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INVERTIBILITY AND SPECTRA OF OPERATORS ON TENSOR PRODUCTS OF HILBERT SPACES

M. I. GIL'
Department of Mathematics, Ben Gurion University of the Negev, P.O. Box 653, Beer-Sheva 84105, Israel
e-mail: gilmi@black.bgu.ac.il

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Abstract. A class of operators on a tensor product of separable Hilbert spaces is considered. It contains various traditional operators. Invertibility, positive invertibility conditions and estimates for the norm of the resolvents are established. In addition, bounds for the spectrum are suggested. Applications to partial integral and integro-differential operators are discussed.

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1. Introduction and statement of the main result. Operators on tensor products of Hilbert spaces arise in various problems of pure and applied mathematics ([**3**], [**11**], and references therein). In many applications, for example in numerical mathematics and stability analysis, conditions for the invertibility and bounds for the spectra of operators on tensor products are very important. But to the best of our knowledge, they have not been investigated in the literature.

In the present paper we consider a class of linear operators on tensor products of Hilbert spaces. It contains various traditional operators. We derive the norm of the resolvent, conditions for the invertibility and positive invertibility, as well as bounds for the spectra. In particular, we suggest estimates for the spectral radius.

A few words about the contents. In this section we formulate the main result of the paper on the invertibility conditions of considered operators. The proof of this result is divided into a series of lemmas which are presented in Sections 2 and 3.

In Section 4, an estimate for the norm of the resolvent is established. By that estimate we investigate bounds for the spectrum. In Sections 5 and 6 we specialize our results in the cases of Hilbert-Schmidt operators and Neumann-Schatten ones, respectively. In Section 7 we suggest the conditions that provide the positive invertibility. In Section 8 we discuss applications of our results to partial integral operators and integro-differential operators.

Let E_1 and E_2 be separable Hilbert spaces with the scalar products $\langle .\, , . \rangle_1$ and $\langle .\, , . \rangle_2$, respectively and norms $\|.\|_j = \sqrt{\langle .\, , . \rangle_j}$ $(j = 1, 2)$. Let $H = E_1 \otimes E_2$ be a tensor product of E_1 and E_2 with the scalar product

$$\langle h, h \rangle_H = \langle h_1, h_1 \rangle_1 \ \langle h_2, h_2 \rangle_2 \ \ (h = h_1 \otimes h_2;\ h_1 \in E_1, h_2 \in E_1).$$

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For a linear operator $A, \sigma(A)$ is the spectrum, $Dom\ (A)$ is the domain, $r_s(A)$ denotes the spectral radius, $\alpha\ (A) = \sup\ Re\ \sigma(A)$ and

$$\rho(A, \lambda) := \inf_{t \in \sigma(A)} |t - \lambda|$$

is the distance between $\sigma(A)$ and a $\lambda \in \mathbf{C}$.

A linear operator V is said to be *quasinilpotent if* $\sigma(V) = \{0\}$. V is called a *Volterra operator* if it is quasinilpotent and compact. In addition, $I = I_H$ and I_j mean the unit operator in H and E_j, respectively.

Recall that *a maximal resolution of the identity* (*m.r.i.*) $\tilde{P}(t)$ $(-\infty \leq t \leq \infty)$ is a left continuous orthogonal resolution of the identity, such that any gap $\tilde{P}(t_0 + 0) - \tilde{P}(t_0)$ of $\tilde{P}(t)$ (if it exists) is one-dimensional (cf. the books by Brodskii [**2**], Gohberg and Krein [**7**] and Gil' [**4**, p. 69]).

Let us consider operators of the type

$$A = D + (V_1^+ + V_1^-) \otimes I_2 + I_1 \otimes (V_2^+ + V_2^-) \tag{1.1}$$

where D is a normal generally unbounded operator and $V_j^{\pm}$ are Volterra operators acting in E_j and having the properties

$$P_j(t) V_j^+ P_j(t) = V_j^+ P_j(t);\ P_j(t) V_j^- P_j(t) = P_j(t) V_j^- \quad (t \in \mathbf{R}) \tag{1.2}$$

for m.r.i. $P_j(t)$ in E_j $(j = 1, 2)$. In addition,

$$D = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(t, s) dP(t, s) \tag{1.3}$$

where

$$P(t, s) := P_1(t) \otimes P_2(s) \quad (t, s \in \mathbf{R})$$

and w is a P-measurable scalar-valued function defined on $\mathbf{R}^2$.

Recall that *a norm ideal* Y_j $(j = 1, 2)$ *of compact operators acting* in a E_j is algebraically a two-sided ideal, which is complete in an auxiliary norm $|\cdot|_{Y_j}$ for which $|CB|_{Y_j}$ and $|BC|_{Y_j}$ are both dominated by $\|C\|_j |B|_{Y_j}$ for a bounded operator C in E_j and a $B \in Y_j$, cf. [**7**]. Assume, in addition, that there are positive constants $\theta_k^{(j)}$ $(k \in \mathbf{N})$, with

$$\sqrt[k]{\theta_k^{(j)}} \to 0,$$

for which, for an arbitrary Volterra operator $V \in Y_j$

$$\|V^k\|_j \leq \theta_k^{(j)} |V|_{Y_j}^k \quad (k = 1, 2, \ldots;\ j = 1, 2). \tag{1.4}$$

Below we will check that the Neumann-Schatten ideal has property (1.4).

Denote by $ni(V)$ *the nilpotency index of a nilpotent operator* V, so that $V^{ni(V)} = 0 \neq V^{ni(V)-1}$; if V is quasinilpotent but not nilpotent we write $ni(V) = \infty$.

