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UNIFORM APPROXIMATION BY POLYNOMIAL, RATIONAL AND ANALYTIC FUNCTIONS

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Abstract

Let K and X be compact plane sets such that $K \subseteq X$. Let $P(K)$ be the uniform closure of polynomials on K, let $R(K)$ be the uniform closure of rational functions on K with no poles in K and let $A(K)$ be the space of continuous functions on K which are analytic on $\mathrm{int}(K)$. Define $P(X, K)$, $R(X, K)$ and $A(X, K)$ to be the set of functions in $C(X)$ whose restriction to K belongs to $P(K)$, $R(K)$ and $A(K)$, respectively. Let $S_0(A)$ denote the set of peak points for the Banach function algebra A on X. Let S and T be compact subsets of X. We extend the Hartogs–Rosenthal theorem by showing that if the symmetric difference $S \Delta T$ has planar measure zero, then $R(X, S) = R(X, T)$. Then we show that the following properties are equivalent:

(i) $R(X, S) = R(X, T)$;
(ii) $S \setminus T \subseteq S_0(R(X, S))$ and $T \setminus S \subseteq S_0(R(X, T))$;
(iii) $R(K) = C(K)$ for every compact set $K \subseteq S \Delta T$;
(iv) $R(X, S \cap \overline{U}) = R(X, T \cap \overline{U})$ for every open set U in $\mathbb{C}$;
(v) for every $p \in X$ there exists an open disk D_p with centre p such that

$$R(X, S \cap \overline{D}_p) = R(X, T \cap \overline{D}_p).$$

We prove an extension of Vitushkin's theorem by showing that the following properties are equivalent:

(i) $A(X, S) = R(X, T)$;
(ii) $A(X, S \cap \overline{D}) = R(X, T \cap \overline{D})$ for every closed disk $\overline{D}$ in $\mathbb{C}$;
(iii) for every $p \in X$ there exists an open disk D_p with centre p such that

$$A(X, S \cap \overline{D}_p) = R(X, T \cap \overline{D}_p).$$

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1. Introduction

The algebra of all continuous complex-valued functions on the compact Hausdorff space X is denoted by $C(X)$. The subalgebra $A \subseteq C(X)$ is a Banach function algebra on X if A separates the points of X, contains the constants and is complete under an algebra norm. If the norm of a Banach function algebra is the uniform norm then it is a uniform algebra.

Let A be a Banach function algebra on X. A point $p \in X$ is a peak point for A if there exists $f \in A$ such that $f(p) = 1$ and $|f(x)| < 1$ for every $x \in X$ different from p. The set of all peak points for A is denoted by $S_0(A)$.

Let K, S, T and X be compact subsets of $\mathbb{C}$ such that $K, S, T \subseteq X$, and let $P_0(K)$, $R_0(K)$ be the algebras of all polynomials and rational functions on K with poles off K, respectively. The uniform closures of $P_0(K)$ and $R_0(K)$ are denoted by $P(K)$ and $R(K)$, respectively, which are uniform algebras on K.

The polynomial convex hull of K is

$$\widehat{K} = \{z \in \mathbb{C} : |p(z)| \leq \|p\|_K \text{ for all polynomials } p\}.$$

The set K is polynomially convex if $\widehat{K} = K$. Let m denote the planar measure and $M(X)$ denote the space of all regular complex Borel measures on X. A theorem due to Hartogs and Rosenthal asserts that $R(K) = C(K)$ if K has planar measure zero; see, for example, [2, II.8.4] or [4]. It is also known that $R(K) = C(K)$ if and only if every point of K is a peak point for $R(K)$ [6, 5.3.8]. A stronger result is Bishop's peak point criterion for rational approximation, which asserts that if $m(K) = m(S_0(R(K)))$ then $R(K) = C(K)$ [2, II.11.4]. Moreover, $P(K) = R(K)$ if and only if K is polynomially convex. Also a theorem due to Vitushkin gives criteria for $R(K) = C(K)$; see, for example, [2, VIII.5.1] or [7].

In this work we extend the above results to more general algebras in the theory of uniform algebras. For another extension of Hartogs–Rosenthal to Lipschitz algebras, see [5].

If we take $P_0(X, K) = \{f \in C(X) : f|_K \in P_0(K)\}$ and $R_0(X, K) = \{f \in C(X) : f|_K \in R_0(K)\}$ then it is easy to see that $P(X, K) = \{f \in C(X) : f|_K \in P(K)\}$ and $R(X, K) = \{f \in C(X) : f|_K \in R(K)\}$ are, in fact, the uniform closures of $P_0(X, K)$ and $R_0(X, K)$, respectively. We take $A(X, K) = \{f \in C(X) : f|_K \in A(K)\}$ where $A(K)$ is the algebra of continuous functions on K, which are analytic on $\operatorname{int}(K)$. Note that if K is finite then $P_0(X, K) = R_0(X, K) = C(X)$ and so $P(X, K) = R(X, K) = A(X, K) = C(X)$. Hence, we may assume that K is infinite.

It is easy to show that $P(X, K)$, $R(X, K)$ and $A(X, K)$ are uniform algebras on X. Moreover, $P_0(X, K) = P_0(X)$, $R_0(X, K) = R_0(X)$, $P(X, K) = P(X)$, $R(X, K) = R(X)$ and $A(X, K) = A(X)$ if $K = X$.

