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ON THE CLASSIFICATION OF MANIFOLDS UP TO FINITE AMBIGUITY

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Introduction. In the early 70's Dennis Sullivan applied his theory of minimal models and surgery to the classification of 1-connected closed smooth manifolds of dimension ≥ 5 up to finite ambiguity [Su]. To a diffeomorphism class of such a manifold M he assigns the isomorphism class given by the real minimal model $\mathcal{M}(M)$, the integral structure in form of various lattices and the real Pontryagin classes. If one controls the torsion of the manifolds by some bound, his result is that the map given by the triple above is finite-to-one ([Su], Theorem 13.1). He also proves a realization result for the rational minimal model and the Pontryagin classes but not for the lattices ([Su], Theorem 13.2).

In this paper we show that these invariants can be substantially simplified especially the lattices which are subtle and hard to compute. This can be demonstrated with the following result.

THEOREM 2.2. *Let $n \geq 5$. The diffeomorphism type of a 1-connected closed smooth n-manifold M with formal* $([n/2]+1)$*-skeleton is determined up to finite ambiguity by the isomorphism class of the truncated cohomology ring* $H \equiv \oplus_{i\leq[n/2]+1}H^i(M;\mathbb{Z})$*, the real Pontryagin classes and an element* $\alpha_M \in H^n(\mathcal{M}(H)[n/2])^*$.

Here $\mathcal{M}(H)$ is the real minimal model of H and $\mathcal{M}(H)[n/2]$ is the subalgebra generated by elements of degree $\leq [n/2]$. If $4i \geq [n/2]$ the real Pontryagin classes have to be identified with their image under Poincaré duality in the dual vector space $H^{n-4i}(M;\mathbb{Q})^* \cong H^{n-4i}(\mathcal{M}(H)[n/2])^*$. We recall that a space is *formal* if its minimal model is determined by its cohomology ring. Formality is inherited by the skeleta of a formal space. There are rather general classes of manifolds which fulfill the conditions of Theorem 2.2. For instance it applies to $(r-1)$-connected manifolds of dimension $\leq 6r-5$; or to 1-connected manifolds whose integral cohomology ring is non-trivial only in four dimensions. If one wants to classify these manifolds by Sullivan's theorem one needs the full information of the minimal model and all lattices. This seems to be a very complicated task. We will discuss this in Section 2 and demonstrate the simplification of Sullivan's invariants more explicitly.

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In particular we will apply Theorem 2.2 to the following very natural class of 7-dimensional manifolds. For integers e and p with $p \equiv 2e \bmod 4$ we denote by $M_k^{e,p}$ the total spaces of the linear S^3-bundle over $\#_k S^2 \times S^2$ with Euler and first Pontryagin class e and p resp. (for the existence compare [DW]). If $k = 0$ (base space S^4) and $e = 1$ these are Milnor's exotic 7-spheres [Mil]. If $k = 1$, e even and $p = 2e$ they are homogeneous spaces of the form $SU(2) \times SU(2) \times SU(2)/U(1) \times U(1)$. In general if $e \neq 0$ the non-trivial cohomology groups are $H^0 \cong \mathbb{Z}$, $H^2 \cong \mathbb{Z}^{2k}$ and $H^4 \cong \mathbb{Z}_e$. These manifolds are not formal. We will show that α_M is completely determined by the 3-fold Massey products of elements in H^2 and that these Massey products depend only on e. Thus we obtain

COROLLARY 2.3. *For fixed $k \in \mathbb{N}$ and $e \in \mathbb{Z} - \{0\}$ the number of diffeomorphism types of the non-formal manifolds $M_k^{e,p}$ is finite.*

The classification of Milnor's 7-spheres indicates that the actual diffeomorphism classification of these manifolds is rather complicated.

Theorem 2.2. follows from a more general classification result, Theorem 1.1 below. Recall that the minimal model of a space is an algebraic mirror image of the rational Postnikov tower. The lattices are given by the cohomology and homotopy groups of the integral Postnikov tower on each stage. We weaken these invariants for a manifold M by omitting all information above the middle dimension, i.e., we replace $\mathcal{M}(M)$ by $\mathcal{M}(M)[n/2]$ and consider only the lattices up to dimension $[n/2]$. It is surprising that besides the Pontryagin classes and a torsion bound one has to add only a single invariant α_M to obtain a classification up to finite ambiguity of 1-connected closed oriented manifolds of dimension ≥ 5. The class $\alpha_M \in H^n(\mathcal{M}(M)[n/2])^*$ is the image of the fundamental class in the n-th homology of the $[n/2]$-th stage of a rational Postnikov tower. This is our main classification result—Theorem 1.1. Roughly speaking we do not need the whole minimal model but only "half of it" for our classification. We remark that this is not a formal consequence of Poincaré duality. We also obtain a realization result of our invariants if we omit the lattices, Theorem 1.2.

To derive Theorem 2.2 from Theorem 1.1 one needs to know that the minimal model as well as the lattices of a formal space are determined by the integral cohomology ring. In Section 3 we will prove this result for arbitrary 1-connected formal spaces with finitely generated cohomology groups (Proposition 2.1).