Furthermore, put

$$\psi_j := \min\{\|V_j^+\|_j,\ \|V_j^-\|_j\} \quad (j = 1, 2)$$

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and

$$\tilde{V}_j = \begin{cases} V_j^+ & \text{if } \|V_j^-\|_j \le \|V_j^+\|_j \\ V_j^- & \text{if } \|V_j^+\|_j < \|V_j^-\|_j. \end{cases}$$

Let us suppose that

$$\tilde{V}_j \in Y_j \quad (j = 1, 2). \tag{1.5}$$

Without any loss of generality, assume that

$$ni(\tilde{V}_1) \ge ni(\tilde{V}_2)$$

and put

$$b_m(\tilde{V}_1, \tilde{V}_2) := \sum_{k=m_2}^{m_1} \binom{m}{k} \theta_k^{(1)} \theta_{m-k}^{(2)} |\tilde{V}_1|_{Y_1}^k |\tilde{V}_2|_{Y_2}^{m-k}, \tag{1.6}$$

where $\binom{m}{k} = m!/k!(m-k)!$ are the binomial coefficients,

$$m_1 = \min\{m, ni(\tilde{V}_1) - 1\} \text{ and } m_2 = \max\{0, m - ni(\tilde{V}_2) + 1\}. \tag{1.7}$$

Finally, put

$$\psi_0 := \psi_1 + \psi_2$$

and

$$J(\tilde{V}_1, \tilde{V}_2, y) := \sum_{k=0}^{ni(\tilde{V}_1)-1} \frac{b_k(\tilde{V}_1, \tilde{V}_2)}{y^{k+1}} \quad (y > 0).$$

Everywhere below one can replace $ni(\tilde{V}_j)$ by ∞, $b_m(\tilde{V}_1, \tilde{V}_2)$ by

$$\tilde{b}_m(\tilde{V}_1, \tilde{V}_2) := \sum_{k=0}^{m} \binom{m}{k} \theta_k^{(1)} \theta_{m-k}^{(2)} |\tilde{V}_1|_{Y_1}^k |\tilde{V}_2|_{Y_2}^{m-k}$$

and $J(\tilde{V}_1, \tilde{V}_2, y)$ by

$$\tilde{J}(\tilde{V}_1, \tilde{V}_2, y) = \sum_{k=0}^{\infty} \frac{\tilde{b}_k(\tilde{V}_1, \tilde{V}_2)}{y^{k+1}} \quad (y > 0). \tag{1.8}$$

Now we are in a position to formulate the main result of the paper.

THEOREM 1.1. *Let the conditions* (1.2), (1.3), (1.5),

$$d_0 := \inf |\sigma(D)| > 0 \tag{1.9}$$

and

$$\psi_0 J(\tilde{V}_1, \tilde{V}_2, d_0) < 1 \tag{1.10}$$

hold. Then the operator defined by (1.1) *is invertible. Moreover,*

$$\|A^{-1}\|_H \leq \frac{J(\tilde{V}_1, \tilde{V}_2, d_0)}{1 - \psi_0 J(\tilde{V}_1, \tilde{V}_2, d_0)}. \tag{1.11}$$

2. Powers of quasi-nilpotent operators. Let W_1, W_2 be commuting operators in H. Then, clearly,

$$(W_1 + W_2)^n = \sum_{k=0}^{n} \binom{n}{k} W_1^k W_2^{n-k}. \tag{2.1}$$

Let $c_{jk} := \|W_j^k\|$ and

$$\sqrt[k]{c_{jk}} \to 0 \quad (j = 1, 2;\ k = 1, 2, \ldots).$$

So W_1, W_2 are quasinilpotent operators. Then $W_1 + W_2$ is a quasinilpotent operator. Indeed, due to (2.1),

$$\|(W_1 + W_2)^n\| \leq c_{3n} := \sum_{k=0}^{n} \binom{n}{k} c_{1k} c_{2,n-k}$$

since W_1, W_2 commute. Since, $c_{1k}, c_{2,k}$ are coefficients of some entire functions $f_1(z)$ and $f_2(z)$, and

$$\sum_{k=0}^{n} c_{1k} c_{2,n-k}$$

are coefficients of the entire function $f_1(z) f_2(z)$, taking into account that $\binom{n}{k} \leq 2^n$, we can assert that $\sqrt[n]{c_{3n}} \to 0$. So $W_1 + W_2$ is indeed a quasinilpotent operator.

Let us suppose that

$$W_1 = V_1 \otimes I_2 \quad \text{and} \quad W_2 = I_1 \otimes V_2 \tag{2.2}$$

where V_j are Volterra operators satisfying the condition

$$V_j \in Y_j \ \ (j = 1, 2) \tag{2.3}$$

and where Y_j has the property (1.4). Then

$$\|W_j^k\|_H = \|V_j^k\|_j \leq \theta_k^{(j)} |V_j|_{Y_j}^k \ \ (k = 1, 2, \ldots, ni(V_j) - 1;\ j = 1, 2).$$

Without any loss of the generality, assume that $ni(V_1) \geq ni(V_2)$. Thus,

$$\|(W_1 + W_2)^n\|_H \leq b_n(V_1, V_2) := \sum_{k=n_2}^{n_1} \binom{n}{k} \theta_k^{(1)} \theta_{n-k}^{(2)} |V_1|_{Y_1}^k |V_2|_{Y_2}^{n-k} \tag{2.4}$$

where

$$n_1 = \min\{n, ni(V_1) - 1\} \quad \text{and} \quad n_2 = \max\{0, n - ni(V_2) + 1\}.$$

We thus have proved the following lemma.

