MORPHIC RINGS AS TRIVIAL EXTENSIONS

JIANLONG CHEN
Department of Mathematics, Southeast University, Nanjing, 210096, P. R. China
e-mail: jlchen@seu.edu.cn

and YIQIANG ZHOU
Department of Mathematics and Statistics, Memorial University of Newfoundland
St. John’s, NL A1C 5S7, Canada
e-mail: zhou@math.mun.ca

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Abstract. A ring $R$ is called left morphic if, for every $a \in R$, $R/Ra \cong I(a)$ where $I(a)$ denotes the left annihilator of $a$ in $R$. Right morphic rings are defined analogously. In this paper, we investigate when the trivial extension $R \propto M$ of a ring $R$ and a bimodule $M$ over $R$ is (left) morphic. Several new families of (left) morphic rings are identified through the construction of trivial extensions. For example, it is shown here that if $R$ is strongly regular or semisimple, then $R \propto R$ is morphic; for an integer $n > 1$, $\mathbb{Z}/n^2$ is morphic if and only if $n$ is a product of distinct prime numbers; if $R$ is a principal ideal domain with classical quotient ring $Q$, then the trivial extension $R \propto Q/R$ is morphic; for a bimodule $M$ over $\mathbb{Z}$, $\mathbb{Z} \propto M$ is morphic if and only if $M \cong \mathbb{Z}/n$. Thus, $\mathbb{Z} \propto Q/Z$ is a morphic ring which is not clean. This example settled two questions both in the negative raised by Nicholson and Sánchez Campos, and by Nicholson, respectively.

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§1. All rings here are associative rings with identity. An element $a$ in a ring $R$ is called left morphic if $R/Ra \cong I(a)$, where $I(a)$ denotes the left annihilator of $a$ in $R$; equivalently, $a \in R$ is left morphic if and only if there exists $b \in R$ such that $Ra = I(b)$ and $Rb = I(a)$. See [4, Lemma 1]. The ring $R$ is called left morphic if every element of $R$ is left morphic. Right morphic rings are defined analogously. A left and right morphic ring is simply called a morphic ring. Left morphic rings were first introduced by Nicholson and Sánchez Campos [4] and were discussed in great detail in [4], [5] and [6]. In this paper, we investigate when the trivial extension $R \propto M$ of a ring $R$ and a bimodule $M$ over $R$ is left morphic. The following new families of (left) morphic rings are identified.

If $R$ is a strongly regular ring and $\sigma : R \rightarrow R$ is an endomorphism of $R$ with $\sigma(e) = e$ for every $e^2 = e \in R$, then $R[x; \sigma]/(x^2)$ is a left morphic ring [Theorem 1]; for any semisimple ring $R$, every matrix ring $M_n(R \propto R)$ is morphic [Theorem 7]; for any positive integers $d > 1$ and $m$, $\mathbb{Z}_{md} \propto \mathbb{Z}_d$ is morphic if and only if $d$ and $m$ are relatively prime and $d$ is a product of distinct prime numbers [Theorem 8]; if $R$ is a principal ideal domain with classical quotient ring $Q$, then $R \propto Q/R$ is morphic [Theorem 13]; for a bimodule $M$ over $\mathbb{Z}$, $\mathbb{Z} \propto M$ is morphic if and only if $M \cong \mathbb{Q}/\mathbb{Z}$ [Theorem 14]. Thus, $\mathbb{Z} \propto Q/\mathbb{Z}$ is a morphic ring which is not clean (a ring is clean if every element is the sum of an idempotent and a unit) and does not have stable range 1 (a ring $R$...
has stable range 1 if, whenever \( aR + bR = R \) with \( a, b \in R \), \( a + by \) is a unit for some \( y \in R \). This example settled two questions both in the negative raised by Nicholson and Sánchez Campos [4], and by Nicholson, respectively.

For a ring \( R \) and \( a \in R \), we let \( I_R(a) = \{ r \in R : ra = 0 \} \). Right annihilators are defined analogously. Sometimes, we simply write \( I(a) \) for \( I_R(a) \) and \( r(a) \) for \( r_R(a) \). The \( n \times n \) matrix ring over \( R \) is denoted by \( M_n(R) \). We write \( \mathbb{Z} \) for the ring of integers, \( \mathbb{Q} \) for rational numbers, and \( \mathbb{Z}_n \) for integers modulo \( n \), respectively. Regular rings here mean von Neumann regular rings.

§2. Let \( R \) be a ring and \( M \) a bimodule over \( R \). The trivial extension of \( R \) and \( M \) is \( R \otimes M = \{(a, x) : a \in R, x \in M\} \) with addition defined componentwise and multiplication defined by

\[
(a, x)(b, y) = (ab, ay + xb).
\]

In fact, \( R \otimes M \) is isomorphic to the subring \( \{(a^n, a) : a \in R, x \in M\} \) of the formal \( 2 \times 2 \) matrix ring \( (R^2)^M \), and \( R \otimes R \cong R[x]/(x^2) \). For convenience, we let \( I \propto X = \{(a, x) : a \in I, x \in X\} \) where \( I \) is a subset of \( R \) and \( X \) is a subset of \( M \).

If \( R \) is a ring and \( \sigma : R \to R \) is a ring endomorphism, let \( R[X; \sigma] \) denote the ring of skew polynomials over \( R \); that is all formal polynomials in \( x \) with coefficients from \( R \) with multiplication defined by \( xr = \sigma(r)x \). Note that if \( R(\sigma) \) is the \((R, R)\)-bimodule defined by \( R(\sigma) = R \otimes R \) and \( m \circ r = m\sigma(r) \), for all \( m \in R(\sigma) \) and \( r \in R \), then \( R[X; \sigma]/(x^2) \cong R \otimes R(\sigma) \).

A ring \( R \) is called strongly regular if \( a \in a^2R \), for every \( a \in R \). It is well known that a ring \( R \) is strongly regular if and only if \( R \) is von Neumann regular and every idempotent in \( R \) is central. If \( R \) is strongly regular, then for any \( a \in R \), \( a = ue \) where \( u \) is a unit and \( e \) is an idempotent, so that \( aR = uR(eR = Re) = Re = Ra \). Thus every one-sided ideal of a strongly regular ring is an ideal.

