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ON THE NONWANDERING SETS OF DIFFEOMORPHISMS OF SURFACES

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§ 1. Introduction

Let M be a compact manifold without boundary. Let $f: M \to M$ be a C^1 diffeomorphism. Then the *nonwandering set* $\Omega(f)$ is defined to be the closed invariant set consisting of $x \in M$ such that for any neighborhood U of x, there exists an integer $n \neq 0$ satisfying $f^n(U) \cap U \neq \phi$. In particular, the set $\mathrm{Per}(f)$ of all periodic points is included in $\Omega(f)$.

Generally, in the study of the orbit structure of diffeomorphisms their nonwandering sets play an essential role. Several results relating to the non-wandering sets established in these ten years or so have developed a new aspect of dynamics—the study of the orbit structure of dynamical systems. In his survey [8], Smale set up a concept called *Axiom A*, i.e. (a) $\Omega(f) = \overline{\mathrm{Per}(f)}$, (b) Tf has a hyperbolic structure over $\Omega(f)$, i.e. there exists a Tf-invariant continuous splitting $E^s \oplus E^u$ of $TM|\Omega(f)$—the restriction of the tangent bundle TM to $\Omega(f)$—such that for some constants $C > 0$, $0 < \lambda < 1$,

$$\|Tf^n(v)\| \leq C\lambda^n \|v\| , \qquad \forall v \in E^s,\ \forall n > 0 ,$$
$$\|Tf^{-n}(v)\| \leq C\lambda^n \|v\| , \qquad \forall v \in E^u,\ \forall n > 0 .$$

After that, many important results were obtained in this direction.

On the other hand, Pugh [7] proved a very important theorem about the nonwandering sets. To state it, we shall explain the concept of genericity. Let $\mathrm{Diff}^1(M)$ be the set of all C^1 diffeomorphisms endowed with the C^1 topology. Then a property of diffeomorphisms is called *generic* if the diffeomorphisms having it form a residual subset of $\mathrm{Diff}^1(M)$.

PUGH'S DENSITY THEOREM. *The property $\Omega(f) = \overline{\mathrm{Per}(f)}$ is generic in* $\mathrm{Diff}^1(M)$.

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In this paper we shall study the nonwandering sets of diffeomorphisms of surfaces from the viewpoint of genericity. Our results are as follows: Let M^2 be a compact connected surface without boundary.

THEOREM 1. *The property that* $\operatorname{int}\Omega(f), = \phi$, *or* f *is an Anosov diffeomorphism is generic in* $\operatorname{Diff}^1(M^2)$.

Remark. For a topological space X, the closure and the interior of $A \subset X$ are denoted by $\bar{A}$ and $\operatorname{int} A$ respectively.

A diffeomorphism $f\colon M \to M$ is called *Anosov* if Tf has a hyperbolic structure over M. For surfaces except a torus, there is no Anosov diffeomorphisms ([9], p. 90). So, in this case Theorem 1 is written as follows:

THEOREM 1′. *The property* $\operatorname{int}\Omega(f) = \phi$ *is generic in* $\operatorname{Diff}^1(M^2)$ *if* M^2 *is not a torus.*

A diffeomorphism f is said to be *topologically Ω-stable* if $\Omega(f)$ is homeomorphic to $\Omega(g)$ for all g C^1 near f. We have the following from Theorem 1.

COROLLARY. *If* $f \in \operatorname{Diff}^1(M^2)$ *is topologically Ω-stable, then* $\operatorname{int}\Omega(f) = \phi$ *or* f *is an Anosov diffeomorphism.*

The main stage in proving Theorem 1 is the following. First we shall fix our notation.

DEFINITION. For an open subset U of M, we denote by $\mathscr{H}(U)$ the set of $f \in \operatorname{Diff}^1(M)$ whose periodic points in U are all hyperbolic, and by $\mathscr{D}(U)$ the set of $f \in \operatorname{Diff}^1(M)$ whose periodic points are dense in U.

THEOREM 2. *Let* M^2 *be a compact connected surface. Then for any open subset* U *of* M^2,

$$\mathscr{D}(U) \cap \operatorname{int}\mathscr{H}(U) \subset \mathscr{D}(M^2)\,.$$

Theorem 1 is proved in Section 2 and Theorem 2 in Section 4. Sections 3 and 5 are devoted to two propositions necessary for the proof of Theorem 2. In Appendix we shall prove a lemma about a non-transversal homoclinic point, which is necessary in Section 5.

Throughout this paper except Appendix, 'M' will denote a compact connected surface without boundary.

I would like to thank Professor M. Adachi for his guidance of this

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§ 2. Proofs of Theorem 1 and Corollary

In this section we prove Theorem 1, assuming Theorem 2. We denote by $\mathscr{A}$ the set of all Anosov diffeomorphisms of M.

LEMMA 1. *If* $f \in \operatorname{int} \overline{\mathscr{D}(M)}$, *then* $f \in \bar{\mathscr{A}}$. *Hence* $\mathscr{A}$ *is open and dense in* $\operatorname{int} \overline{\mathscr{D}(M)}$.

Proof. Let $f \in \operatorname{int} \overline{\mathscr{D}(M)}$. First, we suppose $f \notin \operatorname{int} \mathscr{H}(M)$. Some diffeomorphism g near f has a non-hyperbolic periodic point p. Since the dimension of M is 2, it is possible to make p a sink or a source of a small C^1 perturbation g_1 of g, i.e., if n is the period of p, then the eigenvalues of $T_p g_1^n$ have absolute values <1 (or >1). Obviously, $g_1 \notin \overline{\mathscr{D}(M)}$. This contradicts the hypothesis, because g_1 can be chosen sufficiently near f. Thus $f \in \operatorname{int} \mathscr{H}(M)$. We can choose $f_1 \in \operatorname{int} \mathscr{H}(M) \cap \mathscr{D}(M)$ near f. We here apply a theorem of Mañé [3], i.e. $\operatorname{int} \mathscr{H}(M) \cap \mathscr{D}(M) = \mathscr{A}$ if the dimension of M is 2. Hence we have $f_1 \in \mathscr{A}$. Therefore, $f \in \bar{\mathscr{A}}$. q.e.d.

For each point $x \in M$, we define

$$\mathscr{U}_x = \{f \in \operatorname{Diff}^1(M);\ x \notin \operatorname{int} \overline{\operatorname{Per}(f)}\}\ .$$

Then we have

LEMMA 2. *If* $f \notin \mathscr{U}_x$, *then* $f \in \mathscr{D}(M)$ *or* $f \in \overline{\operatorname{int} \mathscr{U}_x}$.

Proof. Let $f \notin \mathscr{U}_x$. By definition, $x \in \operatorname{int} \overline{\operatorname{Per}(f)}$. Let U be a small neighborhood of x in $\overline{\operatorname{Per}(f)}$. When $f \in \operatorname{int} \mathscr{H}(U)$, by Theorem 2, we have $f \in \mathscr{D}(M)$. So it is sufficient to show that $f \in \overline{\operatorname{int} \mathscr{U}_x}$, when $f \notin \operatorname{int} \mathscr{H}(U)$. Then some f_1 near f has a non-hyperbolic periodic point p in U. Similarly, it is possible to make p a sink or a source of some C^1 perturbation f_2 of f_1. Since U is a small neighborhood of x, we can choose $h \in \operatorname{Diff}^1(M)$ with $h(x) = p$ in a small C^1 neighborhood of the identity of M. Put $g = h^{-1} \cdot f_2 \cdot h$. Clearly g is C^1 near f. Naturally $x = h^{-1}(p)$ is a sink or source of g. Hence, for any $g_1 \in \operatorname{Diff}^1(M)$ near g we have $x \notin \operatorname{int} \overline{\operatorname{Per}(g_1)}$, or $g_1 \in \mathscr{U}_x$. This implies $g \in \operatorname{int} \mathscr{U}_x$. Since g is near f, it follows that $f \in \overline{\operatorname{int} \mathscr{U}_x}$. q.e.d.

LEMMA 3. $\operatorname{int} \mathscr{U}_x \cup \operatorname{int} \overline{\mathscr{D}(M)}$ *is dense in* $\operatorname{Diff}^1(M)$.