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LEMMA 2.1. *Let W_1 and W_2 be quasinilpotent and commuting operators. Then the operator $W_1 + W_2$ is quasinilpotent. Moreover, conditions* (2.2) *and* (2.3) *imply inequality* (2.4).

3. Proof of Theorem 1.1. Let us consider the operator

$$A_0 = D + V_1 \otimes I_2 + I_1 \otimes V_2 \tag{3.1}$$

where D is a normal operator defined by (1.3), $V_1 \in Y_1$ and $V_2 \in Y_2$ are Volterra operators in E_1 and E_2, respectively, such that

$$P_j(t)V_jP_j(t) = VP_j(t)\ (j = 1, 2;\ -\infty \leq t \leq \infty). \tag{3.2}$$

Besides, (1.4) holds. Due to Lemma 2.1

$$V_A := V_1 \otimes I_2 + I_1 \otimes V_2$$

is a quasinilpotent operator.

In the sequel, $P(t, s)$, D and V_A will be called *the spectral measure, diagonal part* and *nilpotent part of* A_0, respectively. In addition, the equality

$$A_0 = D + V_A \tag{3.3}$$

is said to be *the triangular representation of* A_0 .

LEMMA 3.1. *Let conditions* (3.1) *and* (3.2) *hold. Then*

$$\|(A_0 - \lambda I)^{-1}\| \leq \sum_{n=0}^{\infty} \frac{b_n(V_1, V_2)}{\rho^{n+1}(D, \lambda)}\ \ (\lambda \notin \sigma(D)) \tag{3.4}$$

where $b_n(V_1, V_2)$ are defined by (2.4).

Proof. Due to the triangular representation (3.3) we have

$$(A_0 - \lambda I)^{-1} = (D + V_A - \lambda I)^{-1} = (I + Q_\lambda)^{-1}(D - \lambda I)^{-1}\ \ (\lambda \notin \sigma(A_0)), \tag{3.5}$$

where

$$Q_\lambda = (D - \lambda I)^{-1} V_A.$$

According to (1.4),

$$(D - I\lambda)^{-1} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (w(t, s) - \lambda)^{-1} dP(t, s)\ \ (\lambda \notin \sigma(D)).$$

Thus

$$(D - I\lambda)^{-1} = \int_{-\infty}^{\infty} dP_1(t) \otimes T_2(t, \lambda) = \int_{-\infty}^{\infty} T_1(s, \lambda) \otimes dP_2(s)$$

where

$$T_1(s, \lambda) = \int_{-\infty}^{\infty} (w(t, s) - \lambda)^{-1} dP_1(t)$$

and

$$T_2(t, \lambda) = \int_{-\infty}^{\infty} (w(t, s) - \lambda)^{-1} dP_2(s).$$

Then $Q_\lambda = B_1(\lambda) + B_2(\lambda)$, where

$$B_1(\lambda) := (D - \lambda)^{-1}(V_1 \otimes I_2) = \int_{-\infty}^{\infty} T_1(s, \lambda) V_1 \otimes dP_2(s)$$

and

$$B_2(\lambda) := (D - \lambda)^{-1}(I_1 \otimes V_2) = \int_{-\infty}^{\infty} dP_1(t) \otimes T_2(t, \lambda) V_2.$$

It can be directly checked that the operators $B_1(\lambda)$ and $B_2(\lambda)$ commute and that

$$B_1^n(\lambda) = \int_{-\infty}^{\infty} (T_1(s, \lambda) V_1)^n \otimes dP_2(s) \quad (n = 1, 2, \ldots).$$

Hence,

$$\|B_1^n(\lambda) h\|_H^2 = \int_{-\infty}^{\infty} \|(T_1(s, \lambda) V_1)^n h_1\|_1^2 d\langle P_2(s) h_2, h_2\rangle_2$$
$$(h = h_1 \otimes h_2, \ h_j \in E_j).$$

Let us use the following result: let a quasinilpotent operator V and a bounded operator A in H have the same m.r.i.. Then the operators AV and VA are quasinilpotent ([**4**, Lemma 3.2.4]).

Since $T_j(s, \lambda)$ and V_j have the same m.r.i. P_j, due to this result $T_j(s, \lambda) V_j$ $(j = 1, 2)$ are quasinilpotent operators. So

$$\|(T_j(s, \lambda) V_j)^n\|_j \leq \theta_n^{(j)} |V_j|_{Y_j}^n \|T_j(s, \lambda)\|_j^n \leq \frac{\theta_n^{(j)} |V_j|_{Y_j}^n}{\rho^n(D, \lambda)}.$$

Consequently,

$$\|B_1^n(\lambda) h\|_H = \left[\int_{-\infty}^{\infty} \|(T(s, \lambda) V_1)^n h_1\|_1^2 d\langle P_2(s) h_2, h_2\rangle_2\right]^{1/2}$$
$$\leq \frac{\theta_n^{(1)} |V_1|_{Y_1}^n}{\rho^n(D, \lambda)} \quad (h = h_1 \otimes h_2, \ \|h\|_H = 1).$$

Therefore,

$$\|B_1^n(\lambda)\|_H \leq \frac{\theta_n^{(1)} |V_1|_{Y_1}^n}{\rho^n(D, \lambda)}.$$

Similarly,

$$\|B_2^n(\lambda)\|_H \leq \frac{\theta_n^{(2)} |V_2|_{Y_2}^n}{\rho^n(D, \lambda)}.$$