**Theorem 1.** If \( R \) is a strongly regular ring and \( \sigma : R \to R \) is a ring endomorphism such that \( \sigma(e) = e \) for all \( e^2 = e \in R \), then \( R[X; \sigma]/(x^2) \) is a left morphic ring.

**Proof.** Let \( S = R[x; \sigma]/(x^2) \). Then \( S = \{ r + sx : r, s \in R \} \) with \( x^2 = 0 \) and \( xt = \sigma(t)x \) for all \( t \in R \).

**Claim 1.** If \( I \) is a left or right ideal of \( S \) and \( r, s \in R \), then \( r + sx \in I \) implies that \( r \in I \) and \( sx \in I \).

In fact, \( Rr = Re \), where \( e^2 = e \in R \), so that \( r = re \) and \( e = tr \) for some \( t \in R \). If \( I \) is a left ideal, then \( e = e^2 = (e - tx)(e + tx) = (e - tx)t(r + sx) \in I \) and so \( r = re \in I \).

Hence \( sx \in I \). Let \( I \) be a right ideal. Since \( rR = eR \) and \( e = r t_0 \) with \( t_0 \in R \), then \( e = e^2 = [e + \sigma(t_0)x][e - \sigma(t_0)x] = (r + sx)t_0[e - \sigma(t_0)x] \in I \), so that \( r = re \in R \) and thus \( sx \in I \). Hence the Claim holds.

Now let \( a + bx \in S \). We need to show that \( a + bx \) is left morphic in \( S \). Write \( Ra = Re \), where \( e^2 = e \in R \).

**Claim 2.** There exists \( g^2 = g \in R \) with \( eg = ge = 0 \) such that \( S(a + bx) = S(e + gx) \).

In fact, by Claim 1, \( S(a + bx) = Sa + Sbx = Se + Sbx = Se + Sb(1 - e)x \). The last equality holds because \( bx = (bx)e + b(1 - e)x \). Let \( b_1 = b(1 - e) \) and write \( Rb_1 = Rf \) where \( f^2 = f \in R \). Thus, \( Sb_1 = Sf \), so \( Se + Sb(1 - e)x = Se + Sb_1x = Se + Sf \). Since \( fe = 0 \), we see that \( g := (1 - e)f \) is an idempotent and \( ge = eg = 0 \). Moreover,
For example, let $fg=(1-e)f=f^2=f$, showing that $Rf=Rg$. Hence $Sf=Sg$. It follows that we have $S(a+bx)=Se+Sf\cdot x=Se+SGx=S(e+gx)$.

**Claim 3.** $S(e+gx)=I((1-e)(1-g)+(1-e)x)$, where $e$ and $g$ are as in Claim 2. For any

$$
c + dx \in I((1-e)(1-g)+(1-e)x),
$$

$$
0 = (c + dx)[(1-e)(1-g)+(1-e)x]
$$

$$
= c(1-e)(1-g) + [c(1-e) + d(1-e)(1-g)]x.
$$

Thus,

$$
c(1-e)(1-g) = 0,
$$

$$
(1)
$$

$$
c(1-e) + d(1-e)(1-g) = 0.
$$

Adding (2) times $g$ to (1) gives $c(1-e)=0$; i.e., $c=ce$. Multiplying (2) by $g$ yields $cg=0$. Moreover, it follows from (2) that $d(1-e)(1-g)=0$ and so $d=de+dg$. Now let $u=e+dg$ and $v=d$. Then $(u+vx)(e+gx)=ue + (ug + ve)x = (c+dg)e + [(c+dg)g + de]x = ce + (dg + de)x = c + dx$. Hence, $c + dx \in S(e+gx)$, so that $I((1-e)(1-g)+(1-e)x) \subseteq S(e+gx)$. However, clearly we have

$$
S(e+gx) \subseteq I((1-e)(1-g)+(1-e)x).
$$

**Claim 4.** $I(a+bx) = S((1-e)(1-g)+(1-e)x)$. Let $b_1$ and $f$ be as in the proof of Claim 2. Since $Ra=Re$, we have $aR=eR$. So $aS=eS$. Thus, by Claim 1, $(a+bx)S = aS + bS = eS + bS = eS + b(1-e)xS = eS + b_1xS = (e+b_1)xS$. Hence, it suffices to show that

$$
I(e+b_1x) = S((1-e)(1-g)+(1-e)x).
$$

Since $b_1 = b_1f$ and $f = fg$, we have $[1-e)(1-g)+(1-e)x][e+b_1x] = (1-e)(1-g)b_1x = (1-e)(1-g)b_1fx = (1-e)(1-g)b_1gx = (1-e)(1-g)ggb_1fx = 0$. Hence $S((1-e)(1-g)+(1-e)x) \subseteq I(e+b_1x)$.

If $r + sx \in I(e+b_1x)$, then

$$
0 = (r + sx)(e + b_1x) = re + (rb_1 + se)x.
$$

Hence $re = 0$ and $rb_1 + se = 0$. Since $b_1 = b_1f = b_1fg$, we have $b_1g = b_1$. Thus, $0 = (rb_1 + se)g = rb_1g = rb_1$, and hence $se = 0$. Since $Rf = Rb_1 = b_1R$, we write $f = b_1t$ with $t \in R$. Then $rg = r(1-e)f = rf = rb_1t = 0$. Hence $r = r(1-e)(1-g)$, $s = s(1-e)$. Therefore, $r + sx \in S((1-e)(1-g)+(1-e)x) \subseteq S((1-e)(1-g)+(1-e)x)$. The proof is complete.

**Corollary 2.** [4, Example 8] If $D$ is a division ring with an endomorphism $\sigma$, then $D[x; \sigma]/(x^2)$ is a left morphic ring.