2. Polynomial and rational approximation in uniform algebras

Throughout this section we always assume that K, S, T and X are compact plane sets such that $K, S, T \subseteq X$, and $\mu \in M(X)$.

LEMMA 2.1. *If $A = \{f \in C(X) : f|_K = 0\}$, then $C_0(X \setminus K) = A|_{X \setminus K}$.*

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PROOF. Clearly, for every $f \in A$, $f|_{X\backslash K} \in C_0(X\backslash K)$.

Let $f_0 \in C_0(X\backslash K)$. We extend f_0 to X by

$$f(x) = \begin{cases} f_0(x), & x \in X\backslash K, \\ 0, & x \in K. \end{cases}$$

We now show that $f \in A$. Let $x_0 \in X$. If $f(x_0) \neq 0$ then $x_0 \in X\backslash K$. Hence, there exists $\delta_1 > 0$ such that $B(x_0; \delta_1) \cap K = \emptyset$. Since $f_0 \in C_0(X\backslash K)$ for every $\varepsilon > 0$ there exists $\delta_2 > 0$ such that for every $x \in X\backslash K$ if $|x - x_0| < \delta_2$ then $|f(x) - f(x_0)| < \varepsilon$. If we take $\delta = \min\{\delta_1, \delta_2\}$, then for every $x \in X$ if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$. This shows that f is continuous at x_0.

If $f(x_0) = 0$ then for every $\varepsilon > 0$ the set $S = \{x \in X\backslash K : f(x) \geq \varepsilon\}$ is compact. Since $x_0 \in X\backslash S$ there exists $\delta > 0$ such that $B(x_0; \delta) \cap S = \emptyset$. Hence, for every $x \in X\backslash S$, the inequality $|f(x) - f(x_0)| < \varepsilon$ holds, and this shows that for every $x \in B(x_0; \delta) \cap X$, $|f(x) - f(x_0)| < \varepsilon$ and so f is continuous at x_0. □

LEMMA 2.2. *Let μ be a regular complex Borel measure on X. If U is an open set in $\mathbb{C}$ such that for almost all $z \in U$, with respect to planar measure, $\int_X d\mu(\zeta)/(\zeta - z) = 0$, then $\mu = 0$ on $U \cap X$.*

PROOF. Since $\mu \in M(X)$ it is enough to show that $|\mu|(Y) = 0$ for every compact subset Y of $U \cap X$. We consider a decreasing sequence of bounded open neighbourhoods $\{U_n\}_{n=1}^{\infty}$ of Y such that $\bigcap_{n=1}^{\infty} U_n = Y$ and $U_1 \subseteq U$. It is known that for every n we can find a continuously differentiable function h_n on the complex plane such that $h_n = 1$ on Y, $0 \leq h_n \leq 1$ and $E_n = \mathrm{supp}(h_n)$ is contained in U_n. Now let $f \in C^1(Y)$, where $C^1(Y)$ is the algebra of all continuously differentiable functions on Y. We can extend f to a function $g \in C^1(\mathbb{C})$ such that it is bounded on the closure of U_1. Now we define $f_n = gh_n$. Clearly $f_n \in C^1(\mathbb{C})$ and it is, in fact, an extension of f. By applying Green's theorem as well as Fubini's theorem,

$$\begin{aligned} \int_X f_n(\lambda)\, d\mu(\lambda) &= \int_X \left\{ \iint_{E_n} \frac{-1}{\pi} (z - \lambda)^{-1} (f_n)_{\bar{z}}\, dx\, dy \right\} d\mu(\lambda) \\ &= \iint_{E_n} \frac{-(f_n)_{\bar{z}}}{\pi} \left(\int_X \frac{d\mu(\lambda)}{z - \lambda} \right) dx\, dy = 0. \end{aligned}$$

Since $\lim_{n \longrightarrow \infty} f_n(x) = \chi_Y(x) g(x)$, for every $x \in X$, then $\int_Y f(\lambda)\, d\mu(\lambda) = 0$ by the dominated convergence theorem. Since the algebra of all such functions f is dense in $C(Y)$, we conclude that μ is the zero measure on Y and hence $|\mu|(Y) = 0$. By the regularity of μ we conclude that $\mu = 0$ on $U \cap X$. □

THEOREM 2.3. *If $m(S\backslash T) = 0$ then $R(X, T) \subseteq R(X, S)$.*

PROOF. Let $\mu \in (R(X, S))^{\perp}$. We prove that $\mu \in (R(X, T))^{\perp}$.

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We first show that $\operatorname{supp}(\mu) \subseteq S$. For every $f_0 \in C_0(X\backslash S)$ the function

$$f(x) = \begin{cases} f_0(x), & x \in X\backslash S, \\ 0, & x \in S, \end{cases}$$

belongs to $R(X, S)$ by Lemma 2.1, hence $\int_X f\, d\mu = \int_{X\backslash S} f_0\, d\mu = 0$. Therefore, $\mu|_{X\backslash S} \in (C_0(X\backslash S))^{\perp}$, that is, $\mu|_{X\backslash S} = 0$. This shows that $\operatorname{supp}(\mu) \subseteq S$.