Now (2.1) implies

$$\|(B_1(\lambda) + B_2(\lambda))^n\|_H = \|Q_\lambda^n\|_H \le \frac{b_n(V_1, V_2)}{\rho^n(D, \lambda)}. \tag{3.6}$$

Relations (3.5) imply

$$\|(A - \lambda I)^{-1}\|_H \le \|(D - \lambda I)^{-1}\|_H \sum_{n=0}^{\infty} \|Q_\lambda^n\|_H.$$

According to (3.6) we get the required result. □

Proof of Theorem 1.1. First assume that

$$\psi_j := \min\{\|V_j^+\|_j, \|V_j^-\|_j\} = \|V_j^+\|_j, \tag{3.7}$$

so that $\tilde{V}_j = V_j^-$. Rewrite A as $A = D + V_1^- \otimes I_2 + I_1 \otimes V_2^- + Z$, where

$$Z = V_1^+ \otimes I_2 + I_1 \otimes V_2^+.$$

Then $\|Z\| = \psi_0$. Due to the previous lemma, taking $A_0 = D + V_1^- \otimes I_2 + I_1 \otimes V_2^-$, we have

$$\|A_0^{-1}\| \le J(\tilde{V}_1, \tilde{V}_2, d_0).$$

Hence condition (1.10) implies the inequality $\|Z\|\|A_0^{-1}\| < 1$ and

$$\|A^{-1}\| \le \|A_0^{-1}\|(1 - \|Z\|\|A_0^{-1}\|)^{-1}.$$

Hence, the required inequality (1.11) follows. Similar reasoning is valid if instead of (3.7) we consider the general case. This finishes the proof. □

4. Localization of the spectrum.

We begin with the following result.

LEMMA 4.1. *Under conditions* (1.2), (1.3) *and* (1.5) *for any* $\lambda \notin \sigma(D)$, *let*

$$\psi_0 J(\tilde{V}_1, \tilde{V}_2, \rho(\lambda, D)) < 1. \tag{4.1}$$

Then λ *is a regular point of operator* A *represented by* (1.1). *Moreover, its resolvent satisfies the inequality*

$$\|R_\lambda(A)\| \le \frac{J(\tilde{V}_1, \tilde{V}_2, \rho(\lambda, D))}{1 - \psi_0 J(\tilde{V}_1, \tilde{V}_2, \rho(\lambda, D))}. \tag{4.2}$$

Proof. Considering the operator $A - \lambda I$ instead of A and taking into account that $\|R_\lambda(D)\|_H = \rho^{-1}(\lambda, D)$, we arrive at the required result, due to Theorem 1.1. □

Lemma 4.1 implies the validity of the following.

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COROLLARY 4.2. *Under conditions* (1.1)–(1.3) *and* (1.5), *for any* $\mu \in \sigma(A)$, *there is a* $\mu_0 \in \sigma(D)$, *such that, either* $\mu = \mu_0$, *or*

$$\psi_0 J(\tilde{V}_1, \tilde{V}_2, |\mu - \mu_0|) \geq 1. \tag{4.3}$$

THEOREM 4.3. *Under conditions* (1.1)–(1.3) *and* (1.5), *let at least one of the relations* $\tilde{V}_1 \neq 0$ *or* $\tilde{V}_2 \neq 0$ *hold. Then the equation*

$$\psi_0 J(\tilde{V}_1, \tilde{V}_2, y) = 1 \tag{4.4}$$

has a unique non-negative root $z(Y_1, Y_2, \tilde{V}_1, \tilde{V}_2)$. *Moreover, for any* $\mu \in \sigma(A)$, *there is a* $\mu_0 \in \sigma(D)$, *such that* $|\mu - \mu_0| \leq z(Y_1, Y_2, \tilde{V}_1, \tilde{V}_2)$.

Proof. Comparing equations (4.4) with inequalities (4.3), we arrive at the result. □

Note that if $\tilde{V}_1 = \tilde{V}_2 = 0$, then $A = D$. Consider the case $ni\ (V_1) = \infty$. Note that

$$J(\tilde{V}_1, \tilde{V}_2, 1/z) = f_1(z) f_2(z),$$

where

$$f_j(z) = \sum_{k=0}^{\infty} z^k \theta_k^{(j)} |\tilde{V}_j|_{Y_j}^k \quad (z \in \mathbf{C};\ j = 1, 2)$$

are entire functions. But the product of entire functions is an entire function, whose Taylor series always converges. This proves that the series, which defines $J(\tilde{V}_1, \tilde{V}_2, y)$ converges.

To estimate $z(Y_1, Y_2, \tilde{V}_1, \tilde{V}_2)$, let us consider the equation

$$\sum_{k=1}^{\infty} a_k z^k = 1, \tag{4.5}$$

where the coefficients a_k, $k = 1, 2, \ldots$ are nonnegative, at least one of them is positive, and they have the property

$$\gamma_0 \equiv 2 \max_k \sqrt[k]{a_k} < \infty.$$

LEMMA 4.4. *The unique nonnegative root* z_0 *of equation* (4.5) *satisfies the estimate* $z_0 \geq 1/\gamma_0$.

For the proof see [**6**, Lemma 3.4]. Lemma 4.4 gives us the inequality

$$z(Y_1, Y_2, \tilde{V}_1, \tilde{V}_2) \leq \delta(\psi_0, \tilde{V}_1, \tilde{V}_2) := 2 \max_{j=1,2,\ldots} \sqrt[j+1]{\psi_0 b_j(\tilde{V}_1, \tilde{V}_2)}.$$

If $\psi_0 = 0$, then $\delta(\psi_0, \tilde{V}_1, \tilde{V}_2) = 0$. Now Theorem 4.3 implies the following corollary.