**Corollary 3.** If $R$ is a strongly regular ring, then $R \cong R$ is a morphic ring.

Note that there exist strongly regular rings $R$ that are not division rings with an endomorphism $\sigma : R \to R$ such that $\sigma \neq 1_R$ and $\sigma(e) = e$, for every idempotent $e \in R$. For example, let $R = D \times D$, where $D$ is a division ring and $f_i(i = 1, 2) : D \to D$ are endomorphisms that are not all identity maps. Let $\sigma : R \to R$ be given by $(d_1, d_2) \mapsto (f_1(d_1), f_2(d_2))$.

Following [6], a ring $R$ is called strongly left morphic if every matrix ring $\mathbb{M}_n(R)$ is left morphic. Strongly right morphic rings are defined analogously. A strongly left and
strongly right morphic ring is called a *strongly morphic ring*. Next, we show that, for a semisimple ring \( R, R \cong R \) is strongly morphic. We need a few lemmas.

For a bimodule \( V \) over a ring \( R \), let \( \mathcal{M}_n(V) \) be the set of \( n \times n \) formal matrices with entries in \( V \). Then \( \mathcal{M}_n(V) \) is a bimodule over \( \mathcal{M}_n(R) \) with the usual multiplication of matrices. If \( V_i \) is a bimodule over the ring \( R_i \), for \( i = 1, \ldots, n \), then \( V_1 \oplus \cdots \oplus V_n \) is a bimodule over \( R_1 \oplus \cdots \oplus R_n \) in a natural way.

**Lemma 4.** If \( V_i \) is a bimodule over the ring \( R_i \), for \( i = 1, \ldots, n \), then
\[
(R_1 \oplus \cdots \oplus R_n) \cong (V_1 \oplus \cdots \oplus V_n) \cong (R_1 \cong V_1) \oplus \cdots \oplus (R_n \cong V_n).
\]

**Proof.** The map \( \theta : (R_1 \oplus \cdots \oplus R_n) \otimes (V_1 \oplus \cdots \oplus V_n) \rightarrow (R_1 \cong V_1) \oplus \cdots \oplus (R_n \cong V_n) \) defined by
\[
((a_1, \ldots, a_n), (v_1, \ldots, v_n)) \mapsto ((a_1, v_1), \ldots, (a_n, v_n))
\]
is the required isomorphism.

**Lemma 5.** If \( V \) is a bimodule over \( R \), then \( \mathcal{M}_n(R \cong V) \cong \mathcal{M}_n(R) \otimes \mathcal{M}_n(V) \).

**Proof.** The map \( \theta : \mathcal{M}_n(R \cong V) \rightarrow \mathcal{M}_n(R) \otimes \mathcal{M}_n(V) \) defined by
\[
((a_{ij}, v_{ij})) \mapsto ((a_{ij}), (v_{ij})).
\]
is the required isomorphism.

**Lemma 6.** If \( D \) is a division ring, then \( D \cong D \) is strongly morphic.

**Proof.** It suffices to show that \( \mathcal{M}_n(D \cong D) \) is morphic. Let \( X \neq 0 \) be given by \( X = (x_{ij}) \in \mathcal{M}_n(D \cong D) \), where \( x_{ij} = (a_{ij}, b_{ij}) \in D \cong D \).

If some \( x_{ij} \in D \cong D \) is a unit, then one can move \( x_{ij} \) to the \((1, 1)\)-entry and further change all the \((1, k)\)- and \((k, 1)\)-entries for \( k > 1 \) to 0, using a series of elementary row and column operations.

If all \( x_{ij} \) are not units of \( D \cong D \), then all \( a_{ij} = 0 \), but \( b_{lm} \neq 0 \) for some \( l \) and \( m \). Again, by a series of elementary operations, \( x_{lm} \) can be moved to the \((1, 1)\)-entry, and all entries in the first row and the first column, except the \((1, 1)\)-entry, can be reduced to 0. Hence \( X \) can be reduced to
\[
Y = \begin{bmatrix} y_{11} & 0 & \cdots & 0 \\ 0 & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & y_{n2} & \cdots & y_{nn} \end{bmatrix}.
\]

Continuing in this way, we can reduce \( Y \) to a diagonal matrix. Therefore, there exist units \( U \) and \( V \) of \( \mathcal{M}_n(D \cong D) \) such that
\[
UXV = \begin{bmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_n \end{bmatrix}.
\]

Since \( D \cong D \) is morphic, by Corollary 3, each \( z_i \) is morphic in \( D \cong D \). Therefore, \( UXV \) is clearly morphic in \( \mathcal{M}_n(D \cong D) \). Since \( U, V \) are units, \( X \) is morphic in \( \mathcal{M}_n(D \cong D) \) by [4, Lemma 3].

\[ \square \]
THEOREM 7. If \( R \) is a semisimple ring, then \( R \cong R \) is strongly morphic.

Proof. Write \( R = R_1 \oplus \cdots \oplus R_s \), where \( R_i \cong M_n(D_i) \), for some division ring \( D_i \), for \( i = 1, \ldots, s \). By Lemma 4, \( R \cong R \cong \bigoplus_{i=1}^s (R_i \cong R_i) \), so that by Lemma 5

\[
\mathbb{M}_n(R \cong R) \cong \bigoplus_{i=1}^s \mathbb{M}_n(R_i \cong R_i) \cong \bigoplus_{i=1}^s \mathbb{M}_n(D_i \cong D_i) \text{ morphic ring if and only if } d \text{ and } m \text{ are relatively prime and } d \text{ is a product of distinct primes.}
\]

which is morphic, because each \( \mathbb{M}_n(D_i \cong D_i) \) is morphic by Lemma 6.

Next, we consider the morphic property of \( R \cong M \), where \( R = \mathbb{Z}_n \) or \( \mathbb{Z} \).

THEOREM 8. Let \( d, m \) be positive integers with \( d > 1 \) and \( n = dm \). Then \( \mathbb{Z}_n \cong \mathbb{Z}_d \) is a morphic ring if and only if \( d \) and \( m \) are relatively prime and \( d \) is a product of distinct primes.