Proof. Suppose $f \notin \overline{\operatorname{int} \mathscr{U}_x}$. It suffices to show $f \in \overline{\mathscr{D}(M)}$. When $f \notin \mathscr{U}_x$, by Lemma 2, we have $f \in \mathscr{D}(M)$. When $f \in \mathscr{U}_x$ hence $f \in \mathscr{U}_x - \overline{\operatorname{int} \mathscr{U}_x}$, there is a sequence $f_n \notin \mathscr{U}_x \cup \overline{\operatorname{int} \mathscr{U}_x}$ converging to f. By Lemma 2, $f_n \in \mathscr{D}(M)$. Hence $f \in \overline{\mathscr{D}(M)}$ follows. q.e.d.

Now Theorem 1 is proved as follows: By Lemmas 1 and 3, $\mathscr{U}_x \cup \mathscr{A}$ is generic in $\operatorname{Diff}^1(M)$. Really it contains an open dense subset of $\operatorname{Diff}^1(M)$. By the Pugh's density theorem, the set

$$\mathscr{C} = \{f \in \operatorname{Diff}^1(M);\ \Omega(f) = \overline{\operatorname{Per}(f)}\}$$

is generic. Let K be a dense countable subset of M. Then

$$\begin{aligned}\mathscr{B} &= \bigcap_{x \in K} (\mathscr{U}_x \cup \mathscr{A}) \cap \mathscr{C} \\ &= \left(\left(\bigcap_{x \in K} \mathscr{U}_x\right) \cap \mathscr{C}\right) \cup \mathscr{A}\end{aligned}$$

is generic in $\operatorname{Diff}^1(M)$. Now we need only check that if $f \in (\bigcap_{x \in K} \mathscr{U}_x) \cap \mathscr{C}$ then $\operatorname{int} \Omega(f) = \phi$. From $f \in \bigcap_{x \in K} \mathscr{U}_x$, we have $\operatorname{int} \overline{\operatorname{Per}(f)} \cap K = \phi$. But, since K is dense in M, $\operatorname{int} \overline{\operatorname{Per}(f)} = \phi$. On the other hand, $f \in \mathscr{C}$ means $\overline{\operatorname{Per}(f)} = \Omega(f)$. Hence $\operatorname{int} \Omega(f) = \phi$ follows. q.e.d.

Proof of Corollary. Let $f \in \operatorname{Diff}^1(M)$ be topologically Ω-stable. First suppose $f \notin \bar{\mathscr{A}}$. By Theorem 1, there is $g \in \operatorname{Diff}^1(M)$ near f such that $\operatorname{int} \Omega(g) = \phi$. By stability, it follows from the theorem of domain invariance that $\operatorname{int} \Omega(f) = \phi$.

Next suppose $f \in \bar{\mathscr{A}}$. There is $f_1 \in \mathscr{A}$ near f. Since $\Omega(f_1) = M$ ([9], p. 89), by stability, we have $\Omega(f) = M$. Hence by stability, $\Omega(g) = M$ for all g near f. By Mañé [3], it follows that f is Anosov. q.e.d.

§ 3. Laminations

In this section we prepare a proposition for the proof of Theorem 2. Let us begin with definitions.

Definition. Let $f \in \operatorname{Diff}^1(M)$. For a hyperbolic periodic point p of f, we denote by $W^s(p;f)$ (resp. $W^u(p;f)$) the stable (resp. unstable) manifold of f at p. We define $E^s(p;f)$ to be the tangent space of $W^s(p;f)$ at p. Likewise $E^u(p;f)$ is defined.

In what follows, we shall drop 'f' in these symbols when it does not give rise to confusion.

Definition. A hyperbolic periodic point is called a *saddle* if it is not a sink nor source. We denote by $\operatorname{Sd}(f)$ the set of all saddles of f.

DEFINITION. *A C^1 lamination* of M is a continuous foliation whose leaves are C^1 immersed submanifolds such that their tangent spaces, as a whole, form a continuous subbundle of TM.

Refer to [1, § 7] for general definitions.

We shall prove the following.

PROPOSITION 1. *Let $f \in \mathrm{Diff}^1(M)$. Let U be an open subset of M such that*:

(1) *U is invariant under f.*

(2) *The periodic points in U are all saddles and are dense in U.*

(3) *There is a continuous splitting $E^s \oplus E^u$ of $TM|U$ whose splitting at $\forall p \in \mathrm{Sd}(f) \cap U$ is $E^s(p;f) \oplus E^u(p;f)$.*

Then there is an f-invariant C^1 lamination W^s on U such that (a) *all laminae are tangent to E^s,* (b) *stable manifolds $W^s(p;f)$, $\forall p \in \mathrm{Sd}(f) \cap U$, are its laminae. Likewise there is an f-invariant lamination W^u on U with the corresponding properties.*

Proof. We want to construct a lamination on a neighborhood of $\forall x_0 \in U$. First, we take a coordinate neighborhood (Q, φ) of x_0 with the following properties.

(4) $Q \subset U$.

(5) $\varphi(Q) = [-1, 1] \times [-1, 1]$.

(6) $\varphi(x_0) = (0, 0)$.

(7) Identify Q with $[-1, 1] \times [-1, 1]$ and E^s with $T\varphi(E^s)$. There is a C^0 map $w: Q \to \boldsymbol{R}$ such that $|w(x)| < 1/4$, and the vector $(1, w(x))$ spans $E^s(x)$, $\forall x \in Q$. $E^s(x)$ is the fiber of E^s at x.

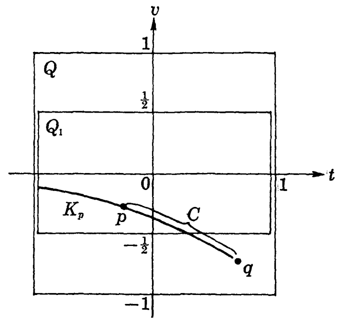
We, first of all, notice that stable manifolds $W^s(p)$, $\forall p \in \mathrm{Sd}(f) \cap U$ are tangent to E^s. Because, if at a point $x \in W^s(p)$, $E^s(x)$ is not tangent to $W^s(p)$, then $E^s(f^{\alpha n}(x)) = Tf^{\alpha n}(E^s(x))$ (α is the period of p) tends to $E^u(p)$ as $n \to \infty$ by hyperbolicity of $T_p f^\alpha$, contradicting continuity of E^s. Likewise unstable manifolds $W^u(p)$, $\forall p \in \mathrm{Sd}(f) \cap U$, are tangent to E^u.

Let $\pi_1: Q \to [-1, 1]$ be the projection on the first factor. Write $Q_1 = [-1, 1] \times [-1/2, 1/2] \subset Q$. For $\forall p \in \mathrm{Sd}(f) \cap Q$, let K_p be the connected component of $W^s(p) \cap Q$ containing p. Let $h_p: K_p \to [-1, 1]$ be the mapping defined by

$$h_p(x) = \pi_1(x) , \qquad \forall x \in K_p .$$

We want to show that h_p is a homeomorphism if $p \in \mathrm{Sd}(f) \cap Q_1$.

First, h_p is one to one, because K_p is an integral curve of the vector field $x \mapsto (1, w(x))$, $\forall x \in Q$, which spans E^s over Q. So we show h_p is onto. We notice that K_p cannot meet the top nor the bottom of Q, because the slope of K_p is less than 1/4. So h_p not being onto implies $\bar{K}_p - K_p \neq \phi$. Let $q \in \bar{K}_p - K_p$. See the figure.



Thus K_p includes one of the components of $W^s(p) - \{p\}$, say C. Since $f^{2\alpha}(C) = C$, clearly we have $f^{2\alpha}(q) = q$, namely $q \in \operatorname{Per}(f)$. Hence, by (2), $q \in \operatorname{Sd}(f)$. For $\forall x \in C$, $f^{-2\alpha n}(x)$ tends to q as $n \to \infty$. This implies $C \subset W^u(q)$. Thus C is tangent to E^s and E^u at once, which contradicts (3). Hence h_p must be onto.