COROLLARY 4.5. *Under conditions* (1.1)–(1.3) *and* (1.5) *for any* $\mu \in \sigma(A)$ *there is a* $\mu_0 \in \sigma(D)$, *such that* $|\mu - \mu_0| \leq \delta(\psi_0, \tilde{V}_1, \tilde{V}_2)$. *In particular,*

$$\alpha\ (A) \leq \alpha\ (D) + \delta(\psi_0, \tilde{V}_1, \tilde{V}_2)$$

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and the spectral radius of operator A represented by (1.1) *satisfies the inequality*

$$r_s(A) \le r_s(D) + \delta(\psi_0, \tilde{V}_1, \tilde{V}_2)$$

provided D is bounded.

We will say that a linear operator A is *stable* if $\alpha(A) < 0$.

So under the hypothesis of Corollary 4.5, operator A represented by (1.1) is stable provided $\alpha(D) + \delta(\psi_0, \tilde{V}_1, \tilde{V}_2) < 0$.

5. Operators with Hilbert-Schmidt off-diagonals. Recall that $\tilde{V}_j$ $(j = 1, 2)$ are defined in Section 1. In this section we assume that $\tilde{V}_j$ are Hilbert-Schmidt quasinilpotent operators:

$$\tilde{V}_j \in \tilde{C}_2 \quad (j = 1, 2) \tag{5.1}$$

where $\tilde{C}_2 = C_2(E_j)$ is the ideal of Hilbert-Schmidt operators in E_j with the Hilbert-Schmidt norm

$$N_2(K) \equiv [\text{Trace } K^*K]^{1/2} \quad (K \in \tilde{C}_2).$$

The asterisk means the adjointness.

We need also the following result: for an arbitrary quasinilpotent Hilbert-Schmidt operator V_0 in H, the inequality

$$\| V_0^k \|_H \le \frac{N_2^k(V_0)}{\sqrt{k!}} \quad (k = 1, 2, \ldots) \tag{5.2}$$

is true ([**4**, Lemma 2.3.1]).

Let W_1, W_2 be defined by (2.2), again. Lemma 2.1 and (5.2) imply the following.

Corollary 5.1. *Under conditions* (2.2) *and* (5.1), *we have* $\|(W_1 + W_2)^n\|_H \le b_n(A, \tilde{C}_2)$ *where*

$$b_n(A, \tilde{C}_2) := \sum_{k=0}^{n} \frac{\binom{n}{k} N_2^k(\tilde{V}_1) N_2^{n-k}(\tilde{V}_2)}{\sqrt{(n-k)!k!}}. \tag{5.3}$$

Under conditions (5.1) put

$$J_2(\tilde{V}_1, \tilde{V}_2, y) := \sum_{n=0}^{\infty} \frac{b_n(A, \tilde{C}_2)}{y^{n+1}} \quad (y > 0). \tag{5.4}$$

Now Theorem 1.1 implies the following result.

Theorem 5.2. *Let the conditions* (1.2), (1.3), (5.1), *and*

$$\psi_0 J_2(\tilde{V}_1, \tilde{V}_2, d_0) < 1$$

hold. Then the operator defined by (1.1) *is invertible. Moreover,*

$$\|A^{-1}\|_H \le \frac{J_2(\tilde{V}_1, \tilde{V}_2, d_0)}{1 - \psi_0 J_2(\tilde{V}_1, \tilde{V}_2, d_0)}.$$

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Since $C_k^n \le 2^n \quad (k \le n)$, due to Lemma 5.1 we have

$$\begin{aligned}\|(W_1 + W_2)^n\|_H &\le \frac{1}{\sqrt{n!}} \sum_{k=0}^{n} \binom{n}{k} \sqrt{\binom{n}{k}} N_2^k(\tilde V_1) N_2^{n-k}(\tilde V_2)\\ &\le \frac{2^{n/2}}{\sqrt{n!}} \sum_{k=0}^{n} \binom{n}{k} N_2^k(\tilde V_1) N_2^{n-k}(\tilde V_2)\\ &= \frac{[\sqrt{2^n}(N_2(\tilde V_1) + N_2(\tilde V_2))]^n}{\sqrt{n!}}.\end{aligned}$$

Hence,

$$J_2(\tilde V_1, \tilde V_2, y) \le \sum_{n=0}^{\infty} \frac{[\sqrt{2}(N_2(\tilde V_1) + N_2(\tilde V_2))]^n}{\sqrt{n!}\,y^{n+1}} \quad (y > 0). \tag{5.5}$$

By the Schwarz inequality

$$\sum_{n=0}^{\infty} \frac{b^n}{\sqrt{n!}\,y^n} = \sum_{n=0}^{\infty} \frac{(\sqrt{2}b)^n}{\sqrt{2^n n!}\,y^n} \le \left[\sum_{n=0}^{\infty} \frac{2^n b^{2n}}{n!\,y^{2n}}\right]^{1/2} \left[\sum_{n=0}^{\infty} 2^{-n}\right]^{1/2} = \sqrt{2}\exp\left[\frac{b^2}{y^2}\right] \ (b, y > 0).$$

This relation and (5.5) imply $J_2(\tilde V_1, \tilde V_2, y) \le \theta_2(\tilde V_1, \tilde V_2, y)$ where

$$\theta_2(\tilde V_1, \tilde V_2, y) := \frac{\sqrt{2}}{y} \exp\left[\frac{2(N_2(\tilde V_1) + N_2(\tilde V_2))^2}{y^2}\right] \quad (y > 0). \tag{5.6}$$

Now Theorem 5.2 implies the following result.