Proof. Let \( S = \mathbb{Z}_n \cong \mathbb{Z}_d \). For \( k \in \mathbb{Z} \), we write \( \bar{k} \in \mathbb{Z}_n \) to mean \( \bar{k} = k + n\mathbb{Z} \) and \( \bar{k} \in \mathbb{Z}_d \) to mean \( \bar{k} = k + d\mathbb{Z} \). This will not cause problems. If \( \bar{k} \in \mathbb{Z}_n \) and \( \bar{l} \in \mathbb{Z}_d \), then \( \bar{k} = \bar{l} \in \mathbb{Z}_d \). Throughout the proof, the greatest common divisor of integers \( m \) and \( n \) is denoted by \( \gcd(m, n) \).

“\( \Rightarrow \)”. Suppose \( \mathbb{Z}_n \cong \mathbb{Z}_d \) is morphic. We first show that \( \gcd(d, m) = 1 \).

Since \((\bar{d}, 0) \in S \) is morphic, there exists \((\bar{a}, \bar{b}) \in S \) such that \( I(\bar{a}, \bar{b}) = S(\bar{d}, 0) \) and \( I(\bar{d}, 0) = S(\bar{a}, \bar{b}) \). (\( (\bar{d}, 0)(\bar{a}, \bar{b}) = (\bar{d}\bar{a}, \bar{d}\bar{b}) = 0 \). Thus \( n|da \); i.e., \( dm|da \) showing that \( m|a \). Write \( a = nm_1 \).

Since \((\bar{m}, 0) \in I(\bar{d}, 0) = S(\bar{a}, \bar{b}) \), \( (\bar{m}, 0) = (\bar{r}, \bar{s})(\bar{a}, \bar{b}) = (\bar{r}\bar{a}, \bar{r}\bar{b} + \bar{s}\bar{a}) \), where \( (\bar{r}, \bar{s}) \in S \). Thus, \( \bar{m} = \bar{r}\bar{a} \), showing that \( n|(m - ra) \) and so \( dm|(m - rm_1) \). This gives \( d|(1 - rm_1) \), so that \( \gcd(d, m_1) = 1 \).

Since \((0, \bar{1}) \in I(\bar{d}, 0) = S(\bar{a}, \bar{b}) \), \( (0, \bar{1}) = (\bar{x}, \bar{y})(\bar{a}, \bar{b}) = (\bar{x}\bar{a}, \bar{x}\bar{b} + \bar{y}\bar{a}) \), where \( (\bar{x}, \bar{y}) \in S \). Thus \( n|x \); i.e., \( dm|xnm1 \) and so \( d|xn \). Since \( \gcd(d, m_1) = 1 \), we have \( d|x \). Write \( x = \bar{d}\bar{m}_1 \). Also, \( \bar{1} = \bar{x}\bar{b} + \bar{y}\bar{a} = \bar{d}\bar{m}_1 \bar{b} + \bar{y}\bar{m}_1 = \bar{y}\bar{m}_1 \). Thus, \( d|(1 - ynm_1) \), showing that \( \gcd(d, m) = 1 \).

Next we show that \( d \) is a product of distinct primes. Since \( d > 1 \), we can write \( d = pd_1 \), so that \( n = dm = pd_1m \).

Since \((0, \bar{p}) \in S \) is morphic, there exists \((\bar{a}, \bar{b}) \in S \) such that \( I(\bar{a}, \bar{b}) = S(0, \bar{p}) \) and \( I(0, \bar{p}) = S(\bar{a}, \bar{b}) \). Thus, \( 0 = (\bar{a}, \bar{b})(0, \bar{p}) = (0, \bar{a}\bar{p}) \). Hence \( d|a \); i.e., \( pd_1|a \). Thus \( d|a \). Write \( a = d_1a_1 \).

Since \((\bar{d}_1, 0) \in I(0, \bar{p}) = S(\bar{a}, \bar{b}) \), \( (\bar{d}_1, 0) = (\bar{x}_1, \bar{y}_1)(\bar{a}, \bar{b}) = (\bar{x}_1\bar{a}, \bar{x}_1\bar{b} + \bar{y}_1\bar{a}) \), where \( (\bar{x}_1, \bar{y}_1) \in S \). Hence \( n|(d_1 - xa) \); i.e., \( pd_1m|(d_1 - xd_1a_1) \). Thus, \( pm|(1 - xa) \), showing that \( \gcd(a_1, pm) = 1 \).

Since \((0, \bar{1}) \in I(0, \bar{p}) = S(\bar{a}, \bar{b}) \), \( (0, \bar{1}) = (\bar{x}_1, \bar{y}_1)(\bar{a}, \bar{b}) = (\bar{x}_1\bar{a}, \bar{x}_1\bar{b} + \bar{y}_1\bar{a}) \), where \( (\bar{x}_1, \bar{y}_1) \in S \). It follows that \( n|x_1a \); i.e., \( pd_1m|x_1d_1a_1 \). Hence \( pm|x_1a_1 \). Because \( \gcd(pm, a_1) = 1 \), we have \( pm|x_1 \). Write \( x_1 = pm_1t_1 \). Also \( \bar{1} = \bar{x}_1\bar{b} + \bar{y}_1\bar{a} = \bar{p}\bar{m}_1\bar{b} + \bar{y}_1\bar{m}_1 \), showing that \( \bar{p} = \bar{p}^2\bar{m}_1\bar{b} + \bar{y}_1\bar{m}_1 \bar{b} \). Thus, \( d|p(1 - pmt_1b) \), so that \( pd_1|p(1 - pmt_1b) \). Hence \( \gcd(d_1, p) = 1 \). Therefore, \( d \) is a product of distinct primes.