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1. **The main theorems.** The rational (real) minimal model of a 1-connected n-dimensional manifold M (or more generally of a 1-connected CW-complex) is an isomorphism class of minimal differential graded algebras (d.g.a.) $\mathcal{M}$ over $\mathbb{Q}(R)$ such that there exists a homomorphism $\mathcal{M} \longrightarrow E_M$ inducing an isomorphism on cohomology. Here E_M is the de Rahm complex over $\mathbb{Q}(R)$ (for details see [Su]).

The rational minimal model is an algebraic picture of the rational homotopy type of M. Especially if we pass to the subalgebra generated by elements of degree $\leq k$ we obtain the minimal model $\mathcal{M}[k]$ of the k-th stage of a rational Postnikov decomposition of M.

There is a homomorphism $\mathcal{M}[k] \longrightarrow E_M$ which induces an isomorphism on cohomology in degree $\leq k$ and an injection on cohomology of degree $k+1$.

We denote the k-th stage of an integral Postnikov tower of M by $M[k]$. Obviously, $M[k]$ is determined by any $(k+1)$-skeleton of M. Thus our problem is closely related to the classification of manifolds of dimension n with prescribed $(k+1)$-skeleton for some $k \geq [n/2]$.

As described in Sullivan's article, the image of the integral homology or homotopy groups determine integer lattices in $H^{r+1}(\mathcal{M}[r-1])$ and $\pi_r(\mathcal{M}[k])$, the vector space of the r-dimensional elements modulo decomposables, for $r \leq k$. We denote the direct sum of these lattices in $\oplus_r\big(H^{r+1}(\mathcal{M}[r-1]) \oplus \pi_r(\mathcal{M}[k])\big)$ by Z_1. Similarly, we have lattices in $H^*(\mathcal{M}[k])$. We denote by Z_2 the direct sum of these lattices in $\oplus_{4r \leq k} H^{4r}(\mathcal{M}[k]) \oplus H^n(\mathcal{M}[k])$ if $n \equiv 0 \ (4)$, and in $\oplus_{4r \leq k} H^{4r}(\mathcal{M}[k]) \oplus_{4r \leq k} H^{n-4r}(\mathcal{M}[k]) \oplus H^n(\mathcal{M}[k])$ if $n \not\equiv 0 \ (4)$.

Now, we suppose that M is oriented. If $\mathcal{M}[k] \longrightarrow E_M$ is a homomorphism with the cohomology properties described above, we denote the image of the fundamental class under the dual of this map by $\alpha(M) \in \big(H^n(\mathcal{M}[k])\big)^*$.

The last invariants we consider are the rational or real Pontryagin classes of M. If $r \leq k$, we can consider them as elements of $H^{4r}(\mathcal{M}[k])$ and if $4r > k$ we replace them by the Poincaré duals $\Delta p_r(M) \in \big(H^{n-4r}(\mathcal{M}[k])\big)^*$.

The isomorphism class of the data $(\mathcal{M}[k]), Z_1, Z_2, p_r(M)\ (4r \leq k), \Delta p_r(M)\ (4r > k), \alpha(M))$ forms an invariant of the diffeomorphism type of M. We call it the k-th *rational invariant* if $\mathcal{M}[k]$ is the rational model and the *k-th real invariant* if it is the real model.

Now, we are ready to formulate our extension of Sullivan's result.

THEOREM 1.1. *i) For given $n \geq 5$, $k \geq [n/2]$ and $N \in \mathbb{N}$ the set of diffeomorphism classes of 1-connected closed smooth oriented n-manifolds with isomorphic k-th invariants and with* $|\operatorname{Tor} H_*(M;Z)| \leq N$ *is finite.*

ii) If $n = 2m > 4$ and $k = m-1$, we obtain the same result if we restrict ourselves to manifolds with $H_m(M;\mathbb{Q}) = \{0\}$.

The proof is deferred to Section 3.

REMARK. If $k = n$, $\alpha(M)$ is contained in the integer lattice of $H^n(\mathcal{M})^*$ which is isomorphic to $\mathbb{Z}$ and is completely determined by this lattice and the orientation of M. Thus we obtain Sullivan's result ([Su], Theorem 13.1) as a special case of ours.

The general problem of the realization of the rational invariants is obviously difficult if we include the lattices Z_1 and Z_2. As in Sullivan's paper we only consider the realization problem for the rest of the data (and only for the rational invariant).