We denote by Π the set of all $p \in \operatorname{Sd}(f) \cap Q$ such that h_p is onto. By the above $\operatorname{Sd}(f) \cap Q_1 \subset \Pi$. Let $\pi_2 \colon Q \to [-1, 1]$ be the projection on the second factor. When we put $V_0 = \{\pi_2 h_p^{-1}(0); p \in \Pi\} \subset [-1, 1]$, it is easy to see that V_0 is dense in $[-1/2, 1/2]$. For $\forall p \in \Pi$, we write $k_u = \pi_2 \cdot h_p^{-1}$, where $u = \pi_2 \cdot h_p^{-1}(0)$. Hence $\operatorname{graph}(k_u) = K_p$. We define a function $v = k(t, u)$, $t \in [-1, 1]$, $u \in [-1/2, 1/2]$ by the following:

$$k(t, u) = \lim_{u' \to u} k_{u'}(t) , \qquad u' \in V_0 .$$

The aim of the following is to prove that curves $t \mapsto (t, k(t, u))$, $u \in [-1/2, 1/2]$, are C^1 differentiable and tangent to E^s, and they form, as a whole, a C^1 lamination on a neighborhood of x_0.

1. *$k(t, u)$ is well-defined*: Let (t, u) be fixed. Take $u_1, u_2 \in V_0$ with $u_1 < u < u_2$. If $p \in \operatorname{Sd}(f) \cap Q$ is in the domain between $\operatorname{graph}(k_{u_1})$ and $\operatorname{graph}(k_{u_2})$, then p belongs to Π. This is proved by the method proving in the above that h_p is onto, and by the fact that subarcs K_p, K_q of different two stable manifolds never meet each other. Remark that this fact also plays an important role in the following.

So it is obvious that $\{K_p; p \in \Pi\}$ meet the vertical segment $\{t\} \times [k_{u_1}(t), k_{u_2}(t)] \subset Q$ densely. That is, the set $\{k_{u'}(t); u' \in V_0\}$ is dense in $[k_{u_1}(t), k_{u_2}(t)]$. Therefore, given $\varepsilon > 0$, there is a finite sequence of numbers $u'_1, u'_2, \cdots, u'_n \in V_0$ such that

(8) $u_1 = u'_1 < u'_2 < \cdots < u'_n = u_2$,

(9) $k'_{u_{i+1}}(t) - k'_{u_i}(t) < \varepsilon, \ \forall 1 \leq i < n$.

Let j be the suffix with $u'_j < u < u'_{j+1}$. By (9), for $\forall u', u'' \in V_0 \cap [u'_j, u'_{j+1}]$,

$$|k_{u'}(t) - k_{u''}(t)| < k_{u'_{j+1}}(t) - k_{u'_j}(t) < \varepsilon .$$

Hence $\{k_{u'}(t); u' \to u, u' \in V_0\}$ is a Cauchy sequence. q.e.d.

2. *The convergence $k_{u'}(t) \to k(t, u)$ is C^0 uniform*: Given $\varepsilon > 0$, choose a finite sequence of numbers $t_1, t_2, \cdots, t_n \in [-1, 1]$ such that

(10) $-1 = t_1 < t_2 < \cdots < t_n = 1$,

(11) $t_{i+1} - t_i < \varepsilon/2, \ \forall 1 \leq i < n$.

We can take $u_1, u_2 \in V_0$ such that

(12) $u_1 < u < u_2$

(13) $k_{u_2}(t_i) - k_{u_1}(t_i) < \varepsilon, \ \forall 1 \leq i \leq n$.

By the way, if $|t - t_i| < \varepsilon$, by (7) we have

$$\begin{aligned} |k_{u_1}(t) - k_{u_1}(t_i)| &= \left|\int_{t_i}^{t} \frac{d}{dt} k_{u_1}(t)\, dt\right| \\ &= \left|\int_{t_i}^{t} w(t, k_{u_1}(t)) dt\right| \\ &\leq |t - t_i|/4 < \varepsilon/4 . \end{aligned}$$

Likewise $|k_{u_2}(t) - k_{u_2}(t_i)| < \varepsilon/4$. Let $u' \in V_0$, $u_1 < u' < u_2$. For $\forall t \in [-1, 1]$, choose t_i with $|t_i - t| < \varepsilon$. Then

$$\begin{aligned} |k(t, u) - k_{u'}(t)| &\leq k_{u_2}(t) - k_{u_1}(t) \\ &\leq |k_{u_2}(t) - k_{u_2}(t_i)| + |k_{u_2}(t_i) - k_{u_1}(t_i)| \\ &\quad + |k_{u_1}(t) - k_{u_1}(t_i)| < \varepsilon/4 + \varepsilon/2 + \varepsilon/4 = \varepsilon . \end{aligned}$$

Thus we have $|k(\cdot, u) - k_{u'}(\cdot)| < \varepsilon$ if $u' \in V_0$, $|u' - u| < \delta$, where $\delta = \min\{|u_1 - u|, |u_2 - u|\}$. q.e.d.

3. *$\{(d/dt)k_{u'}; u' \to u, u' \in V_0\}$ is uniformly convergent*: Because

$$\frac{d}{dt} k_{u'}(t) = w(t, k_{u'}(t)) ,$$

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and $k_{u'}(t)$ is uniformly convergent. q.e.d.

Therefore, $v = k(t, u)$, $(t, u) \in [-1, 1] \times [-1/2, 1/2]$, is C^1 differentiable in t and satisfies the differential equation $dv/dt = w(t, v)$.

It is easy to see that the mapping $H: [-1, 1] \times [-1/2, 1/2] \to Q$ defined by $H(t, u) = (t, k(t, u))$ is a homeomorphism (into). So we can define a C^1 lamination on a neighborhood of x_0 by letting its laminae be curves $t \mapsto H(t, u)$, $u \in [-1/2, 1/2]$. To guarantee the existence of a global lamination W^s on U, we need only check that two local laminations thus defined are always consistent with each other. But, otherwise, there must be a pair of stable manifolds having an intersection by the construction of laminae.

Clearly the lamination W^s satisfies the desired conditions. q.e.d.

§ 4. Theorem 2

For simplicity we denote by U_f the f orbit of $U \subset M$. The following proposition plays a basic role in proving Theorem 2.

PROPOSITION 2. *Let U be an open subset of M. If $f \in \operatorname{int} \mathscr{H}(U)$, then there is a continuous splitting $E^s \oplus E^u$ of $TM | \overline{\operatorname{Sd}(f) \cap U_f}$ whose splitting at $\forall p \in \operatorname{Sd}(f) \cap U_f$ is $E^s(p; f) \oplus E^u(p; f)$.*

The proof will be given in the next section. Now we prove Theorem 2.

THEOREM 2. *For any open subset U of M, we have*

$$\mathscr{D}(U) \cap \operatorname{int} \mathscr{H}(U) \subset \mathscr{D}(M) .$$

Proof. Let $f \in \mathscr{D}(U) \cap \operatorname{int} \mathscr{H}(U)$. Clearly $\operatorname{Per}(f) \cap U_f \subset \operatorname{Sd}(f)$. So, $\operatorname{Sd}(f)$ is dense in U_f. Applying Proposition 2, we have a splitting $E^s \oplus E^u$ of $TM | \overline{U}_f$ whose splitting at $\forall p \in \operatorname{Sd}(f) \cap U_f$ is $E^s(p; f) \oplus E^u(p; f)$. Hence, by Proposition 1, there are f-invariant laminations W^s and W^u such that $W^s(p; f)$ and $W^u(p; f)$, $\forall p \in \operatorname{Sd}(f) \cap U_f$, are respectively their laminae.

It is sufficient to show $\overline{U}_f = M$, because $\operatorname{Per}(f)$ is dense in U_f. For this, we need only prove that for $\forall x_0 \in \overline{U}_f$, there is a neighborhood of x_0 included in $\overline{U}_f$. Let us write $\Sigma = \operatorname{Sd}(f) \cap U_f$. We claim

(1) *Let $p \in \Sigma$. Let $\varphi: \boldsymbol{R} \to W^s(p)$, $\varphi(0) = p$, be a parametrization of $W^s(p)$. Then $\varphi(\infty) = \lim_{t \to \infty} \varphi(t)$ never exists.*

Proof of (1). Suppose there exists $\varphi(\infty)$. Let α be the period of p. First, $\varphi(\infty) \notin U_f$, because by Proposition 1 $W^s(p)$ is a lamina of W^s. It is

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also clear that $f^{2\alpha}(\varphi(\infty))=\varphi(\infty)$. Since the laminations W^s, W^u are transversal, we have $q\in\Sigma$ with $\varphi\{(0,\infty)\}\cap W^u(q)\neq\phi$. Let $y\in\varphi\{(0,\infty)\}\cap W^u(q)$. Denote by β the period of q. Since $y\in W^u(q)$, $f^{-2\alpha\beta n}(y)\to q$ as $n\to\infty$. Since $y\in\varphi\{(0,\infty)\}$, $f^{-2\alpha\beta n}(y)\to\varphi(\infty)$ as $n\to\infty$. Hence $q=\varphi(\infty)$. This is a contradiction, because $\varphi(\infty)\notin U_f$. q.e.d.