There exists a bounded open set U such that $U \cap T = \emptyset$ and $X\backslash T \subseteq U$. For every $\alpha \in X\backslash(S \cup T)$, there exists a function f in $R(X, S)$ such that $f|_S = (z-\alpha)^{-1}$. Hence,

$$\int_X (z-\alpha)^{-1}\, d\mu = \int_S (z-\alpha)^{-1}\, d\mu = \int_S f\, d\mu = \int_X f\, d\mu = 0.$$

Since $m(S\backslash T) = 0$, for almost all $\alpha \in U$, $\int_X (z-\alpha)^{-1}\, d\mu(z) = 0$. Hence, by Lemma 2.2, $\mu = 0$ on $X \cap U = X\backslash T$ and so $\operatorname{supp}(\mu) \subseteq T$. This shows that $\operatorname{supp}(\mu) \subseteq S \cap T$.

Now suppose that $\alpha \in \mathbb{C}\backslash T$. Since $m(S\backslash T) = 0$, $S\backslash T$ has no interior. Hence, there is a sequence $\{\alpha_n\}$ in $\mathbb{C}\backslash S$ such that $\lim_{n\longrightarrow\infty} \alpha_n = \alpha$.

By hypothesis, $\int_{S\cap T} (z-\alpha_n)^{-1}\, d\mu = 0$ for every n. By the dominated convergence theorem,

$$\int_{S\cap T} (z-\alpha)^{-1}\, d\mu = \lim_{n\longrightarrow\infty} \int_{S\cap T} (z-\alpha_n)^{-1}\, d\mu = 0.$$

On the other hand, for every $g \in R_0(X, T)$, $g|_T \in R_0(T)$. Since $g|_T$ is the limit of a sequence of rational functions with poles off S, by the same argument as above we conclude that $\int_{S\cap T} g\, d\mu = 0$, and hence

$$\int_X g\, d\mu = \int_{S\cap T} g\, d\mu = 0.$$

Thus for every $g \in R(X, T)$, $\int_X g\, d\mu = 0$, that is, $\mu \in (R(X, T))^{\perp}$. Therefore, $R(X, T) \subseteq R(X, S)$. $\square$

COROLLARY 2.4. *If $m(S \Delta T) = 0$ then $R(X, S) = R(X, T)$.*

COROLLARY 2.5. *If $m(K) = 0$ then $R(X, K) = C(X)$. In particular, if $m(X) = 0$ then $R(X) = C(X)$, which is the Hartogs–Rosenthal theorem.*

PROOF. Take $S = K$ and $T = \{z_0\}$ for some $z_0 \in X$, in Corollary 2.4. $\square$

COROLLARY 2.6. *If $m(X) = m(K)$ then $R(X, K) = R(X)$.*

PROOF. Take $S = K$ and $T = X$ in Corollary 2.4. $\square$

THEOREM 2.7. *$R(X, T) \subseteq R(X, S)$ if and only if $\operatorname{supp}(\mu) \subseteq S \cap T$ for every $\mu \in R(X, S)^{\perp}$.*

PROOF. Let $R(X, T) \subseteq R(X, S)$ and $\mu \in R(X, S)^{\perp}$. For every $f_0 \in C_0(X \backslash S)$, the function

$$f(x) = \begin{cases} f_0(x), & x \in X \backslash S, \\ 0, & x \in S, \end{cases}$$

is continuous on X by Lemma 2.1, and hence $f \in R_0(X, S)$. Therefore, $\int_{X \backslash S} f_0 \, d\mu = \int_X f \, d\mu = 0$, which shows that $\mu|_{X \backslash S} \in C_0(X \backslash S)^{\perp}$ and so $\mu|_{X \backslash S} = 0$, that is, $\mathrm{supp}(\mu) \subseteq S$. Since $R(X, S)^{\perp} \subseteq R(X, T)^{\perp}$, $\mu \in R(X, T)^{\perp}$. Hence, by the same argument as above, $\mathrm{supp}(\mu) \subseteq T$. Therefore, $\mathrm{supp}(\mu) \subseteq S \cap T$.

For the converse, we first show that $\mathrm{int}(S \backslash T) = \emptyset$ if $\mathrm{supp}(\mu) \subseteq S \cap T$ for all $\mu \in R(X, S)^{\perp}$. Suppose on the contrary that there exists a closed disk $D \subseteq \mathrm{int}(S \backslash T)$. Since $R(D) \neq C(D)$ there exists $\lambda \in R(D)^{\perp}$ such that $\lambda \neq 0$. We define the measure $\mu \in M(X)$ by $\mu(E) = \lambda(E \cap D)$, which is not the zero measure. If $f \in R(X, S)$, then

$$\int_X f \, d\mu = \int_D f \, d\mu = \int_D f \, d\lambda = 0$$

since $f|_D \in R(D)$. Therefore, $\mu \in R(X, S)^{\perp}$ while $\mathrm{supp}(\mu) \subseteq D \subset S \backslash T$, which is in contradiction with our hypothesis.