COROLLARY 5.3. *Let the conditions* (1.2), (1.3), (5.1) *and*

$$\psi_0 \theta_2(\tilde V_1, \tilde V_2, d_0) < 1 \tag{5.7}$$

hold. Then the operator defined by (1.1) *is invertible. Moreover,*

$$\|A^{-1}\|_H \le \frac{\theta_2(\tilde V_1, \tilde V_2, d_0)}{1 - \psi_0 \theta_2(\tilde V_1, \tilde V_2, d_0)}. \tag{5.8}$$

Theorem 4.3 and relation (5.2) yield the next theorem.

THEOREM 5.4. *Under conditions* (1.1)–(1.3) *and* (5.1), *the equation*

$$\psi_0 J_2(\tilde V_1, \tilde V_2, y) = 1 \tag{5.9}$$

has a unique non-negative root $z(\tilde C_2)$. Moreover, for any $\mu \in \sigma(A)$, there is a $\mu_0 \in \sigma(D)$ such that $|\mu - \mu_0| \le z(\tilde C_2)$.

According to Lemma 4.4,

$$z(\tilde C_2) \le 2 \max_{j=1,2,\dots} \sqrt[j+1]{\psi_0 b_j(A, \tilde C_2)} \ \ (\psi_0 \ne 0).$$

If $\psi_0 = 0$, then $z(\tilde C_2) = 0$. We need the following simple lemma.

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LEMMA 5.5. *The unique positive root* z_0 *of the equation*

$$ze^z = a(a = const. > 0) \tag{5.10}$$

satisfies the estimate

$$z_0 \geq \ln[1/2 + \sqrt{1/4 + a}].$$

If, in addition, the condition $a \geq e$ *holds, then* $z_0 \geq \ln a - \ln\ln a$.

For the proof see [**6**, Lemma 4.6].

According to (5.5), we have $z(\tilde{C}_2) \leq z_0(\tilde{C}_2)$ where $z_0(\tilde{C}_2)$ is the unique positive root of the equation

$$\frac{\psi_0\sqrt{2}}{y} \exp\left[\frac{2[N_2(\tilde{V}_1) + N_2(\tilde{V}_2)]^2}{y^2}\right] = 1. \tag{5.11}$$

Clearly, this equation is equivalent to the following one:

$$\frac{2\psi_0^2}{y^2} \exp\left[\frac{4(N_2(\tilde{V}_1) + N_2(\tilde{V}_2))^2}{y^2}\right] = 1.$$

Write

$$z = \frac{4(N_2(\tilde{V}_1) + N_2(\tilde{V}_2))^2}{y^2}.$$

Then we have equation (5.10). Now Lemma 5.5 gives us the inequality

$$z_0(\tilde{C}_2) \leq \delta_2\,(A) \tag{5.12}$$

where

$$\delta_2\,(A) \equiv \frac{2(N_2(\tilde{V}_1) + N_2(\tilde{V}_2))}{\ln^{1/2}\left[1/2 + \sqrt{1/4 + \frac{2(N_2(\tilde{V}_1) + N_2(\tilde{V}_2))^2}{\psi_0^2}}\right]}. \tag{5.13}$$

Furthermore, Theorem 5.4 implies the following corollary.

COROLLARY 5.6. *Under conditions* (1.1)–(1.3)*, and* (5.1)*, for any* $\mu \in \sigma(A)$*, there is a* $\mu_0 \in \sigma(D)$*, such that,* $|\mu - \mu_0| \leq \delta_2(A)$.

So operator A *is stable, provided* $\alpha(D) + \delta_2(A) < 0$.

If, in addition, D *is bounded, then* $r_s(A) \leq \delta_2(A) + r_s(D)$.

6. Operators with Neumann-Schatten off diagonals. Suppose now for some integer $p > 1$,

$$\tilde{V}_j \in \tilde{C}_{2p} = C_{2p}(E_j) \;\; (j = 1, 2) \tag{6.1}$$

where $\tilde{C}_{2p}$ is the Neumann-Schatten ideal in E_j. That is,

$$N_{2p}(K) := \;[\text{Trace}\;(K^*K)^p]^{1/2p} < \infty \;\; (K \in \tilde{C}_{2p}).$$

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According to (5.2), for any quasinilpotent operator $V \in \tilde{C}_{2p}$ in E_j,

$$\| V^{mp} \|_j \leq \frac{N_2^m(V^p)}{\sqrt{m!}} \leq \frac{N_{2p}^{pm}(V)}{\sqrt{m!}} \ (m = 1, 2, \ldots).$$

Take into account that, for an arbitrary operator $K \in C_{2p}(E_j)$,

$$\|K\|_j^2 = \max_k |\lambda_k(K^*K)| \leq N_{2p}(K) := \ [\text{Trace } (K^*K)^p]^{1/p},$$

where $\lambda_k(K^*K)$, $k = 1, 2, \ldots$ are the eigenvalues of K^*K.

