	2	4	−8	3	0	*ijk iij*	0	1	7	−10	3	0

Indices	Coefficients of n^{-1}	n^{-2}	n^{-3}	n^{-4}	n^{-5}	n^{-6}	Indices	Coefficients of n^{-1}	n^{-2}	n^{-3}	n^{-4}	n^{-5}	n^{-6}
iki iij	0	1	4	9	−34	24	*jik iji*	0	1	6	−8	2	0
kii iij	0	1	8	−12	4	0	*jki iji*	0	1	4	5	−20	12
kii iji	0	1	4	9	−34	24	*kji iji*	0	1	7	−9	2	0
iij iik	0	1	11	−22	12	0	*kij iji*	0	1	6	−8	2	0
iji iki	0	1	5	14	−49	30	*ikj iji*	0	1	4	5	−20	12
jii kii	0	1	11	−22	12	0	*ijk iji*	0	1	7	−9	2	0
iii jjk	0	2	3	−6	2	0	*jik jii*	0	1	7	−9	2	0
iii jkj	0	2	3	−5	1	0	*jki jii*	0	1	5	16	−64	60
iii kjj	0	2	3	−6	2	0	*kji jii*	0	1	7	−10	3	0
iij jik	0	1	4	1	−8	3	*kij jii*	0	1	4	1	−8	3
iij jki	0	1	4	2	−8	2	*ikj jii*	0	1	4	2	−8	2
iij kji	0	1	4	4	−18	12	*ijk jii*	0	1	4	4	−18	12
iij kij	0	1	7	−9	2	0	*jij iik*	0	1	4	1	−9	3
iij ikj	0	1	5	16	−64	60	*ijj iik*	0	1	4	9	−34	24
iij ijk	0	1	7	−10	3	0	*jji iik*	0	1	4	4	−14	4
iji jik	0	1	6	−8	2	0	*ijj iki*	0	1	4	5	−19	12
iji jki	0	1	4	5	−20	12	*jij iki*	0	1	4	4	−18	10
iji kji	0	1	7	−9	2	0	*jji iki*	0	1	4	5	−19	12
iji kij	0	1	6	−8	2	0	*ijj kii*	0	1	4	4	−14	4
iji ikj	0	1	4	5	−20	12	*jij kii*	0	1	4	1	−9	3
iji ijk	0	1	7	−9	2	0	*jji kii*	0	1	4	9	−34	24
jii jik	0	1	7	−9	2	0	*iij kjk*	0	1	4	2	−13	6
jii jki	0	1	5	16	−64	60	*iij jkk*	0	1	4	0	−3	−2
jii kji	0	1	7	−10	3	0	*iji jkk*	0	1	4	2	−13	6
jii kij	0	1	4	1	−8	3	*kjk iij*	0	1	4	2	−13	6
jii ikj	0	1	4	2	−8	2	*jkk iij*	0	1	4	0	−3	−2
jii ijk	0	1	4	4	−18	12	*jkk iji*	0	1	4	2	−13	6
iik jij	0	1	4	1	−9	3	*ijk ijk*	1	2	−2	0	0	0
iik ijj	0	1	4	9	−34	24	*ijk jki*	0	1	6	−7	1	0
iik jji	0	1	4	4	−14	4	*ijk kij*	0	1	6	−7	1	0
iki ijj	0	1	4	5	−19	12	*ijk jik*	0	1	4	8	−32	24
iki jij	0	1	4	4	−18	10	*ijk ikj*	0	1	4	8	−32	24
iki jji	0	1	4	5	−19	12	*ijk kji*	0	1	4	0	−5	1
kii ijj	0	1	4	4	−14	4	*iij kkj*	0	1	4	8	−30	24
kii jij	0	1	4	1	−9	3	*iji kjk*	0	1	4	0	−7	1
kii jji	0	1	4	9	−34	24	*ijj ikk*	0	1	4	8	−30	24

CLASS 5

Indices	n^{-1}	n^{-2}	n^{-3}	n^{-4}	n^{-5}	n^{-6}	Indices	n^{-1}	n^{-2}	n^{-3}	n^{-4}	n^{-5}	n^{-6}
iii iij	0	4	−3	0	0	0	*jii iji*	0	1	8	−12	4	0
iii iji	0	2	8	−17	10	0	*jii iij*	0	1	8	−8	0	0
iii jii	0	4	−3	0	0	0	*iji iji*	1	2	−2	0	0	0
iij iii	0	4	−3	0	0	0	*iij iij*	1	2	−2	0	0	0
iji iii	0	2	8	−17	10	0	*jii jii*	1	2	−2	0	0	0
jii iii	0	4	−3	0	0	0	*iii jjj*	0	4	−3	0	0	0
iii ijj	0	2	5	−10	4	0	*iij jji*	0	1	4	8	−24	16
iii jij	0	2	5	−10	4	0	*iji jij*	0	3	0	−2	0	0
iii jji	0	2	5	−10	4	0	*jii ijj*	0	1	4	8	−24	16
ijj iii	0	2	5	−10	4	0	*iij jij*	0	1	7	−9	2	0
jij iii	0	2	5	−10	4	0	*iij ijj*	0	1	11	−19	10	0
jji iii	0	2	5	−10	4	0	*iji ijj*	0	1	7	−9	2	0
iji iij	0	1	8	−12	4	0	*jij iij*	0	1	7	−9	2	0
iji jii	0	1	8	−12	4	0	*ijj iij*	0	1	11	−19	10	0
iij jii	0	1	8	−8	0	0	*ijj iji*	0	1	7	−9	2	0
iij iji	0	1	8	−12	4	0							

CLASS 6

Indices	n^{-1}	n^{-2}	n^{-3}	n^{-4}	n^{-5}	n^{-6}
iii iii	2	−1	0	0	0	0

https://doi.org/10.4153/CJM-1967-079-3 Published online by Cambridge University Press

Acknowledgments. I am indebted to Professor E. G. Straus, Professor H. S. Wilf, and Dr. R. F. King for their comments on the manuscript and to Professors Straus and Wilf for permission to refer to their unpublished work. I also wish to thank the United States Atomic Energy Commission and The Australian National University for the financial support with which this work was carried out.

References

1. A. C. Climescu, *Etudes sur la théorie des systèmes multiplicatifs uniformes. I—L'indice de non-associativité*, Bull. Ecole Polytech. Jassy, *2* (1947), 347–371.
2. E. G. Straus and H. S. Wilf, *Combinatorial aspects of the associative law of arithmetic* (unpublished).

Argonne National Laboratory,
Argonne, Illinois