By continuity of $E^s\oplus E^u$, we may choose a coordinate neighborhood (Q,ψ) of x_0 satisfying the following (2) ~ (4).

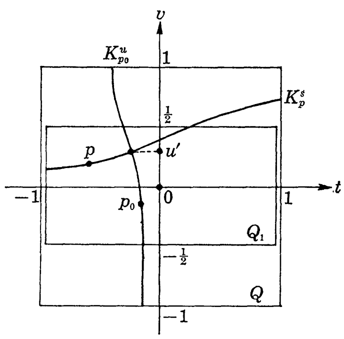
(2) $\psi(Q)=[-1,1]\times[-1,1]$

(3) $\psi(x_0)=(0,0)$

(4) Identify Q with its image by ψ and E^s, E^u with $T\psi(E^s)$, $T\psi(E^u)$ respectively. Then we have C^0 functions $w_s, w_u\colon Q\cap\bar{U}_f\to[-1/4,1/4]$ such that $(1,w_s(x)), (w_u(x),1)\in T_xQ$ span respectively $E^s(x)$, $E^u(x)$ for $\forall x\in Q\cap\bar{U}_f$.

Let $p\in\Sigma\cap Q$. We denote by K_p^s (resp. K_p^u) the connected component of $W^s(p)\cap Q$ (resp. $W^u(p)\cap Q$) containing p. We express the coordinate system in Q as (t,v). Noting that K_p^s is an integral curve of the vector field $x\mapsto(1,w_s(x))$ $(x\in Q\cap U_f)$, we have a function $v=k_p(t)$ with graph $(k_p)=K_p^s$. Let Π be the set of all $p\in\Sigma\cap Q$ such that the domain of k_p is $[-1,1]$. Put $Q_1=[-1,1]\times[-1/2,1/2]\subset Q$. As in the previous section, we can prove $\Sigma\cap Q_1\subset\Pi$ by virtue of (1).

Let us fix a point $p_0\in[-1/4,1/4]\times[-1/4,1/4]\cap\Sigma$. Similarly as above, we have a function $t=h(v)$, $v\in[-1,1]$ with graph $(h)=K_{p_0}^u$. For $\forall p\in\Pi$, $K_p^s\cap K_{p_0}^u$ consists of just a point. Let $\pi_2(t,v)=v$ be the projection. Define $V_0=\{\pi_2(K_p^s\cap K_{p_0}^u); p\in\Pi\}$. Since $\Sigma\cap Q_1\subset\Pi$, V_0 is dense in $[-1/2,1/2]$. For $\forall u'\in V_0$, we put $k(t,u')=k_p(t)$, where $\pi_2(K_p^s\cap K_{p_0}^u)=u'$. See the figure.



Now we define a function $v=k(t,u)$, $(t,u)\in[-1,1]\times[-1/2,1/2]$ by

$$\underline{k}(t,u)=\lim_{u'\uparrow u}k(t,u')\,,\qquad u'\in V_0\,.$$

First, this is well-defined, because $k(t, u')$ is monotonuous in $u' \in V_0$. As in the previous section, we have similarly that this convergence is C^1 uniform in $t \in [-1, 1]$.

Likewise we define another function $v = \bar{k}(t, u)$, $(t, u) \in [-1, 1] \times [-1/2, 1/2]$ by

$$\bar{k}(t, u) = \lim_{u' \downarrow u} k(t, u') , \qquad u' \in V_0 .$$

We want to show $\underline{k} = \bar{k}$. Suppose that for some t_1, u_1 $\underline{k}(t_1, u_1) \neq \bar{k}(t_1, u_1)$. Let D be the region in Q between the graphs of $\underline{k}(\cdot, u_1)$ and $\bar{k}(\cdot, u_1)$. First we have $D \cap U_f = \phi$. If not, we can take two points $p_1, p_2 \in \Sigma \cap D$. By (1), they belong to Π. So the region in Q between $K^s_{p_1}$ and $K^s_{p_2}$ is included in D. But this is impossible, because $\underline{k}(t_2, u_1) = \bar{k}(t_2, u_1)$ where $(t_2, u_1) \in K^u_{p_0}$. Thus $D \cap U_f = \phi$.

We also have $D \cap U_f \neq \phi$. This is shown as follows. Put $x_1 = (t_1, \underline{k}(t_1, u_1))$. We notice that the graphs of $k(\cdot, u')$, $u' \in V_0$, are included in U_f. So, $x_1 = \lim (t_1, k(t_1, u'))$ $(u' \uparrow u_1, u' \in V_0)$ is contained in $\bar{U}_f$. Hence we can choose a point $p \in \Sigma$ near x_1. Then K^u_p meets the graph of $\underline{k}(\cdot, u_1)$ at a point near x_1. So it meets D, too. Since $K^u_p \subset U_f$, we have $D \cap U_f \neq \phi$.

Thus we have a contradiction. Therefore, $\underline{k} = \bar{k}$. Hereafter we write $k = \underline{k} = \bar{k}$.

It is easily shown that the mapping $H\colon [-1, 1] \times [-1/2, 1/2] \to Q$ defined by $H(t, u) = (t, k(t, u))$ is a homeomorphism (into). Moreover, its image is in $\bar{U}_f$. So it is sufficient to show that $\operatorname{Im}(H) \supset [-1/2, 1/2] \times [-1/4, 1/4]$.

By (4), $K^u_{p_0}$ meets the segments $[-1/2, 1/2] \times \{1/2\}$, and $[-1/2, 1/2] \times \{-1/2\} \subset Q$. Let these intersections be y_1, y_2 respectively. By definition, graph $(k(\cdot, 1/2))$ goes through y_1, and graph $(k(\cdot, 1/2))$ through y_2. Hence it follows from $|(\partial/\partial t)k(t, u)| = |w_s(t, k(t, u)| < 1/4$ that for $\forall t \in [-1/2, 1/2]$, $k(t, 1/2) > 1/4$ and $k(t, -1/2) < -1/4$. Hence, as u goes from $-1/2$ to $1/2$ with $t \in [-1/2, 1/2]$ fixed, $k(t, u)$ varies from $k(t, -1/2) < -1/4$ to $k(t, 1/2) > 1/4$. By continuity of k, it follows that for $\forall t \in [-1/2, 1/2]$, $\{t\} \times [-1/4, 1/4] \subset \operatorname{Im}(H)$. That is, $[-1/2, 1/2] \times [-1/4, 1/4] \subset \operatorname{Im}(H)$. Hence $x_0 = (0, 0) \in \operatorname{int} \bar{U}_f$.

Thus we have proved Theorem 2. q.e.d.

§ 5. Proposition 2

In the proof of Theorem 2, Proposition 2 still remains to be proved.

PROPOSITION 2. *Let U be an open subset of M. If $f \in \operatorname{int} \mathscr{H}(U)$, then there is a continuous splitting $E^s \oplus E^u$ of $TM|\overline{\mathrm{Sd}(f) \cap U_f}$ whose splitting at $\forall p \in \mathrm{Sd}(f) \cap U_f$ is $E^s(p;f) \oplus E^u(p;f)$.*

Proof. We state two assertions, which will be proved later, and using them, we obtain the proof of Proposition 2.

Let GM be the bundle over M whose fiber at x consists of all 1-dimensional subspaces of T_xM. Let d be the metric on GM induced from a Riemann metric on M.