Now let $f \in R_0(X, T)$ be such that $f|_T = 1/(z - z_0)$ where $z_0 \in \mathbb{C} \backslash T$. If $z_0 \in S \backslash T$ then there exists $\{z_n\} \subset X \backslash (S \cup T)$ such that $\lim_{n \longrightarrow \infty} z_n = z_0$, since $\mathrm{int}(S \backslash T) = \emptyset$. By the dominated convergence theorem,

$$\lim_{n \longrightarrow \infty} \int_{S \cap T} (z - z_n)^{-1} \, d\mu(z) = \int_{S \cap T} (z - z_0)^{-1} \, d\mu(z).$$

For every $n \in \mathbb{N}$ there exists $g_n \in R_0(X, S)$ such that $g_n|_S = (z - z_n)^{-1}$. Since

$$0 = \int_X g_n \, d\mu = \int_{S \cap T} (z - z_n)^{-1} \, d\mu,$$

it follows that $\int_{S \cap T} (z - z_0)^{-1} \, d\mu = 0$ and so

$$\int_X f \, d\mu = \int_{S \cap T} (z - z_0)^{-1} \, d\mu = 0.$$

If f is an arbitrary element of $R_0(X, T)$ then $f|_{T \cap S}$ is the limit of a sequence of rational functions with poles off $S \cup T$. Hence, by the above discussion and the dominated convergence theorem, $\int_X f \, d\mu = 0$. This shows that $\mu \in R(X, T)^{\perp}$ and so $R(X, T) \subseteq R(X, S)$. □

COROLLARY 2.8. *$R(X, S) = R(X, T)$ if and only if* $\mathrm{supp}(\mu) \subseteq S \cap T$ *for every* $\mu \in R(X, S)^{\perp} \cup R(X, T)^{\perp}$.

THEOREM 2.9. *$R(X, T) \subseteq R(X, S)$ if and only if $S \setminus T \subseteq S_0(R(X, S))$.*

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PROOF. Let $R(X, T) \subseteq R(X, S)$ and $z_0 \in S \setminus T$. Let U be an arbitrary neighbourhood of z_0 and let V be a bounded neighbourhood of z_0 which is contained in U and moreover, $V \cap T = \emptyset$. There exists a neighbourhood W of z_0 such that its closure is contained in V. By Urysohn's lemma there exists a continuous function f on X such that

$$f(x) = \begin{cases} 1, & x \in W, \\ 0, & x \in \mathbb{C} \setminus V. \end{cases}$$

Since f is zero on T it follows that $f \in R_0(X, T)$. Moreover, $|f| < 1/4$ on $X \setminus U$ and $f(z_0) = \|f\| = 1$. Thus z_0 is a peak point for $R(X, T)$, by [6, 4.7.22]. Hence, $S \setminus T \subseteq S_0(R(X, S))$.

For the converse, we first note that $\mathrm{int}(S \setminus T) = \emptyset$ by the hypothesis. By Theorem 2.7 it is sufficient to show that $\mathrm{supp}(\mu) \subseteq S \cap T$ for every $\mu \in R(X, S)^{\perp}$. By the same argument as in the proof of Theorem 2.7, it follows that $\mathrm{supp}(\mu) \subseteq S$. We now show that $\mu|_{S \setminus T} = 0$. For every compact subset Y of $S \setminus T$ there exists a bounded neighbourhood U of Y such that $\overline{U} \cap T = \emptyset$. For every $z_0 \in U \setminus S$ there exists an $f \in R_0(X, S)$ such that $f|_S = 1/(z - z_0)$. Since $\mathrm{supp}(\mu) \subseteq S$,

$$\int_X \frac{d\mu(z)}{z - z_0} = 0.$$

If $m(S \setminus T) = 0$ then for almost all $z_0 \in U$,

$$\int_X \frac{d\mu(z)}{z - z_0} = 0.$$

Hence, by Lemma 2.2, $\mu = 0$ on $U \cap X$ and so $\mu = 0$ on Y. This implies that $\mu|_{S \setminus T} = 0$.

Now let $m(S \setminus T) > 0$. There is a bounded neighbourhood U of $S \setminus T$ such that $U \cap T = \emptyset$. For every

$$z_0 \in U \setminus S, \qquad \int_X \frac{d\mu(z)}{z - z_0} = 0.$$

If for almost all

$$z_0 \in U, \qquad \int_X \frac{d\mu(z)}{z - z_0} = 0$$

then, by Lemma 2.2, $\mu = 0$ on $U \cap X$ and hence $\mu|_{S \setminus T} = 0$. Suppose, on the contrary, there exists a compact subset Y of U such that $m(Y) > 0$ and for every

$$z_0 \in Y, \qquad \int_X \frac{d\mu(z)}{z - z_0} \neq 0.$$

Hence, by [6, 5.3. Lemma 1], there exists $z_0 \in S \setminus T$ such that

$$\int_X |z - z_0|^{-1} d|\mu|(z) < \infty \quad \text{and} \quad \int_X (z - z_0)^{-1} \, d\mu(z) \neq 0.$$

We may assume that $\int_X (z - z_0)^{-1} \, d\mu(z) = 1$.