“\( \Leftarrow \)”. Suppose that \( \gcd(d, m) = 1 \) and \( d = p_1 \cdots p_s \) is a product of distinct primes. The \( \mathbb{Z}_d \cong \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_s} \) is semisimple and \( \mathbb{Z}_n \cong \mathbb{Z}_d \oplus \mathbb{Z}_m \). Because \( \mathbb{Z}_d \cong \mathbb{Z}_d \) is morphic (Theorem 7) and \( \mathbb{Z}_m \cong \mathbb{Z}_m \) is morphic [4, Example 12], \( \mathbb{Z}_n \cong \mathbb{Z}_d \oplus \mathbb{Z}_m \cong \mathbb{Z}_d \oplus \mathbb{Z}_m \cong \mathbb{Z}_d \oplus \mathbb{Z}_m \) is morphic.
Corollary 9. For \( n \geq 2, \mathbb{Z}_n \cong \mathbb{Z}_n \) is morphic if and only if \( n \) is a product of distinct primes.

Remark 10. Corollary 9 shows that \( \mathbb{Z}_4 \cong \mathbb{Z}_4 \) is not morphic. Thus, a trivial extension of a morphic ring by itself is not morphic. Since \( \mathbb{Z}_4 \) is strongly \( \pi \)-regular (a ring \( R \) is strongly \( \pi \)-regular if, for any \( a \in R \), the chain \( aR \supseteq a^2R \supseteq \cdots \) terminates), this example also shows that Theorem 1 cannot be extended to a strongly \( \pi \)-regular ring.

If \( R \) is a commutative domain and \( Q \) is the classical quotient ring of \( R \), then \( Q = \{ \frac{t}{s} : s, 0 \neq t \in R \} \) and so every element of the \( R \)-module \( Q/R \) can be expressed as \( \frac{t}{s} = \frac{t}{s} + R \).

Lemma 11. Let \( R \) be a commutative domain with classical quotient ring \( Q \) and let \( T = R \cong Q/R \). Then every \( (r, x) \in T \) with \( r \neq 0 \) is morphic in \( T \).

Proof. By direct computation, we have

\[
\mathbf{I}_T \left( 0, \frac{T}{r} \right) = rR \cong \frac{Q}{R} \quad \text{and} \quad \mathbf{I}_T(r, x) = \{(0, y) \in T : ry = 0 \}.
\]

It is clear that

\[
T(r, x) \subseteq \mathbf{I}_T \left( 0, \frac{T}{r} \right) \quad \text{and} \quad T \left( 0, \frac{T}{r} \right) \subseteq \mathbf{I}_T(r, x).
\]

Write \( x = \frac{t}{s} \). For any \((rm, \frac{T}{a}) \in \mathbf{I}_T(0, \frac{T}{r})\), \((rm, \frac{T}{a}) = (m, \frac{\frac{t}{s}}{r} - \frac{m}{a}) (r, \frac{T}{r}) \in T(r, x)\). Hence \( T(r, x) = \mathbf{I}_T(0, \frac{T}{r}) \).

If \( y \in \frac{Q}{R} \) with \( ry = 0 \), we write \( y = \frac{c}{d} \), so that \( rc \in dR \). Hence \( rc = dm \), for some \( m \in R \). Then \( y = \frac{\frac{t}{s}}{a} = \frac{\frac{t}{s}}{a} = \frac{\frac{t}{s}}{a} = \frac{t}{s} \), and so \((0, y) = (m, 0)(0, \frac{T}{r}) \in T(0, \frac{T}{r}) \). Hence \( T(0, \frac{T}{r}) = \mathbf{I}_T(r, x) \). Therefore, \((r, x)\) is morphic in \( T \). \( \square \)

For an ideal \( I \) of a ring \( R \) and \( s \in R \), let \( s^{-1}I = \{ r \in R : sr \in I \} \).

Lemma 12. Let \( R \) be a commutative domain with classical quotient ring \( Q \) and let \( T = R \cong \frac{Q}{R} \). For nonzero elements \( s, t \in R \), \((0, \frac{T}{r}) \) is morphic in \( T \) if and only if there exists \( k \in R \) such that \( s^{-1}(tR) = kR \) and, for any \( c, d \in R \), \( ck \in dR \) implies \( ct \in d(sR + tR) \).

Proof. Let \( a = (0, \frac{T}{r}) \) and \( b = (k, y) \in T \). Then \( \mathbf{I}_T(a) = I \cong \frac{Q}{R} \), where \( I = s^{-1}(tR) \), and \( T^b = \{(km, kz + my) : m \in R, z \in \frac{Q}{R} \} \). If \( \mathbf{I}_T(a) = T^b \), then \( s^{-1}(tR) = kR \). Conversely, if \( s^{-1}(tR) = kR \), then \( k \neq 0 \) because \( t \in kR \), so that

\[
T^b = \left\{ (km, kz + my) : m \in R, z \in \frac{Q}{R} \right\} = kR \cong \frac{Q}{R}.
\]

Thus, \( T^b = \mathbf{I}_T(a) \) and so\n
\[
\mathbf{I}_T(a) = T^b \iff s^{-1}(tR) = kR.
\]

We now assume that \( \mathbf{I}_T(a) = T^b \); i.e., \( s^{-1}(tR) = kR \). Then \( k \neq 0 \) and so \( \mathbf{I}_T(b) = \{(0, z) \in T : k^z = 0 \} = \{(0, \frac{T}{a}) : c, 0 \neq d \in R, ck \in dR \} \). On the other hand, \( T^a = \{(0, \frac{T}{a}) : m \in R \} \). Since \( s^{-1}(tR) = kR \), we have \( T^a \subseteq \mathbf{I}_T(b) \). Note that \( \frac{T}{a} = \frac{\frac{t}{s}}{a} \), for some \( m \in R \), if and
only if \( ct \in d(sR + tR) \). Therefore, \( I_T(a) = Tb \) and \( I_T(b) = Ta \) if and only if \( s^{-1}(tR) = kR \) and, for any \( c, d \in R, ck \in dR \) implies that \( ct \in d(sR + tR) \). The proof is complete. \( \square \)

**Theorem 13.** Let \( R \) be a principal ideal domain and \( Q \) the classical quotient ring of \( R \). Then \( R \cong Q/R \) is a commutative morphic ring.