ASSERTION 1. *There is a C^1 neighborhood $\mathscr{U}$ of f such that*

$$\inf\{d(E^s(p;g), E^u(p;g));\ g \in \mathscr{U},\ p \in \mathrm{Sd}(g) \cap U_g\} > 0\,.$$

ASSERTION 2. *There is a positive integer ν such that*

$$\| Tf^\nu | E^s(p) \| / \| Tf^\nu | E^u(p) \| \leq 1/2\,, \qquad \forall p \in \mathrm{Sd}(f) \cap U_f\,.$$

Now Proposition 2 is proved as follows: Let $x \in \overline{\mathrm{Sd}(f) \cap U_f}$. Let $p_n, q_n \in \mathrm{Sd}(f) \cap U_f$, $n = 1, 2, \cdots$ be two sequences converging to x such that $E^s(p_n), E^u(p_n); E^s(q_n), E^u(q_n)$ have a limit. Denote their limits by F^s, F^u; G^s, G^u respectively. It is sufficient to prove $F^s = G^s$ and $F^u = G^u$. Suppose $F^s \neq G^s$, for example. It follows from Assertion 1 that $F^s \neq F^u$, $G^s \neq G^u$. Our argument is divided into three cases.

1. The case $F^s \neq G^u$. It follows from Assertion 2 that

$$\| T_x f^{k\nu} | F^s \| / \| T_x f^{k\nu} | F^u \| \leq 1/2^k\,, \qquad \forall k > 0\,.$$

Since $G^s \neq F^s$ and $G^u \neq F^s$, we have by this that given $\varepsilon > 0$, there is $k > 0$ such that

$$d(T_x f^{k\nu}(G^s), T_x f^{k\nu}(F^u)) < \varepsilon\,,$$
$$d(T_x f^{k\nu}(G^u), T_x f^{k\nu}(F^u)) < \varepsilon\,.$$

Hence we have

$$d(T_x f^{k\nu}(G^s), T_x f^{k\nu}(G^u)) < 2\varepsilon\,.$$

This clearly contradicts Assertion 1.

2. The case $F^u \neq G^s$. This is the same with the case 1, if F and G are interchanged.

3. The case $F^s = G^u$ and $F^u = G^s$. By Assertion 2, we have

$$\|T_x f^\nu | F^s\| / \|T_x f^\nu | F^u\| \leq 1/2 ,$$
$$\|T_x f^\nu | G^s\| / \|T_x f^\nu | G^u\| \leq 1/2 .$$

The above inequalities contradict each other, because $F^s = G^u$ and $F^u = G^s$. Thus we have derived a contradiction from the assumption $F^s \neq G^s$. Hence we have Proposition 2. q.e.d.

To prove Assertions 1, 2 we prepare the following.

ASSERTION 3. *For some small C^1 neighborhood $\mathscr{U}_1$ of f, there is a constant $0 < \lambda < 1$ such that for $\forall g \in \mathscr{U}_1$, $\forall p \in \mathrm{Sd}(g) \cap U_g$*

$$\|Tg^{\alpha(p)} | E^s(p; g)\| < \lambda ,$$
$$\|Tg^{-\alpha(p)} | E^u(p; g)\| < \lambda ,$$

where $\alpha(p)$ means the g period of p.

Proof of Assertion 3. Suppose otherwise. We may assume without loss of generality that for any $\varepsilon > 0$, there exists g in the $\varepsilon - C^1$ neighborhood of f with $\|Tg^{\alpha(p)} | E^s(p; g)\| > 1 - \varepsilon$ for some $p \in \mathrm{Sd}(g) \cap U$. Let $\varepsilon_1 = 1 - \|Tg^{\alpha(p)} | E^s(p; g)\|$. Clearly $0 < \varepsilon_1 < \varepsilon$.

By Lemma B_2 in Appendix, we have a $C\varepsilon - C^1$ perturbation h of the identity of M (C is a constant as in that lemma) such that

(1) $h(p) = p$.

(2) $T_p h = (1 - \varepsilon_1)^{-1} I_p$ where $I_P: T_p M \hookleftarrow$ is the identity.

(3) $h(x) = x$ for x outside a small neighborhood of p.

We define $g_1 = h \cdot g \in \mathrm{Diff}^1(M)$. By (1), (3), $g_1 = g$ on the orbit of p. Clearly $E^s(p; g)$ is invariant under $T_p g^\alpha(p)$. But we have

$$\begin{aligned}\|T_p g_1^{\alpha(p)} | E^s(p; g)\| &= \|T_p h \cdot T_p g^{\alpha(p)} | E^s(p; g)\| \\ &= (1 - \varepsilon_1)^{-1} \|T_p g^{\alpha(p)} | E^s(p; g)\| = 1 .\end{aligned}$$

Since the dimension of $E^s(p; g)$ is one, it follows that p is not hyperbolic for g_1. By construction, g_1 is near f in $\mathrm{Diff}^1(M)$, so $f \notin \mathrm{int}\,\mathscr{H}(U)$. This is a contradiction. q.e.d.

Proof of Assertion 1. Suppose it is not true. Then, for any $\varepsilon > 0$ we have $g \in \mathscr{U}_1$ with

$$\tan d(E^s(p; g), E^u(p; g)) < 2^{-1}\varepsilon(1 - \lambda)$$

for some $p \in \mathrm{Sd}(g) \cap U_g$, where $\mathscr{U}_1$ and λ are the ones given in Assertion 3. Let α be the period of p. Take a small neighborhood Q of p with

(1) $Q \not\ni g^n(p),\ \forall 1 \leq n \leq \alpha - 1$.

We denote by $W_r^s(p; g)(W_r^u(p; g))$ the local stable (unstable) manifold of size $r > 0$. We choose orthogonal coordinates (t, v) in Q with origin at p such that the t-axis is $W_r^s(p; g)$.

The function $v = \psi(t)$ representing $W_r^u(p; g)$ has the form:

(2) $\psi(t) = c \cdot t + R(t),\ t \in (-r, r)$.

(3) $|c| < 2^{-1}\varepsilon(1 - \lambda)$.

(4) $R(0) = R'(0) = 0$.

By (4), taking r small, we may assume:

(5) $|R'(t)| < 2^{-1}\varepsilon(1 - \lambda),\ \forall t \in (-r, r)$.

So we have

(6) $|R(t)| < 2^{-1}\varepsilon(1 - \lambda)r,\ \forall t \in (-r, r)$.

Noting $p = g^\alpha(p) = (0, 0)$, we define C^1 mappings $h_1, h_2 \colon (-r, r) \to (-r, r)$ respectively by

(7) $h_1(t) = \pi_1 g^\alpha(t, 0)$,

(8) $h_2(t) = \pi_1 g^{-\alpha}(t, \psi(t))$, where π_1 is the projection on the first factor.

Since $|h_1'(0)| < \lambda$, $|h_2'(0)| < \lambda$, by taking r small enough we have

(9) $|h_1(t)| \leq \lambda |t|,\ |h_2(t)| \leq \lambda |t|,\ \forall t \in (-r, r)$.

Put $b = r/2$ and $\delta = (1 - \lambda)b$. For $\forall t \in (-r, r)$ we have

(10) $|\psi(t)| \leq |c|\, r + |R(t)| < \varepsilon r(1 - \lambda) = 2\varepsilon\delta$,

(11) $|\psi'(t)| \leq |c| + |R'(t)| < \varepsilon(1 - \lambda) < \varepsilon$.

Let $x_1 = (b, 0)$, $x_2 = (b, \psi(b))$. Then

$$|\pi_1 g^\alpha(x_1) - b| = |h_1(b) - b| > b - \lambda b = \delta\ ,$$
$$|\pi_1 g^\alpha(x_2) - b| = |h_2(b) - b| > b - \lambda b = \delta\ .$$

Hence we have

(12) $\|g^\alpha(x_1) - x_1\| > \delta$,

(13) $\|g^{-\alpha}(x_2) - x_1\| > \delta$.

We define a C^1 mapping $k \colon Q \to Q$ as follows: Let $\phi \colon \boldsymbol{R} \to \boldsymbol{R}$ be a C^1 function with $\phi(-\infty, 1/2] = 1$, $\phi[1, \infty) = 0$.

(14) $k(t, v) = (t, v - \phi(\{(t - b)^2 + v^2\}/\delta^2) \cdot \psi(t))$.

Then the following holds:

(15) $k(t, \psi(t)) = (t, 0)$, if $|t - b|$ is sufficiently small.

(16) $k(x) = x$, if $\|x - x_1\| > \delta$.

(17) k is near the identity of Q in the C^1 sense when ε is small.