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For every

$$f \in R_0(X, S), \qquad \frac{f(z) - f(z_0)}{z - z_0} \in R_0(S).$$

By the Tietze extension theorem there exists $F \in R_0(X, S)$ such that for every $z \in S$,

$$F(z) = \frac{f(z) - f(z_0)}{z - z_0},$$

and hence

$$\int_X \frac{f(z)}{z - z_0}\, d\mu(z) = \int_X \frac{f(z_0)}{z - z_0}\, d\mu(z) = f(z_0).$$

Therefore, for every

$$f \in R(X, S), \qquad \int_X \frac{f(z)}{z - z_0}\, d\mu(z) = f(z_0)$$

by the density of $R_0(X, S)$ in $R(X, S)$. Since z_0 is a peak point for $R(X, S)$, there exists $g \in R(X, S)$ such that $g(z_0) = 1$ and $|g| < 1$ on $X \backslash \{z_0\}$. For every $n \in N$,

$$\int_X \frac{g^n(z)}{z - z_0}\, d\mu(z) = g^n(z_0) = 1.$$

Since for every $z \in X \backslash \{z_0\}$, $\lim_{n \longrightarrow \infty} g^n(z) = 0$, by the dominated convergence theorem,

$$\lim_{n \longrightarrow \infty} \int_X \frac{g^n(z)}{z - z_0}\, d\mu(z) = 0,$$

which is a contradiction. Hence, $\mu|_{S \backslash T} = 0$, which implies that $\mathrm{supp}(\mu) \subseteq S \cap T$. □

COROLLARY 2.10. *$R(X, S) = R(X, T)$ if and only if $S \setminus T \subseteq S_0(R(X, S))$ and $T \setminus S \subseteq S_0(R(X, T))$.*

COROLLARY 2.11. *$R(X, K) = R(X)$ if and only if $X \setminus K \subseteq S_0(R(X))$.*

PROOF. In Corollary 2.10 we take $S = K$ and $T = X$. □

COROLLARY 2.12. *$R(X, K) = C(X)$ if and only if $K \subseteq S_0(R(X, K))$. In particular, $R(X) = C(X)$ if and only if $X = S_0(R(X))$.*

PROOF. In Corollary 2.10 we take $S = K$ and $T = \emptyset$. □

COROLLARY 2.13. *If $R(X, S) = R(X, T)$ then $R(X, S) = R(X, T) = R(X, S \cap T)$.*

PROOF. This is immediate. □

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THEOREM 2.14. *$P(X, T) \subseteq P(X, S)$ if and only if* $\text{supp}(\mu) \subseteq S \cap T$ *for every* $\mu \in P(X, S)^{\perp}$.

PROOF. Let $P(X, T) \subseteq P(X, S)$. By the same method as in the proof of Theorem 2.7, $\text{supp}(\mu) \subseteq S \cap T$ for every $\mu \in P(X, S)^{\perp}$.

For the converse, let $\mu \in P(X, S)^{\perp}$. By hypothesis, $\text{supp}(\mu) \subseteq S \cap T$. For every $f \in P_0(X, T)$, $f|_{S\cap T} = p$ where $p \in P_0(S \cap T)$. If we consider p as an element of $P_0(X, S)$ then

$$\int_X f\, d\mu = \int_{S\cap T} f\, d\mu = \int_{S\cap T} p\, d\mu = \int_X p\, d\mu = 0.$$

This shows that $\mu \in P(X, T)^{\perp}$. Hence, $P(X, T) \subseteq P(X, S)$. □

COROLLARY 2.15. *$P(X, S) = P(X, T)$ if and only if* $\text{supp}(\mu) \subseteq S \cap T$ *for every* $\mu \in P(X, S)^{\perp} \cup P(X, T)^{\perp}$.

COROLLARY 2.16. *$P(X, K) = P(X)$ if and only if* $\text{supp}(\mu) \subseteq K$ *for every* $\mu \in P(X)^{\perp}$.

PROOF. In Corollary 2.15 take $S = K$ and $T = X$. □

COROLLARY 2.17. *If $P(X, S) = P(X, T)$ then $P(X, S) = P(X, T) = P(X, S \cap T)$.*

PROOF. We prove that $P(X, S) = P(X, S \cap T)$. Obviously $P(X, S) \subseteq P(X, S \cap T)$. Let $\mu \in P(X, S)^{\perp}$. By Corollary 2.15, $\text{supp}(\mu) \subseteq S \cap T$. For every $f \in P_0(X, S \cap T)$, $f|_{S\cap T} = p$ where $p \in P_0(S \cap T)$. If we consider p as an element of $P_0(X, S)$, then $\int_X f\, d\mu = \int_{S\cap T} f\, d\mu = \int_{S\cap T} p\, d\mu = \int_X p\, d\mu = 0$.

This shows that $\mu \in P(X, S \cap T)^{\perp}$ and hence $P(X, S \cap T) \subseteq P(X, S)$. □

THEOREM 2.18. *$P(X, S) = R(X, T)$ if and only if $S \cap T$ is polynomially convex, $T \backslash S \subseteq S_0(R(X, T))$ and* $\text{supp}(\mu) \subseteq S \cap T$ *for every* $\mu \in P(X, S)^{\perp}$.

PROOF. Let $P(X, S) = R(X, T)$. So $T \backslash S \subseteq S_0(P(X, S)) = S_0(R(X, T))$. If $\mu \in P(X, S)^{\perp}$ then $\text{supp}(\mu) \subseteq S$ and, since $P(X, S) = R(X, T)$, then $\mu \in R(X, T)^{\perp}$, which implies that $\text{supp}(\mu) \subseteq S \cap T$. By Corollaries 2.15 and 2.8, $P(X, S) = P(X, S \cap T)$ and $R(X, T) = R(X, S \cap T)$ so that $P(X, S \cap T) = R(X, S \cap T)$ and hence $P(S \cap T) = R(S \cap T)$. Therefore $S \cap T$ is polynomially convex.