**Proof.** Let \( T = R \cong Q/R \) and let \( 0 \neq (r, x) \in T \). If \( r \neq 0 \), then \( (r, x) \) is morphic in \( T \) by Lemma 11.

If \( r = 0 \), then \( x \neq 0 \). Write \( x = \frac{r}{t} \) where \( 0 \neq s, 0 \neq t \in R \). Since \( R \) is a principal ideal domain, we can assume that the greatest common factor of \( s \) and \( i \) is 1, and hence \( R = sR + tR \). Let \( k = t \). Then \( s^{-1}(tR) = kR \) and, for any \( c, d \in R, ck \in dR \) automatically implies that \( ct \in d(sR + tR) \). Thus, by Lemma 12, \( (0, x) = (0, \frac{r}{t}) \) is morphic in \( T \), so that \( T \) is a morphic ring. \( \square \)

**Theorem 14.** Let \( M \) be a bimodule over \( \mathbb{Z} \). Then \( \mathbb{Z} \cong M \) is a morphic ring if and only if \( M \cong \mathbb{Q}/\mathbb{Z} \).

**Proof.** If \( M \cong \mathbb{Q}/\mathbb{Z} \), then \( \mathbb{Z} \cong M \) is a morphic ring, by the previous theorem.

Conversely, suppose that \( R = \mathbb{Z} \cong M \) is a morphic ring. We first show that \( M \) is torsion. If \( x \in M \) is a nonzero torsionfree element, let \( a = (0, x) \in R \). Then \( aR = 0 \cong \mathbb{Z}x \) and \( I(a) = 0 \cong M \), so that \( r(I(a)) = r(0 \cong M) = 0 \cong M \). Since \( a \) is morphic, \( aR = r(I(a)) = 0 \cong M \). This implies that \( M = \mathbb{Z}x \cong \mathbb{Z} \). Hence \( R \cong \mathbb{Z} \cong \mathbb{Z} \). But, \( \mathbb{Z} \cong \mathbb{Z} \) is not morphic. Therefore, \( M \) is torsion, and so \( M = \oplus \{ \tau_p(M) : p \) is a prime\}, where \( \tau_p(M) \) is the \( p \)-torsion component of \( M \).

If \( \tau_p(M) = 0 \), for some prime \( p \), let \( b = (p, 0) \in R \). Then \( I(b) = 0 \) and so \( r(I(b)) = r(0) = R \neq bR \). Thus \( b \) is not morphic. Thus \( \tau_p(M) \neq 0 \), for every prime number \( p \).

For any \( 0 \neq n \in \mathbb{Z} \), let \( c = (n, 0) \in R \). Since \( c \) is morphic, there exists \( d = (m, x) \in R \) such that \( Rc = I(d) \) and \( Rd = I(c) \). We have \( I(c) = \{ (0, z) : nz = 0 \} \) and \( Rc = \{ (kn, nz) : k \in \mathbb{Z}, z \in M \} \). Hence \( d = (0, x) \) with \( nx = 0 \). Now \( I(d) = I(x) \cong M \).

From \( Rc = I(d) \), it follows that \( nM = M \) and so \( M \) is divisible or injective over \( \mathbb{Z} \). Since every injective module over a noetherian ring is a direct sum of indecomposable injective modules, \( M = \oplus M_i \), where each \( M_i \) is an indecomposable torsion injective module over \( \mathbb{Z} \). But \( \{ \mathbb{Z}_{p^\infty} : p \) is a prime\} are all indecomposable torsion injective modules over \( \mathbb{Z} \). Hence every \( M_i \) is isomorphic to \( \mathbb{Z}_{p^\infty} \), for some prime number \( p \). If \( M_i \cong M_j \cong \mathbb{Z}_{p^\infty} \), for some \( i \) and \( j \) with \( i \neq j \), where \( p \) is a prime number, then there exist \( 0 \neq v \in M_i \) and \( 0 \neq w \in M_j \) such that \( pv = pw = 0 \). Then \( R(0, v) = 0 \cong \mathbb{Z}v \) and \( I(0, v) = p\mathbb{Z} \cong M \).

Hence \( (0, v) \notin r(I(0, v) \setminus R(0, v)) \). Thus, \( (0, v) \) is not morphic. Therefore, \( M_i \cong M_j \) if and only if \( i = j \). Furthermore, for every prime \( p \), since \( \tau_p(M) \neq 0 \), \( \mathbb{Z}_{p^\infty} \cong M_j \) for some \( j \). Hence \( M \cong \bigoplus \{ \mathbb{Z}_{p^\infty} : p \) is a prime\} \cong \mathbb{Q}/\mathbb{Z} \). \( \square \)

**Corollary 15.** Let \( R = \mathbb{Z} \cong \mathbb{Q}/\mathbb{Z} \). Then \( R \) is strongly morphic.

**Proof.** Let \( 0 \neq A = (a_{ij}) \in \mathbb{M}_n(R) \). It suffices to show that \( A \) is morphic in \( \mathbb{M}_n(R) \).

Write \( a_{ij} = (n_{ij}, q_{ij}) \in R \).

**Case 1.** \( n_{ij} \neq 0 \) for some \( i \) and \( j \). Then there exists a positive integer \( k \) which is smallest with respect to the property that there exist units \( U \) and \( V \) of \( \mathbb{M}_n(R) \) and \( \overline{q} \in \mathbb{Q}/\mathbb{Z} \) such that \( (k, \overline{q}) \) is the \((1, 1)\)-entry of \( UAV \). Since \( A \) is morphic if and only if \( UAV \) is too, by [4, Lemma 3], we may assume that \( a_{11} = (k, \overline{q}) \). For any \( j \) with \( 1 \leq j \leq n \), \( n_{ij} = sk + r \), for some \( s, r \in \mathbb{Z} \) with \( 0 \leq r < k \). Thus, \( a_{ij} = a_{11}(s, 0) + (r, \overline{q}_{ij} - s\overline{q}) \). Now subtracting the first column times \((s, 0)\) from the \( j \)th column and then interchanging