The last is shown as follows. By (10), (17) is true in the C^0 sense. By (10), (11) and the fact that $\phi(\{(t - b)^2 + v^2\}/\delta^2) = 0$ if $|t - b| > \delta$, we have

$$\left\|\frac{\partial}{\partial t}\{k(t, v) - (t, v)\}\right\| = |\phi'(\{(t-b)^2 + v^2\}/\delta^2)\cdot 2\delta^{-2}(t-b)\cdot\psi(t)$$
$$+ \phi(\{(t-b)^2 + v^2\}/\delta^2)\cdot\psi'(t)|$$
$$\leq |\phi'|\cdot 2\delta^{-2}\delta\cdot 2\varepsilon\delta + |\phi|\,\varepsilon$$
$$= (4|\phi'| + |\phi|)\varepsilon \to 0\,.$$

We also have the same result about $\partial/\partial v$. Thus (17) follows.

We extend k to a mapping of $M \to M$ by letting $k(x) = x$ for x outside Q. By (17) we make k a diffeomorphism of M. Then we define $g_1 = k\cdot g \in \mathrm{Diff}^1(M)$. By (1), (16), we have

(18) $g_1^n(p) = g^n(p)$, $\forall n \in \boldsymbol{Z}$.

In particular, p is a periodic point of g_1.

By (1) and (16) we have

$$g_1^{-\alpha}(x_1) = (g^{-1}k^{-1})^\alpha(x_1) = g^{-\alpha}(x_2)\,.$$

So it follows from (13), (9) that

$$g_1^{-n\alpha}(g^{-\alpha}(x_2)) = g^{-n\alpha}(g^{-\alpha}(x_2))\,, \qquad \forall n \geq 1\,.$$

Hence we have

(19) $g_1^{-n\alpha}(x_1) = g^{-n\alpha}(x_2)$, $\forall n \geq 1$.

This implies that $x_1 \in W^u(p; g_1)$, because $g_1^{-n\alpha}(x_1)$ approaches p as $n \to \infty$. By (15), we can prove similarly that any point of the form $(t, 0)$ with $|t - b|$ small enough is contained in $W^u(p; g_1)$.

By (12) and (16), we have

$$g_1^\alpha(x_1) = (k\cdot g)^\alpha(x_1) = k\cdot g^\alpha(x_1) = g^\alpha(x_1)\,.$$

Similarly we have

(20) $g_1^{n\alpha}(x_1) = g^{n\alpha}(x_1)$, $\forall n \geq 1$.

This implies that $x_1 \in W^s(p; g_1)$. Also, we can prove similarly that any point of the form $(t, 0)$ with $|t - b|$ small enough is contained in $W^s(p; g_1)$.

Thus it is proved that x_1 is a non-transversal homoclinic point of g_1. It is clear that the g_1 orbit of x_1 meets U and hence g_1 has a non-transversal homoclinic point in U. By Lemma A in Appendix, we have a small perturbation of g_1 with a non-hyperbolic periodic point in U. This contradicts the hypothesis, i.e. $f \in \mathrm{int}\,\mathscr{H}(U)$. q.e.d

For the proof of Assertion 2 we first prove the following. For $\forall p \in \mathrm{Sd}(f) \cap U_f$, we define $N(p)$ to be the smallest positive integer n such that

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$$\| Tf^n | E^s(p;f) \| / \| Tf^n | E^u(p;f) \| < \lambda .$$

Clearly, $N(p)$ does not exceed the period of p.

ASSERTION 4. $\sup \{N(p); p \in \mathrm{Sd}(f) \cap U_f\} < \infty$.

Proof. Given $\varepsilon > 0$, take a positive integer n_0 such that

(1) $(1-\lambda)^{n_0} < \lambda\varepsilon^2$.

Suppose the above is not true. Then there is $p \in \mathrm{Sd}(f) \cap U_f$ with $N(p) \geq n_0 + 3$. Let τ be the greatest integer such that $2\tau + 2 \leq N(p)$. Let α be the period of p. Then,

(2) $n_0 + 2 \leq 2\tau + 2 \leq N(p) \leq \alpha$.

We take unit vectors $V^s \in E^s(p;f)$, $V^u \in E^u(p;f)$. In what follows, we simply write

(3) $p_n = f^{n-1}(p)$,

(4) $V_n^s = Tf^{n-1}(V^s)$, $V_n^u = Tf^{n-1}(V^u)$, $\forall n \in \boldsymbol{Z}$.

Note that $p_\alpha = p_0$ but $V_\alpha^s \neq V_0^s$, $V_\alpha^u \neq V_0^u$.

By Lemma B_2 in Appendix we construct $h = h_\varepsilon \in \mathrm{Diff}^1(M)$ with the following properties (5) $\sim$ (11) in such a way that h approaches the identity in the C^1 sense as $\varepsilon \to 0$.

(5) $h(p_n) = p_n$, $\forall 0 \leq n < \alpha$.

(6) $h(x) = x$, $\forall x$ outside a small neighborhood of $\{p_n; 0 \leq n < \alpha\}$.

(7) $T_{p_1}h(V_1^s) = V_1^s$, $T_{p_1}h(V_1^u) = V_1^u + \varepsilon V_1^s$.

(8) $\forall 2 \leq n \leq \tau + 1$;

$$T_{p_n}h(V_n^s) = (1-\varepsilon)^{-1} V_n^s , \qquad T_{p_n}h(V_n^u) = (1-\varepsilon) V_n^u .$$

(9) $\forall \tau + 2 \leq n \leq 2\tau + 1$;

$$T_{p_n}h(V_n^s) = (1-\varepsilon) V_n^s , \qquad T_{p_n}h(V_n^u) = (1-\varepsilon)^{-1} V_n^u .$$

(10) $\forall 2\tau + 2 \leq n \leq \alpha - 1$; $T_{p_n}h \colon T_{p_n}M \hookleftarrow$ is the identity.

(11) $T_{p_\alpha}h(V_\alpha^s) = V_\alpha^s$, $T_{p_\alpha}h(V_\alpha^u) = V_\alpha^u - \varepsilon V_\alpha^s$.

Then we define $g = h \cdot f \in \mathrm{Diff}^1(M)$. By (5),

(12) $g^n(p) = f^n(p)$, $\forall n \in \boldsymbol{Z}$.

It follows from (8), (9), (10) that

(13) $T_{p_1}g^n = T_{p_1}f^n$, $\forall 2\tau \leq n \leq \alpha - 2$.

Now we want to show that

(14) $T_{p_0}g^\alpha = T_{p_0}f^\alpha$.

For this, it is sufficient to show the following:

(15) $T_{p_0}g(V_0^s) = V_\alpha^s$, $T_{p_0}g(V_0^u) = V_\alpha^u$.

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The first is easily shown, so we check the latter:

$$\begin{aligned} T_{p_0}g^{\alpha}(V_0^u) &= T_{p_{\alpha-1}}gT_{p_1}g^{\alpha-2}T_{p_0}g(V_0^u) \\ &= (T_{p_\alpha}hT_{p_{\alpha-1}}f)\cdot(T_{p_1}f^{\alpha-2})\cdot(T_{p_1}hT_{p_0}f)(V_0^u) \qquad \text{(by (13))} \\ &= T_{p_\alpha}hT_{p_1}f^{\alpha-1}T_{p_1}h(V_1^u) \\ &= T_{p_\alpha}hT_{p_1}f^{\alpha-1}(V_1^u + \varepsilon V_1^s) \qquad \text{(by (7))} \\ &= T_{p_\alpha}h(V_\alpha^u + \varepsilon V_\alpha^s) \qquad \text{(by (4))} \\ &= (V_\alpha^u - \varepsilon V_\alpha^s) + \varepsilon V_\alpha^s \qquad \text{(by (11))} \\ &= V_\alpha^u\,. \end{aligned}$$

It follows from (14) that g^{α} is hyperbolic at p_0 and

(16) $E^u(p_0; g) = E^u(p_0; g)$.

It is also clear by the construction of h that

(17) $E^s(p_n; g) = E^s(p_n; f)$, $\forall 0 \leq n < \alpha$.

Now we are in a position to conclude the proof. We estimate $d(E^s(p_{\tau+1}; g), E^u(p_{\tau+1}; g))$. By virtue of (16) and (17), this is equal to the angle θ between $T_{p_0}g^{\tau+1}(V_0^u)$ and $E^s(p_{\tau+1}; f)$.