Conversely, by Corollaries 2.15 and 2.10, $P(X, S) = P(X, S \cap T)$ and $R(X, T) = R(X, S \cap T)$. Since $S \cap T$ is polynomially convex, $P(X, S \cap T) = R(X, S \cap T)$ and hence $P(X, S) = R(X, T)$. □

COROLLARY 2.19. *$P(X, K) = R(X)$ if and only if $\widehat{K} = K$ and $X \backslash K \subseteq S_0(R(X))$.*

PROOF. In Theorem 2.18 take $S = K$ and $T = X$. □

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COROLLARY 2.20. *$P(X) = R(X, K)$ if and only if $\widehat{K} = K$, and for every $\mu \in P(X)^{\perp}$, $\mathrm{supp}(\mu) \subseteq K$.*

PROOF. This is immediate. □

COROLLARY 2.21. *If $P(X, K) = R(X)$ and S is a compact subset of K such that $m(S) = m(K)$ then $\widehat{S} = S$.*

PROOF. By Corollary 2.19, $\widehat{K} = K$. So $P(X, K) = R(X, K)$. By Corollary 2.4, $R(X, K) = R(X, S)$, hence $P(X, K) = R(X, S)$. Therefore, by Theorem 2.18, $\widehat{S} = \widehat{(K \cap S)} = K \cap S = S$. □

LEMMA 2.22. *If $P(X, S) = P(X, T)$ then $R(X, S) = R(X, T)$.*

PROOF. If $P(X, S) = P(X, T)$ then, by the same argument as in the first part of the proof of Theorem 2.9, $S \backslash T \subseteq S_0(P(X, S)) \subseteq S_0(R(X, S))$ and $T \backslash S \subseteq S_0(P(X, T)) \subseteq S_0(R(X, T))$. By Corollary 2.10, $R(X, S) = R(X, T)$. □

However, the converse of the above lemma is not true. For example, if $X = \{z \in \mathbb{C} : |z| = 1\}$ and $S = \{z \in X : \mathrm{Re}\, z \geq 0\}$ and $T = X$ then $R(X, S) = R(X, T)$, but $P(X, S) = C(X) \neq P(X) = P(X, T)$.

3. Extension of Vitushkin's theorem

In 1967 Vitushkin obtained criteria for the equality $R(X) = C(X)$ involving analytic capacity. See [7] or, for example, [2, VIII.5.1]. In this section we extend Vitushkin's theorem.

The following lemma is known; see, for example, [2, p. 64].

LEMMA 3.1. *If $\{K_n\}_{n=1}^{\infty}$ is a sequence of compact plane sets such that $R(K_n) = C(K_n)$ for all n, and $K = \bigcup_{n=1}^{\infty} K_n$ is compact, then $R(K) = C(K)$.*

THEOREM 3.2. *The following assertions are equivalent:*

(i) *$R(X, S) = R(X, T)$;*

(ii) *$R(K) = C(K)$ for every compact subset $K \subseteq S \Delta T$;*

(iii) *for every compact subset $K \subseteq S \Delta T$, and for every open set D, $\gamma(D \setminus K) = \gamma(D)$, where γ is the analytic capacity;*

(iv) *for every compact subset $K \subseteq S \Delta T$, and for almost all $z \in K$ (with respect to the planar measure),*

$$\limsup_{r \to 0^+} \frac{\gamma(\Delta(z; r) \setminus K)}{r} > 0,$$

where $\Delta(z; r)$ is the closed disk with centre z and radius r.

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PROOF. (i) $\longrightarrow$ (ii) Let $R(X, S) = R(X, T)$ and K be a compact subset of $S\Delta T$. Then $S\backslash T \subseteq S_0(R(X, S))$ and $T\backslash S \subseteq S_0(R(X, T))$. We take $K_1 = K \cap S$ and $K_2 = K \cap T$. Then $K_1 \subseteq S_0(R(X, S)) \subseteq S_0(R(X, K_1))$ and $K_2 \subseteq S_0(R(X, T)) \subseteq S_0(R(X, K_2))$. Therefore, $K_1 \subseteq S_0(R(K_1))$ and $K_2 \subseteq S_0(R(K_2))$. Hence, $R(K_1) = C(K_1)$ and $R(K_2) = C(K_2)$ by Corollaries 2.10 and 2.12. By the above lemma, $R(K) = C(K)$.

(ii) $\longrightarrow$ (iii) and (iii) $\longrightarrow$ (iv) are immediate by Vitushkin's theorem [7].

(iv) $\longrightarrow$ (i) We prove that $R(X, S) = R(X, S \cap T)$. So it is sufficient to show that $R(S) = R(S, S \cap T)$. Let $z_0 \in S\backslash T$ and U be a neighbourhood of z_0 such that $\overline{U} \cap T = \emptyset$. If we take $K = \overline{U} \cap S$ then

$$\limsup_{r\to 0^+} \frac{\gamma(\Delta(z_0; r) \setminus K)}{r} > 0.$$

There exists $r_0 > 0$ such that for every $0 < r \leq r_0$, $\Delta(z_0; r) \subseteq U$, so $\Delta(z_0; r)\backslash K = \Delta(z_0; r)\backslash S$. Hence,

$$\limsup_{r\to 0^+} \frac{\gamma(\Delta(z_0; r) \setminus S)}{r} > 0.$$

By Curtis's criterion [2, VIII.4.1], z_0 is a peak point for $R(S)$. Hence, $S\backslash T \subseteq S_0(R(S))$. Therefore, $R(S) = R(S, S \cap T)$ by Corollary 2.11. By the same argument as above $R(X, T) = R(X, S \cap T)$ and hence (i) follows. □

In the above theorem we proved that $R(X, S) = R(X, T)$ if and only if for every compact subset $K \subseteq S\Delta T$, $R(K) = C(K)$. The following example shows that this result and Corollary 2.10 are not true if we replace R by P.