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the 1th and jth columns will bring \((r, \overline{qj} - s\overline{qj})\) to the \((1, 1)\)-entry. The minimality of \(k\) shows that \(r = 0\). Thus, the elementary column operation changes \(a_{ij}\) to the new \((1, j)\)-entry \((0, \overline{qj} - s\overline{qj})\). Similarly, an elementary row operation changes \(a_{i1}\) to the new \((j, 1)\)-entry \((0, \ast)\). Therefore, without loss of generality, we can assume that \(n_{ij} = 0 = n_{j1}\) for \(j = 2, \ldots, n\). Since \(\mathbb{Q}/\mathbb{Z}\) is divisible, \(\overline{qj} = k\overline{s_j}\) for \(j = 2, \ldots, n\) with \(s_j \in \mathbb{Q}\). Thus, \((0, \overline{qj}) = (0, \overline{s_j})a_{11}\). Hence subtracting the first column times \((0, \overline{s_j})\) from the jth column brings \(a_{j1}\) to the new \((1, j)\)-entry 0, and similarly an elementary row operation brings \(a_{j1}\) to the new \((j, 1)\)-entry 0 as well. Hence a series of elementary operations change \(A\) to

\[
B = \begin{bmatrix}
    b_{11} & 0 & \cdots & 0 \\
    0 & b_{22} & \cdots & b_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & b_{n2} & \cdots & b_{nn}
\end{bmatrix}.
\]

**Case 2.** \(n_{ij} = 0\), for all \(i\) and \(j\). Write \(a_{ij} = (0, \frac{j}{s})\) with \(t_{ij}, 0 < s \in \mathbb{Z}\). Since \(A \neq 0\), \(\frac{qj}{s} \neq 0\) for some \(i\) and \(j\). Then there exists a positive integer \(t\) which is smallest with respect to the property that there exist units \(U\) and \(V\) in \(\mathbb{M}_n(R)\) such that \(0 \neq (0, \frac{t}{s})\) is the \((1, 1)\)-entry of \(UAV\). As above, we may assume that \(a_{11} = (0, \frac{t}{s})\). By the Division Algorithm, for any \(j\) with \(2 \leq j \leq n\), \(t_{ij} = tk + r\) where \(k, r \in \mathbb{Z}\) with \(0 \leq r < t\). Thus, \(a_{ij} = (k, 0)a_{11} + (0, \frac{t}{s})\). Hence subtracting the first column times \((k, 0)\) from the jth column yields the new \((1, j)\)-entry \((0, \frac{t}{s})\); and then interchanging the 1th and jth columns will bring \((0, \frac{t}{s})\) to the \((1, 1)\)-entry. The minimality of \(t\) shows that \(\frac{t}{s} = 0\). Thus, an elementary column operation will change \(a_{j1}\) to the new \((1, j)\)-entry 0. Similarly, an elementary row operation changes \(a_{j1}\) to the new \((j, 1)\)-entry 0 as well. Hence a series of elementary operations will change \(A\) to a matrix of the same form as \(B\) above.

Thus, continuing in this way, we can change \(A\) to a diagonal by elementary transformations. Therefore, there exist units \(U_1\) and \(V_1\) of \(\mathbb{M}_n(R)\) such that

\[
U_1AV_1 = \begin{bmatrix}
    a_1 & 0 & \cdots & 0 \\
    0 & a_2 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & a_n
\end{bmatrix},
\]

where \(a_i \in R\), for \(i = 1, \ldots, n\). (In fact, \(U_1\) and \(V_1\) are products of certain elementary matrices over \(R\).) Since each \(a_i\) is morphic, by the previous theorem, \(U_1AV_1\) is morphic. Therefore, \(A\) is morphic.

**Remark 16.** A ring is called **clean** if every element is the sum of an idempotent and a unit. Because every unit regular ring is clean by Camillo-Yu [2] (or by Camillo-Khurana [1]) and because every unit regular ring is morphic (see [4, Example 4]), it is asked in [4, Question, p. 393] whether every morphic ring is clean. The ring \(\mathbb{Z} \simeq \mathbb{Q}/\mathbb{Z}\) is commutative strongly morphic by Corollary 15, but it is not clean because its image \(\mathbb{Z}\) is not clean. This example answers the above question in the negative.

**Remark 17.** A ring \(R\) is said to have **stable range 1** if, whenever \(aR + bR = R\) with \(a, b \in R\), \(a + by\) is a unit for some \(y \in R\). In the Fourth China-Japan-Korea International Symposium on Ring Theory [June 24–28, 2004, Nanjing], Nicholson
asked whether a morphic ring always has stable range 1. The answer to this question is "No". Let \( R = \mathbb{Z} \cong \mathbb{Q}/\mathbb{Z} \). Then \( R \) is morphic by Theorem 14. Since \( R/J(R) \cong \mathbb{Z} \) does not have stable range 1, \( R \) does not have stable range 1.

**Proposition 18.** If \( S = R \cong V \) with \( R \) an integral domain and \( V \) a nonzero bimodule over \( R \), then \( S \cong S \) is not left morphic.

**Proof.** Let \( T = S \cong S \). Take \( 0 \neq v \in V \) and let \( x = (0, v) \in S \). We show next that \((0, x)\) is not left morphic in \( T \). Suppose that this is not the case. Then there exists \((b, c) \in T\) such that \( l(0, x) = T(b, c) \). Since \((0, 1) \in l(0, x)\), \((0, 1) = (u, v)(b, c)\) with \((u, v) \in T\).