Hence, for any $k = i + mp(i = 0, \ldots, p-1;\ m = 0, 1, 2, \ldots)$, we have

$$\| V^k \|_j = \| V^{i+pm} \|_j \leq \frac{\| V^i \|_j N_2^m(V^p)}{\sqrt{m!}} \leq \frac{N_{2p}^{i+pm}(V)}{\sqrt{m!}}.$$

This inequality can be written as

$$\| V^k \|_j \leq \frac{N_{2p}^k(V)}{\sqrt{[k/p]!}} \ \ (V \in \tilde{C}_{2p};\ k = 1, 2, \ldots), \tag{6.2}$$

where $[x]$ means the integer part of a number $x > 0$. Under conditions (2.2) and (6.1), by Lemma 2.1 we have

$$\|(W_1 + W_2)^n\|_H \leq \sum_{k=0}^{n} \binom{n}{k} \frac{N_{2p}^k(\tilde{V}_1) N_{2p}^{n-k}(\tilde{V}_2)}{\sqrt{[k/p]![(n-k)/p]!}}. \tag{6.3}$$

Under (6.1), put

$$J_{2p}(\tilde{V}_1, \tilde{V}_2, y) := \sum_{n=0}^{\infty} \frac{b_n(A, \tilde{C}_{2p})}{y^{n+1}} \ \ (y > 0)$$

with

$$b_n(A, \tilde{C}_{2p}) := \sum_{k=0}^{n} \frac{\binom{n}{k} N_{2p}^k(\tilde{V}_1) N_{2p}^{n-k}(\tilde{V}_2)}{\sqrt{[(n-k)/p]![k/p]!}}.$$

Now Theorem 1.1 implies the following result.

THEOREM 6.1. *Let the conditions* (1.2), (1.3), (6.1), *and*

$$\psi_0 J_{2p}(\tilde{V}_1, \tilde{V}_2, d_0) < 1 \tag{6.4}$$

hold. Then the operator defined by (1.1) *is invertible. Moreover,*

$$\|A^{-1}\|_H \leq \frac{J_{2p}(\tilde{V}_1, \tilde{V}_2, d_0)}{1 - \psi_0 J_{2p}(\tilde{V}_1, \tilde{V}_2, d_0)}. \tag{6.5}$$

Theorem 4.3 and relation (6.3) yield the following result.

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THEOREM 6.2. *Under conditions* (1.1)–(1.3) *and* (6.1), *the equation*

$$\psi_0 J_{2p}(\tilde{V}_1, \tilde{V}_2, y) = 1$$

has a unique non-negative root $z(\tilde{C}_{2p})$. *Moreover, for any* $\mu \in \sigma(A)$, *there is a* $\mu_0 \in \sigma(D)$, *such that* $|\mu - \mu_0| \leq z(\tilde{C}_{2p})$.

According to Lemma 4.4,

$$z(\tilde{C}_2) \leq 2 \max_{j=1,2,\ldots} \sqrt[j+1]{\psi_0 b_j(A, \tilde{C}_{2p})}.$$

7. Positive invertibility. Let E_1, E_2 be Hilbert lattices (cf. **[9]**). Again consider operators of the type (1.1). It is assumed that D is invertible and the operator

$$D^{-1} \text{ is } \textit{positive} \text{ and } V_1^{\pm}, V_2^{\pm} \text{ are non-negative operators}. \tag{7.1}$$

THEOREM 7.1. *Under conditions* (1.2), (1.3), (1.5), *operator A defined by* (1.1) *is positively invertible and the inequalities* (1.11) *and*

$$A^{-1} \geq D^{-1} > 0 \tag{7.2}$$

are true, provided relations (1.10) *and* (7.1) *hold.*

Proof. Inequality (1.11) follows from Theorem 1.1. Put

$$T_{\pm} = -V_1^{\pm} \otimes I_2 - I_1 \otimes V_2^{\pm}.$$

Due to (1.10) and Lemma 3.1, at least one of the following relations:

$$\|(D - T_-)^{-1} T_+\| < 1 \tag{7.3}$$

or $\|(D - T_+)^{-1} T_-\| < 1$ is valid. Without loss of generality, assume that (7.3) holds. Since $V_j^{\pm} \leq 0$, we have $T_{\pm} \geq 0$. According to (1.1),

$$A = D - T_- - T_+ = (D - T_-)(I - (D - T_-)^{-1} T_+).$$

Hence

$$A^{-1}(I - (D - T_-)^{-1} T_+)(D - T_-)^{-1}. \tag{7.4}$$

But $D - T_- = D(I - D^{-1} T_-)$ and $D^{-1} T_-$ is quasinilpotent and nonnegative. So

$$(D - T_-)^{-1} = \sum_{k=0}^{\infty} (D^{-1} T_-)^k D^{-1} \geq D^{-1}.$$

Moreover, under condition (7.3), operator $I(D - T_-)^{-1} T_+$ is invertible, and

$$(I - (D - T_-)^{-1} T_+)^{-1} = \sum_{k=0}^{\infty} ((D - T_-)^{-1} T_+)^k \geq I.$$

This and (7.4) proves the required inequality (7.2). □

Now let $\tilde{V}_j$ be Hilbert-Schmidt quasinilpotent operators. Recall that $\theta_2(\tilde{V}_1, \tilde{V}_2, d_0)$ is defined by (5.6). Theorem 7.1 and Corollary 5.3 imply the following corollary.

COROLLARY 7.2. *Under conditions* (1.2), (1.3), (5.1), (5.7) *and* (7.1), *operator* A *defined by* (1.1) *is positively invertible. Moreover the inequalities* (5.8) *and* (7.2) *are true.*

If $\tilde{V}_j$ are Neumann-Schatten as in Theorem 6.1, Theorem 7.1 implies the following.