EXAMPLE 3.3. Let $S = C(0; 1)$, $T = C(2; 1)$ be two circles in the plane and $X = S \cup T$. For every compact subset K of $S\Delta T$, K is polynomially convex and $m(K) = 0$. Hence $P(K) = C(K)$. Moreover, $S\backslash T \subseteq S_0(P(X, S))$ and $T\backslash S \subseteq S_0(P(X, T))$. If $P(X, S) = P(X, T)$ then $P(X, S) = P(X, S \cap T) = R(X, S \cap T) = R(X, S)$. This implies that $\widehat{S} = S$, which is not true. Therefore, $P(X, S) \neq P(X, T)$.

COROLLARY 3.4. *$R(X, K) = R(X)$ if and only if $R(Y) = C(Y)$ for every compact plane set Y where $Y \subseteq X\backslash K$.*

PROOF. In Theorem 3.2 take $S = K$ and $T = X$. □

The next result is, in fact, an extension of a theorem due to Gauthier for compact plane sets; see [3] or [1].

THEOREM 3.5. *The following conditions are equivalent:*

(i) $R(X, S) = R(X, T)$;

(ii) *for every $f \in R(X, S)(R(X, T))$ and for each $\varepsilon > 0$ there exists a function $g \in R_0(X, T)(R_0(X, S))$ such that $\|f - g\|_{S\cup T} < \varepsilon$;*

(iii) *for every open set U in $\mathbb{C}$, $R(X, S \cap \overline{U}) = R(X, T \cap \overline{U})$;*

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(iv) *for every open disk* D *in* $\mathbb{C}$, $R(X, S \cap \overline{D}) = R(X, T \cap \overline{D})$;
(v) *for every* $p \in X$ *there exists an open disk* D_p *with centre* p *such that*

$$R(X, S \cap \overline{D}_p) = R(X, T \cap \overline{D}_p).$$

PROOF. (i) $\longrightarrow$ (ii) is immediate.

(ii) $\longrightarrow$ (i) Let $f \in R(X, S)$ and $\varepsilon > 0$. There exists $g \in R_0(X, T)$ such that $\|f - g\|_{S \cup T} < \varepsilon/2$. We can extend $(f - g)|_{S \cup T}$ to a continuous function h on X such that $\|h\|_X < \varepsilon$. We define $G = f - h$. Then $G \in R_0(X, T)$ and $\|f - G\|_X = \|h\|_X < \varepsilon$. So $f \in R(X, T)$ and hence $R(X, S) \subseteq R(X, T)$. By a similar method, $R(X, T) \subseteq R(X, S)$.

(i) $\longrightarrow$ (iii) $(S \cap \overline{U})\backslash(T \cap \overline{U}) \subseteq S\backslash T$ and $(T \cap \overline{U})\backslash(S \cap \overline{U}) \subseteq T\backslash S$. But $S \setminus T \subseteq S_0(R(X, S)) \subseteq S_0(R(X, S \cap \overline{U}))$ and $T \setminus S \subseteq S_0(R(X, T)) \subseteq S_0(R(X, T \cap \overline{U}))$. By Corollary 2.10, $R(X, S \cap \overline{U}) = R(X, T \cap \overline{U})$.

(iii) $\longrightarrow$ (iv) and (iv) $\longrightarrow$ (v) are immediate.

(v) $\longrightarrow$ (i) We first assume that K is a compact plane set in $S\backslash T$ and then show that $R(K) = C(K)$. For every $p \in K$ there exists D_p with a small radius such that $R(K_p) = C(K_p)$ where $K_p = K \cap \overline{D}_p$. Hence, there are points $p_1, p_2, \ldots, p_n$ in K such that $K \subseteq \bigcup_{i=1}^n D_{p_i}$ and so $K = \bigcup_{i=1}^n K_{p_i}$. Since $R(K_{p_i}) = C(K_{p_i})$ for $i = 1, 2, \ldots, n$, by Lemma 3.1, $R(K) = C(K)$. Therefore, by Theorem 3.2, $R(X, S) = R(X, S \cap T)$. By the same argument as in the first part of the proof, $R(X, T) = R(X, S \cap T)$. Therefore, $R(X, S) = R(X, T)$. □

THEOREM 3.6. $A(X, S) = R(X, T)$ *if and only if* $\mathrm{int}(S\backslash T) = \emptyset$, $T\backslash S \subseteq S_0(R(X, T))$ *and* $A(S \cap T) = R(S \cap T)$.

PROOF. Let $A(X, S) = R(X, T)$. Clearly $T\backslash S \subseteq S_0(R(X, T))$ by Theorem 2.9. If $\mathrm{int}(S\backslash T) \neq \emptyset$ then there exists an open disk D such that $\overline{D} \subseteq S\backslash T$. For every $f \in C(\overline{D})$ there exists an extension $F \in C(X)$ of f such that $F = 0$ on T. Clearly $F \in R(X, T) = A(X, S)$ and hence $f \in A(\overline{D})$, which is not true.