Write $T_{p_0}g^{\tau+1}(V_0^u) = (w_s, w_u)$ regarding $E^s(p_{\tau+1}; f) \oplus E^u(p_{\tau+1}; f)$. Let us compute w_s, w_u.

$$\begin{aligned} T_{p_0}g^{\tau+1}(V_0^u) &= T_pg^{\tau}(\varepsilon V_1^s + V_1^u) \qquad \text{(by (7))} \\ &= \varepsilon T_pg^{\tau}(V_1^s) + T_pg^{\tau}(V_1^u) \\ &= \varepsilon(1-\varepsilon)^{-\tau}T_pf^{\tau}(V_1^s) + (1-\varepsilon)^{\tau}T_pf^{\tau}(V_1^u)\,. \end{aligned}$$

Hence

(18) $w_s = \varepsilon(1-\varepsilon)^{-\tau}T_pf^{\tau}(V_1^s)$, $w_u = (1-\varepsilon)^{\tau}T_pf^{\tau}(V_1^u)$.

By (2) and the definition of $N(p)$, it follows that

$$\begin{aligned} \|w_u\|/\|w_s\| &= \varepsilon^{-1}(1-\varepsilon)^{2\tau}\|T_pf^{\tau}(V_1^u)\|/\|T_pf^{\tau}(V_1^s)\| \qquad \text{(by (18))} \\ &< \varepsilon^{-1}\lambda^{-1}(1-\varepsilon)^{2\tau} \\ &< \varepsilon^{-1}\lambda^{-1}(1-\varepsilon)^{n_0} < \varepsilon \qquad \text{(by (1))}\,. \end{aligned}$$

Hence it follows that

$$\cos\theta = (w_s + w_u)\cdot w_u/\|w_s + w_u\|\,\|w_u\| > (1-\varepsilon)/(1+\varepsilon) \longrightarrow 1\,,$$

as $\varepsilon \to 0$.

Therefore, θ approaches 0 as $\varepsilon \to 0$, which contradicts Assertion 1.

q.e.d.

Proof of Assertion 2. By Assertion 4, let

$$N = \sup\{N(p); p \in \mathrm{Sd}(f) \cap U_f\} < \infty .$$

Put $C = \|Tf\|\,\|Tf^{-1}\|$. We take a positive integer m with $C^N\lambda^m < 1/2$. Let $\nu = (m+1)N$.

For $\forall p \in \mathrm{Sd}(f) \cap U_f$, we define $q_1, q_2, \cdots, q_{r+1} \in \mathrm{Sd}(f)$ as follows:

(1) $q_1 = p$.

(2) $q_{i+1} = f^{N(q_i)}(q_i)$, $1 \leq i \leq r$.

(3) $\nu - N \leq \sum_{i=1}^{r} N(q_i) < \nu$.

Since $N(q_i) \leq N$, $\forall 1 \leq i \leq r$, it follows that $rN \geq \nu - N = mN$ and hence $r \geq m$.

Noting E^s, E^u are 1 dimensional, we have

$$\begin{aligned}\frac{\|Tf^\nu | E^s(p)\|}{\|Tf^\nu | E^u(p)\|} &\leq C^N \prod_{i=1}^{r} \frac{\|Tf^{N(q_i)} | E^s(q_i)\|}{\|Tf^{N(q_i)} | E^u(q_i)\|} \\ &\leq C^N\lambda^r \leq C^N\lambda^m < 1/2 .\end{aligned}$$

(The second inequality follows from the definition of $N(q_i)$.) q.e.d.

§6. Appendix

Let M be a compact manifold without boundary. Let $f\colon M \to M$ be a C^1 diffeomorphism. The purpose here is to prove the following.

LEMMA A. *If $z \in M$ is a non-transversal homoclinic point of f, then f can be approximated by a diffeomorphism with z as a non-hyperbolic periodic point.*

Remark. A similar result was proved by Newhouse [4] in a different way.

We will apply the perturbation lemmas below to the proof of Lemma A. We fix a metric d on M and a C^1 metric d^1 on a neighborhood of I in $\mathrm{Diff}^1(M)$, where I is the identity of M.

LEMMA B$_1$. *There are constants $C > 0$, $\eta > 0$ depending only on d and d^1 with the following property: Let $x_1, x_2 \in M$. If $d(x_1, x_2) < \varepsilon\delta$ for $0 < \varepsilon < \eta$, $0 < \delta < \eta$, then we have a $(C\varepsilon) - C^1$ perturbation k of I, i.e. $d^1(k, I) < C\varepsilon$, such that $k(x_1) = x_2$, and if $d(y, x_1) > \delta$, $k(y) = y$.*

LEMMA B$_2$. *There are constants $C > 0$, $\eta > 0$ depending only on d^1 with the following property: Let $x \in M$ and let $L_x\colon T_xM \hookleftarrow$ be a linear mapping. Let I_x be the identity of T_xM. If $\|L_x - I_x\| < \varepsilon$ for $0 < \varepsilon < \eta$, then*

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for any $\delta > 0$ we have a $(C\varepsilon) - C^1$ perturbation k of I such that $k(x) = x$, $T_x k = L_x$, and if $d(y, x) > \delta$, $k(y) = y$.

These facts are well-known and can be proved easily, so we omit their proofs.

Proof of Lemma A. It is sufficient to consider the case where $z \in W^s(p) \cap W^u(p)$ for some fixed point p because the other cases can be treated similarly. For convenience we denote by s, u the dimension of $W^s(p)$ and $W^u(p)$ respectively. In what follows, D^s (resp. D^u) denotes the unit disc of $\boldsymbol{R}^s$ (resp. $\boldsymbol{R}^u$) centered at 0, and $B_r(x)$ the ball neighborhood of x of radius $r > 0$ in M.

We take a coordinate neighborhood (U, ψ) of p with the following properties (1) ~ (4).

(1) $\psi(U) = D^s \times D^u$.

From now on, we identify U with $D^s \times D^u$.

(2) $D^s \times \{0\} \subset W^s(p)$, $\{0\} \times D^u \subset W^u(p)$.

(3) $\exists 0 < \lambda < 1$;

$$\|T_p f(v, 0)\| \leq \lambda \|v\| , \qquad \forall v \in \boldsymbol{R}^s ,$$
$$\|T_p f^{-1}(0, w)\| \leq \lambda \|w\| , \qquad \forall w \in \boldsymbol{R}^u .$$

(Note $T_p U \approx \boldsymbol{R}^s \times \boldsymbol{R}^u$. $\|\cdot\|$ means the Euclidean norm.)

(4) $\forall x \in U \cap f(U) \cap f^{-1}(U)$;

$$\|T_x f - T_p f\| < \alpha , \qquad \|T_x f^{-1} - T_p f^{-1}\| < \alpha ,$$

where $\alpha = (1 - \lambda)/4$.

Remark. As regards (3), refer to Nitecki [5], pp. 71 ~ 73.

Let $x \in U \cap f(U) \cap f^{-1}(U)$. For $(v, w) \in \boldsymbol{R}^s \times \boldsymbol{R}^u$, we write $(v_1, w_1) = T_x f(v, w)$, $(v_2, w_2) = T_x f^{-1}(v, w)$. Then we have (5) ~ (8) below.

(5) If $\|v\|/\|w\| \leq 1/2$, $\|v_1\|/\|w_1\| \leq 1/2$.

Proof. Let $\pi_1 : \boldsymbol{R}^s \times \boldsymbol{R}^u \to \boldsymbol{R}^s$, $\pi_2 : \boldsymbol{R}^s \times \boldsymbol{R}^u \to \boldsymbol{R}^u$ be projections.

$$\begin{aligned} v_1 &= \pi_1 T_x f(v, w) \\ &= \pi_1 (T_x f - T_p f)(v, w) + \pi_1 T_p f(v, 0) + \pi_1 T_p f(0, w) . \end{aligned}$$

Hence we have

$$\|v_1\| = \alpha(\|v\| + \|w\|) + \lambda \|v\| \leq (\lambda/2 + \alpha/2 + \alpha) \|w\| \leq \|w\|/2 .$$

Similarly

$$w_1 = \pi_2 T_x f(v, w)$$
$$= \pi_2(T_x f - T_p f)(v, w) + \pi_2 T_p f(v, 0) + \pi_2 T_p f(0, w) .$$

and hence

$$\|w_1\| \geq \lambda^{-1}\|w\| - \alpha(\|v\| + \|w\|) \geq (\lambda^{-1} - \alpha/2 - \alpha)\|w\| \geq \|w\| .$$

Thus $\|v_1\|/\|w_1\| \leq 1/2$ follows. q.e.d.