It follows that

\[
0 = ub \quad \text{and} \quad 1 = uc + vb.
\]

Write \( u = (u_0, u_1), v = (v_0, v_1), b = (b_0, b_1) \) and \( c = (c_0, c_1) \), where \( u_0, v_0, b_0, c_0 \in R \) and \( u_1, v_1, b_1, c_1 \in V \). It follows from (1) that

\[
0 = u_0 b_0 \quad \text{and} \quad 1 = u_0 c_0 + v_0 b_0.
\]

If \( b_0 = 0 \), then \( 1 = u_0 c_0 \), so that \( u_0 \) is a unit in \( R \). Thus \( u \) is a unit in \( S \), and so \( b = 0 \) by (1). Hence \((x, 0) \in l(0, x) = T(0, c)\), showing that \( x = 0 \), a contradiction, so that \( b_0 \neq 0 \). Since \( R \) is an integral domain, it must be that \( u_0 = 0 \) and \( 1 = v_0 b_0 \), by (2). Hence \( b_0 \) is a unit in \( R \). Now from \((b, c) \in l(0, x)\), it follows that \( bx = 0 \) in \( S \); i.e., \((b_0, b_1)(0, v) = 0\). We have \( b_0 v = 0 \) and so \( v = 0 \), a contradiction. \( \Box \)

For a morphic ring \( R, R \cong M \) may not be morphic (e.g., \( R = \mathbb{Z}_4 \) and \( M = \mathbb{Z}_4 \)). On the other hand, \( \mathbb{Z} \cong \mathbb{Q}/\mathbb{Z} \) is morphic, but \( \mathbb{Z} \) is not. We have been unable to completely determine when \( R \cong M \) is (left) morphic. But, the next result shows that \( R \) being left morphic just means that certain elements in \( R \cong R \) are left morphic.

**Theorem 19.** Let \( R \) be a ring and \( a \in R \). Then the following are equivalent.

1. \( a \in R \) is left morphic.
2. \((a, 0) \in R \cong R \) is left morphic.
3. \((a, a) \in R \cong R \) is left morphic.

**Proof.** Let \( S = R \cong R \). Since \((a, a)(1, -1) = (a, 0) \) and \((1, -1) \) is a unit in \( S \), we have that (2) \( \iff \) (3) by [4, Lemma 3].

(1) \( \Rightarrow \) (2). Since \( a \in R \) is left morphic, there exists \( b \in R \) such that \( l(a) = Ra \) and \( l(b) = Ra \). It can be verified that \( l(a, 0) = S(b, 0) \) and \( l(b, 0) = S(a, 0) \). Hence \((a, 0) \in S \)

is left morphic.

(2) \( \Rightarrow \) (1). If \((a, 0) \in S \) is left morphic, then there exists \((b, c) \in S \) such that \( l(a, 0) = S(b, c) \) and \( l(b, c) = S(a, 0) \). It can be verified that \( l(a) = Ra \) and \( l(b) = Ra \), so that \( a \in R \)

is left morphic. \( \Box \)

**Proposition 20.** Let \( R \) be a ring and let \( S = R \cong R \).

1. If \( e^2 = e \in R \) and \( u \) is a unit of \( R \), then \((0, eu), (0, ue) \in S \) are morphic.
2. If \((0, a) \in S \) is left morphic, then \( a \) is left morphic in \( R \).
3. \( \mathbb{Z}_4 \) is morphic, but \((0, \mathbb{Z}_4) \) is not morphic in \( \mathbb{Z}_4 \cong \mathbb{Z}_4 \).

**Proof.** (1). It can be verified that \( S(0, e) = l(1 - e, 1) \) and \( S(1 - e, 1) = l(0, e) \). Hence \((0, e) \in S \) is morphic. Then \((0, ue) = (u, 0)(0, e) \) and \((0, eu) = (0, e)(u, 0) \). Since \((u, 0) \) is a unit in \( S \), \((0, ue) \) and \((0, eu) \) are morphic in \( S \).
(2). If \((0, a)\) is left morphic in \(S\), there exists \((b, c) \in S\) such that \(S(0, a) = l(b, c)\) and \(S(b, c) = l(0, a)\). Then it can be verified that \(Ra = l(b)\) and \(Rb = l(a)\).

(3). Let \(S = \mathbb{Z}_4 \propto \mathbb{Z}_4\). Suppose that \(l(0, \overline{2}) = S(a, b)\), where \((a, b) \in S\). Since \(l(0, \overline{2}) = 2\mathbb{Z}_4 \propto \mathbb{Z}_4\), we must have \(a = \overline{2}\). Since \((0, \overline{1}) \in l(0, \overline{2})\), there exists \((c, d) \in S\) such that \((0, \overline{1}) = (c, d)(\overline{2}, b) = (\overline{2}c, 2d + cb)\). Hence \(\overline{2}c = 0\) and \(\overline{1} = 2d + cb\). It follows that \(\overline{2} = \overline{2}(2d + cb) = 0\), a contradiction. □

**Corollary 21.** Let \(R\) be a ring. If \(R \propto R\) is left morphic, then so is \(R\).

The converse of Corollary 21 is not true since \(\mathbb{Z}_4\) is morphic but \(\mathbb{Z}_4 \propto \mathbb{Z}_4\) is not.

If \(R = \prod R_i\), where each \(R_i\) is either strongly regular or semisimple, then the trivial extension \(R \propto R = (\prod R_i) \propto (\prod R_i) \cong \prod (R_i \propto R_i)\). The last equality can be proved as in Lemma 4 and so, by Corollary 3 and Theorem 7, \(R \propto R\) is morphic. Note that the ring \(R\) is unit regular; i.e., for any \(a \in R\), \(a = au\), for some unit \(u\) of \(R\).

If \(R = \text{End}(V_D)\), where \(V\) is a vector space of countably infinite dimension over a division ring \(D\), then \(R\) is regular, right self-injective, but not unit regular. So \(R\) is not one-sided morphic by [4, Proposition 5], and hence \(R \propto R\) is not one-sided morphic by Corollary 21. Thus there exist regular, right self-injective rings \(R\) and strongly \(\pi\)-regular rings \(R\) (e.g., \(R = \mathbb{Z}_4\)) such that \(R \propto R\) is not left morphic.

By Corollary 21, if \(R\) is regular and \(R \propto R\) is left morphic, then \(R\) is left morphic and hence is unit regular. But we have been unable to answer the question whether \(R\) being unit regular always implies that the trivial extension \(R \propto R\) is a morphic ring.

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