COROLLARY 7.3. *Under conditions* (1.2), (1.3), (6.1), (6.4) *and* (7.1), *operator* A *defined by* (1.1) *is positively invertible. Moreover inequalities* (6.5) *and* (7.2) *are true.*

8. Examples.

8.1. A partial integral operator. Let us consider in the complex space $H \equiv L^2([0, 1] \times [0, 1])$ the operator A defined by

$$(Au)(x, y) = a(x, y)u(x, y) + \int_0^1 K_1(x, x_1)u(x_1, y)\, dx_1 + \int_0^1 K_2(y, y_1)u(x, y_1)\, dy_1 \quad (8.1)$$

where K_1, K_2 are scalar Hilbert-Schmidt kernels, and $a(x, y)$ is a real bounded measurable function defined on $[0, 1]^2$. Such operators arise in various applications ([**1**], [**8**], [**10**]). In the considered case $E_1 = E_2 = L^2[0, 1]$.

For $0 \leq t \leq 1$ and $u \in L^2[0, 1]$, define $P_1(t)$ and $P_2(t)$ by

$$(P_1(t)u)(x) = (P_2(t)u)(x) = \begin{cases} 0 & \text{if } t < x \leq 1 \\ u(x) \text{ for } 0 \leq x < t & \text{if } 0 \leq x < t. \end{cases} \quad (8.2)$$

In addition, put $P_j(t) = I_j$ for $t > 1$ and $P_j(t) = 0$ for $t < 0; j = 1, 2$. Take $(Dv)(x, y) = a(x, y)v(x, y)\ \ (v \in H)$,

$$(V_j^+ u)(x, y) = \int_0^x K_j(x, x_1)u(x_1)\, dx_1$$

and

$$(V_j^- u)(x, y) = \int_x^1 K_j(x, x_1)u(x_1)\, dx_1 \quad (u \in L^2[0, 1]).$$

Then condition (1.2) holds. Besides,

$$N_2(V_j^+) = \left[\int_0^1 \int_0^x |K_j(x, x_1)|^2\, dx_1\, dx\right]^{1/2},$$

$$N_2(V_j^-) = \left[\int_0^1 \int_x^1 |K_j(x, x_1)|^2\, dx_1\, dx\right]^{1/2}, \quad (8.3)$$

and

$$\sigma(D) = \{z \in \mathbf{C} : z = a(x, y),\ 0 \leq x, y \leq 1\}.$$

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Recall that ψ_0 and $\tilde{V}_j^{\pm}$ are defined in Section 1. Due to Theorem 5.4 and Corollary 5.6

$$\sigma(A) \subset \{z \in \mathbf{C} : |z - a(x, y)| \le z_2(A) \le \delta_2(A),\ 0 \le x, y \le 1\}$$

where $z_2(A)$ is the unique positive root of (5.9) and $\delta_2(A)$ is defined by (5.13). Hence,

$$r_s(A) \le \max_{x,y} |a(x, y)| + \delta_2(A) \text{ and } \alpha(A) \le \max_{x,y} a(x, y) + \delta_2(A).$$

Thus, the operator defined by (6.1) is stable, provided $a(x, y) + \delta_2(A) < 0$ for all x, $y \in [0, 1]$.

Suppose now

$$a(x, y) \ge d_0 > 0 \text{ and } K_j(x, y) \le 0 \quad (x, y \in [0, 1]).$$

In addition suppose condition (5.7) holds. Then the operator defined by (8.1) is positively invertible. Moreover, the inequality

$$(A^{-1}u)(x, y) \ge \frac{u(x, y)}{a(x, y)} \quad (x, y \in [0, 1])$$

is valid for any non-negative function $u \in H$.

8.2. An integro-differential operator. Let us consider in $H \equiv L^2([0, 1] \times [0, 1])$ the operator

$$(Au)(x, y) := \frac{\partial^2 u(x, y)}{\partial y^2} + \int_0^1 K_1(x, x_1)u(x_1, y)dx_1 \quad (u \in \mathrm{Dom}(A)) \tag{8.4}$$

with

$$\mathrm{Dom}\ (A) = \left\{u \in H : \frac{\partial^2 u}{\partial y^2} \in H;\ u(x, 0) = u(x, 1) = 0\right\}.$$

Here K_1 is a Hilbert-Schmidt kernel.

In this case condition (1.2) holds with $V_1^{\pm}$ defined as in the previous subsection, $V_2^{\pm} = 0$ and

$$(Du)(x, y) = \frac{\partial^2 u(x, y)}{\partial y^2} \quad (u \in \mathrm{Dom}\ (A)).$$

Take P_1 as in (8.2) and

$$(P_2(t)v)(y) = (P_2(n)v))(y) = 2\sum_{k=1}^{n} \sin(k\pi y) \int_0^1 v(y_1) \sin(k\pi y_1)v(y_1)\, dy_1$$

$(n = 1, 2, \ldots)$. Then condition (1.2) holds. In this case

$$\psi_0 = \psi_1 = \min\{\|V_1^+\|_{L^2}, \|V_1^-\|_{L^2}\} \quad \text{and } \psi_2 = 0.$$

Clearly, $\sigma(D) = \{-\pi^2 k^2;\ k = 1, 2, \ldots\}$. Then due to Corollary 5.6

$$\sigma(A) \subset \{z \in \mathbf{C} : |z + \pi^2 m^2| \le z_2(A) \le \delta_2(A),\ m = 1, 2, \ldots.\},$$

$z_2(A)$ is the unique positive root of equation (5.9) and $\delta_2(A)$ is defined by (5.13) with (8.13) taken into account. In particular,

$$\alpha(A) \leq -\pi^2 + \tilde{z}_2(A) \leq -\pi^2 + \delta_2(A).$$

Thus, operator A defined by (8.4) is stable, provided $-\pi^2 + \delta_2(A) < 0$.

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