(6) If $\|w\|/\|v\| \leq 1/2$, $\|w_2\|/\|v_2\| \leq 1/2$.

The proof is similar to (5).

(7) If $\|w\|/\|v\| \leq 1/2$, $\|v_1\| \leq \lambda_1\|v\|$ where $\lambda_1 = (1 + \lambda)/2$.

Proof. Decompose v_1 as in (5). Then we estimate

$$\|v_1\| \leq \alpha(\|v\| + \|w\|) + \lambda\|v\| \leq (\lambda + 2\alpha)\|v\| \leq \lambda_1\|v\| .$$

Thus we have (7). q.e.d.

(8) If $\|v\|/\|w\| \leq 1/2$, $\|w_2\| \leq \lambda_1\|w\|$.

The proof is similar to (7).

We choose integers n_1, n_2 such that $f^{n_1}(z) \in D^s \times \{0\}$, $f^{-n_2}(z) \in \{0\} \times D^u$ respectively. Remark that these sets really imply their inverse images by ψ. Take $\delta > 0$ so small that

(9) $f^n(z) \notin B_\delta(z_1) \cup B_\delta(z_2)$, $\forall n; -n_2 < n < n_1$

where $z_1 = f^{n_1}(z)$, $z_2 = f^{-n_2}(z)$.

Regarding $U \approx D^s \times D^u$, we write

(10) $z_1 = (a_1, 0)$, $z_2 = (0, a_2)$.

Let $\varepsilon > 0$ be arbitrary. We define

$$F^u = \{(a_1, w) \in D^s \times D^u; \|w\| < \varepsilon\delta\} ,$$
$$F^s = \{(v, a_2) \in D^s \times D^u; \|v\| < \varepsilon\delta\} .$$

If n_3 is sufficiently large, then $f^{-n_3}(F^s)$, $f^{n_3}(F^u)$ are represented by C^1 mappings $h_1: D^s \to D^u$, and $h_2: D^u \to D^s$ respectively. Furthermore, we can assume

(11) $\|h_1\| < \varepsilon\delta$, $\|h_2\| < \varepsilon\delta$

(12) $\|Th_1\| < \varepsilon/2$, $\|Th_2\| < \varepsilon/2$.

Let V be a nonzero vector in $T_z W^s(p) \cap T_z W^u(p)$. We put

(13) $V_1 = T_z f^{n_1}(V)$, $V_2 = T_z f^{-n_2}(V)$.

Clearly V_1 has the form $(v_1, 0)$ with $v_1 \in \boldsymbol{R}^s$, and V_2 has the form $(0, w_2)$ with $w_2 \in \boldsymbol{R}^u$. We put

(14) $x_1 = (a_1, h_1(a_1))$, $x_2 = (h_2(a_2), a_2)$.

Since $x_1 = f^{-n_3}(F^s) \cap F^u$, $x_2 = F^s \cap f^{n_3}(F^u)$, it follows that

(15) $x_2 = f^{n_3}(x_1)$.

We put

(16) $w_1' = T_{a_1}h_1(v_1)$.

By the definition of h_1 there is $v_2 \in \boldsymbol{R}^s$ such

(17) $(v_1, w_1') = T_{x_2}f^{-n_3}(v_2, 0)$.

Let us write $(v_i^*, w_i^*) = T_{x_2}f^{-i}(v_2, 0)$, $i = 0, 1, \cdots, n_3$. Since $\|w_0^*\|/\|v_0^*\| = 0 < 1/2$, it follows inductively by (6) that $\|w_i^*\|/\|v_i^*\| \leq 1/2$, $i = 0, 1, \cdots, n_3$. Hence it follows by (7) that

(18) $\|v_2\| \leq \lambda_1^{n_3}\|v_1\|$.

Likewise we put

(19) $v_2' = T_{a_2}h_2(w_2)$.

By the definition of h_2 there is $w_1 \in \boldsymbol{R}^u$ such that

(20) $(v_2', w_2) = T_{x_1}f^{n_3}(0, w_1)$.

Applying (5) and (8) as above, we have

(21) $\|w_1\| < \lambda_1^{n_3}\|w_2\|$.

By (18), (21), for sufficiently large n_3 we have

(22) $\|w_1\|/\|v_1\| < \varepsilon/2$,

(23) $\|v_2\|/\|w_2\| < \varepsilon/2$.

We define

(24) $V_1' = (v_1, w_1 + w_1')$, $V_2' = (v_2 + v_2', w_2)$.

Then we have

$$\begin{aligned} T_{x_1}f^{n_3}(V_1') &= T_{x_1}f^{n_3}(v_1, w_1 + w_1') \\ &= T_{x_1}f^{n_3}(v_1, w_1') + T_{x_1}f^{n_3}(0, w_1) \\ &= (v_2, 0) + (v_2', w) \qquad \text{(by (17), (20))} \\ &= V_2' \,. \end{aligned}$$

That is,

(25) $T_{x_1}f^{n_3}(V_1') = V_2'$.

By (24), (13), (22), (23) and (12), we estimate

(26) $\|V_1 - V_1'\|/\|V_1\| < \varepsilon$,

(27) $\|V_2 - V_2'\|/\|V_2'\| < \varepsilon$.

By (26) we have a linear mapping $L_1\colon \boldsymbol{R}^m \to \boldsymbol{R}^m$ ($m = \dim M$) such that

(28) $L_1(V_1) = V_1'$,

(29) $\|L_1 - I\| < \varepsilon$, where I is the identity of $\boldsymbol{R}^m$.

For example, take an orthogonal basis $\{V_1, e_2, \cdots, e_m\}$, and define L_1 by

$$L_1(t_1V_1 + t_2e_2 + \cdots + t_me_m) = t_1V_1' + t_2e_2 + \cdots + t_me_m \,, \qquad \forall t_i \in \boldsymbol{R};\ 1 \leq i \leq m \,.$$

Similarly, by (27) we have a linear mapping $L_2: \boldsymbol{R}^m \to \boldsymbol{R}^m$ such that

(30) $L_2(V_2') = V_2$,

(31) $\|L_2 - I\| < \varepsilon$.

By the way, we defined $z_1 = f^{n_1}(z)$, $z_2 = f^{-n_2}(z)$. By (10), (11) and (14) we have

(32) $\|z_1 - x_1\| < \varepsilon\delta$,

(33) $\|z_2 - x_2\| < \varepsilon\delta$.

By (29), (31), (32) and (33) we can apply Lemmas B_1 and B_2 to constructing $k \in \mathrm{Diff}^1(M)$ such that

(34) $k(z_1) = x_1$, $T_{z_1}k = L_1$.

(35) $k(x_2) = z_2$, $T_{x_2}k = L_2$.

(36) $k(x) = x$, $\forall x \notin B_\delta(z_1) \cup B_\delta(z_2)$.

(37) k is a $(C\varepsilon) - C^1$ perturbation of the identity of M (C is the one in Lemmas B_1 and B_2).

Now we shall conclude the proof. Define $g = k\cdot f \in \mathrm{Diff}^1(M)$. First, it follows easily from (9), (15), (34), (35) and (36) that z is a periodic point of g of period $n_1 + n_2 + n_3$. We show that

$$T_z g^{n_1+n_2+n_3}(V) = V\,,$$

which implies that z is not hyperbolic.

$$\begin{aligned} T_z g^{n_1+n_2+n_3}(V) &= T_{x_1} g^{n_2+n_3} T_{z_1} k T_z f^{n_1}(V) && \text{(by (9), (36))} \\ &= T_{x_1} g^{n_2+n_3} L_1(V_1) && \text{(by (13), (34))} \\ &= T_{x_1} g^{n_2+n_3}(V_1') && \text{(by (28))} \\ &= T_{z_2} g^{n_2} L_2(V_2') && \text{(by (25), (35))} \\ &= T_{z_2} f^{n_2}(V_2) && \text{(by (9), (36); (30))} \\ &= V && \text{(by (13))}\,. \end{aligned}$$

Clearly g is near f in $\mathrm{Diff}^1(M)$ by virtue of (37). q.e.d.

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