Since $\mathrm{int}(S\backslash T) = \emptyset$ and $T\backslash S \subseteq S_0(R(X, T))$, then $A(X, S) = A(X, S \cap T)$ and $R(X, T) = R(X, S \cap T)$. So $A(X, S \cap T) = R(X, S \cap T)$, which implies that $A(S \cap T) = R(S \cap T)$.

Conversely, since $\mathrm{int}(S\backslash T) = \emptyset$, then $A(X, S) = A(X, S \cap T)$, and moreover, $R(X, T) = R(X, S \cap T)$ by Corollary 2.10. Since $A(S \cap T) = R(S \cap T)$ it follows that $A(X, S \cap T) = R(X, S \cap T)$. Therefore, $A(X, S) = R(X, T)$. □

COROLLARY 3.7. $A(X, S) = R(X, T)$ *if and only if for every compact set* $K \subseteq T\backslash S$, $R(K) = C(K)$, *for every compact set* $K \subseteq S\backslash T$, $A(K) = C(K)$ *and* $A(S \cap T) = R(S \cap T)$.

PROOF. By Theorem 3.2, $R(X, S \cap T) = R(X, T)$ if and only if $R(K) = C(K)$ for all compact sets $K \subseteq T\backslash S$. On the other hand $R(X, S \cap T) = R(X, T)$ if and only if $T\backslash S \subseteq S_0(R(X, T))$, by Corollary 2.10. Since $\mathrm{int}(S\backslash T) = \emptyset$ if and only

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if $A(K) = C(K)$ for all compact sets $K \subseteq S \backslash T$, the result follows from the above theorem. □

We are now ready to extend a result due to Boivin and Jiang [1, Theorem 2] for compact plane sets.

THEOREM 3.8. *The following assertions are equivalent.*

(i) $A(X, S) = R(X, T)$;

(ii) *for every closed disk* $\overline{D}$ *in* $\mathbb{C}$, $A(X, S \cap \overline{D}) = R(X, T \cap \overline{D})$;

(iii) *for every* $p \in X$ *there exists a closed disk* $\overline{D}_p$ *in* $\mathbb{C}$ *with centre* p *such that* $A(X, S \cap \overline{D}_p) = R(X, T \cap \overline{D}_p)$.

PROOF. (i) $\longrightarrow$ (ii) By Theorem 3.6, $\text{int}((S \cap \overline{D}) \backslash (T \cap \overline{D})) = \emptyset$, and by Corollary 3.7, $R(K) = C(K)$ for every compact set $K \subseteq (T \cap \overline{D}) \backslash (S \cap \overline{D})$. Since $A(S \cap T) = R(S \cap T)$, it follows from [1, Theorem 2] that

$$A((S \cap \overline{D}) \cap (T \cap \overline{D})) = R((S \cap \overline{D}) \cap (T \cap \overline{D})).$$

So by Corollary 3.7, $A(X, S \cap \overline{D}) = R(X, T \cap \overline{D})$.

(ii) $\longrightarrow$ (iii) is immediate.

(iii) $\longrightarrow$ (i) By Theorem 3.6, for each $p \in X$, $A((S \cap \overline{D}_p) \cap (T \cap \overline{D}_p)) = R((S \cap \overline{D}_p) \cap (T \cap \overline{D}_p))$ for a closed disk $\overline{D}_p$ in $\mathbb{C}$ with centre p. So $A((S \cap T) \cap \overline{D}_p) = R((S \cap T) \cap \overline{D}_p)$. By [2, II.10.5], $A(S \cap T) = R(S \cap T)$.

Now we prove that $\text{int}(S \backslash T) = \emptyset$. If $p \in \text{int}(S \backslash T)$ then there exists a closed disk $\overline{D}_p$ in $\mathbb{C}$ with centre p such that $A(X, S \cap \overline{D}_p) = R(X, T \cap \overline{D}_p)$. Therefore, $\text{int}((S \cap \overline{D}_p) \backslash (T \cap \overline{D}_p)) = \emptyset$ by Theorem 3.6. We may take D_p with small enough radius such that $\overline{D}_p \subseteq S \backslash T$ so that $\overline{D}_p \subseteq (S \cap \overline{D}_p) \backslash (T \cap \overline{D}_p)$, which is in contradiction with $\text{int}((S \cap \overline{D}_p) \backslash (T \cap \overline{D}_p)) = \emptyset$. Hence, $A(X, S) = A(X, S \cap T)$. By $A(X, (S \cap T) \cap \overline{D}_p) = A(X, S \cap \overline{D}_p) = R(X, T \cap \overline{D}_p)$, it follows that $R(X, (S \cap T) \cap \overline{D}_p) \subseteq R(X, T \cap \overline{D}_p)$, so $R(X, (S \cap T) \cap \overline{D}_p) = R(X, T \cap \overline{D}_p)$. Hence, $R(X, S \cap T) = R(X, T)$ by Theorem 3.5. On the other hand, $A(X, S \cap T) = R(X, S \cap T)$ since $A(S \cap T) = R(S \cap T)$. Therefore, $A(X, S) = R(X, T)$. □

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