In our situation we first collect some necessary conditions. Let M be a 1-connected oriented closed n-manifold and $\mathcal{M}[k]$ as above. We have a commutative diagram

$$\begin{array}{ccc} H^i(\mathcal{M}[k]) & \longrightarrow & H^i(M;\mathbb{Q}) \\ \big\downarrow{\scriptstyle \cap\alpha(M)} & & \big\downarrow{\scriptstyle \cap[M]} \\ \big(H^{n-i}(\mathcal{M}[k])\big)^* & \longleftarrow & H_{n-i}(M;\mathbb{Q}), \end{array}$$

where $\cap\alpha\colon H^i(\mathcal{M}[k]) \longrightarrow H^{n-i}(\mathcal{M}[k])^*$ is given by $x \longmapsto \big(y \longmapsto \alpha(x \cup y)\big)$. As the horizontal maps are isomorphisms in degree $\leq k$, and in degree $k+1$ the upper one is injective and the lower one is surjective, Poincaré duality of M implies that $\cap\alpha(M)$ is an isomorphism if $n-k \leq i \leq k$, injective if $i = k+1$ and surjective if $i = n-k-1$. If $\mathcal{N}$ is a rational minimal d.g.a. generated by elements of degree $\leq k$ we say a class $\alpha \in H_n(\mathcal{N})$ is a *k-partial Poincaré duality class* of $\mathcal{N}$ if the homomorphism $\cap\alpha$ fulfills the properties above (compare [Kr], Section 5).

If $n \not\equiv 0$ (4) we will see that if α is a k-partial Poincaré duality class all rational data (excluding the lattices) can be realized but if $n \equiv 0$ (4) we have other necessary conditions. By the diagram above we know that if $\mathcal{N} = \mathcal{M}[k]$ and $\alpha \in H_n(\mathcal{N})$ are invariants of a manifold, the quadratic form on $H^{n/2}(\mathcal{N})$ given by $y \longmapsto \alpha(y \cup y)$ is the tensor product of a symmetric unimodular integer form with $\mathbb{Q}$. Thus it is equivalent to $\sum \pm X_i^2$.

Another necessary condition for $n \equiv 0$ (4) comes from the fact that the Pontryagin numbers of a closed oriented n-manifold have to fulfill the congruences of a cobordism class [St]. If $\langle p_{i_1}(M)\dots p_{i_r}(M), [M]\rangle$, $i_1 < \cdots < i_r$, is a Pontryagin number and $4i_r \leq k$ we can consider the classes $p_{i_1}(M), \dots, p_{i_r}(M)$ as elements of $H^*(\mathcal{M}[k])$ and this number is equal to $\alpha(M)\big(p_{i_1}(M) \cup \cdots \cup p_{i_r}(M)\big)$. If $4i_r > k$ the number is equal to $\Delta p_{i_r}(M)\big(p_{i_1}(M) \cup \cdots \cup p_{i_{r-1}}(M)\big)$, where $\Delta p_{i_r}(M)$ is considered as element of $H^{n-4i_r}(\mathcal{M}[k])^*$. This indicates how for given classes $a_i \in H^{4i}(\mathcal{N})$, $4i \leq k$, $b_{n-4i} \in H^{n-4i}(\mathcal{N})^*$, $4i > k$, and $\alpha \in H^n(\mathcal{N})^*$ one can introduce the notion of characteristic numbers.

The characteristic numbers of a closed smooth oriented manifold fulfill the relation of the Hirzebruch signature theorem [Hi]. Thus if a_i, b_j and α as above are characteric classes and fundamental class respectively of a manifold, the signature of the quadratic form $y \longmapsto \alpha(y \cup y)$ is equal to $L(a_i, b_j)$ where L is the Hirzebruch L-poloynomial. As the coefficient of b_0 (b_0 corresponds to the top Pontryagin class $p_n(M)$) is non-trivial this formula determines $b_0 \in \mathbb{Q}$ in terms of the a_i, the b_j for $j > 0$ and α. Thus for given classes a_i, b_j ($j > 0$) and α we can define their characteristic numbers if we add to these classes the class b_0 determined by the a_i, b_j and α. For $n \equiv 0$ (4), we say that a_i, b_j ($j > 0$) and α can be realized by a manifold if these classes together with the class b_0 determined by the other classes can be realized.

These are the obvious conditions which the invariants coming from manifolds have to fulfill. They are also sufficient.

THEOREM 1.2. *Let $\mathcal{N}$ be a minimal differential graded algebra over $\mathbb{Q}$ generated by elements of degree $\leq k$ with $H^1(\mathcal{N}) = \{0\}$. Let $[n/2] \leq k$ and $n \geq 5$. Then $\mathcal{N}$*

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and classes $a_r \in H^{4r}(\mathcal{N})$ for $0 < 4r \leq k$, $b_{n-4r} \in H^{n-4r}(\mathcal{N})^*$ for $k < 4r < n$ and $\alpha \in H^n(\mathcal{N})^*$ can be realized as the k-th rational invariant (excluding the lattices) of a 1-connected closed smooth n-manifold iff

i) α is a k-partial Poincaré duality class and $n \not\equiv 0 \ (4)$,

ii) for $n \equiv 0 \ (4)$, α is a k-partial Poincaré duality class,

so that the quadratic form on $H^{n/2}(\mathcal{N})$, $y \mapsto \alpha(y \cup y)$, is equivalent to a sum of squares $\pm x_i$ and the characteristic numbers of a_i and b_j with respect to α satisfy the congruences of a cobordism class [St].

Also if $n \equiv 2 \ (4)$, $n \equiv 2 \ (k+1)$ and $H^{n/2}(\mathcal{N}) = \{0\}$ the data above can be realized by a 1-connected closed smooth n-manifold M with $H^{n/2}(M; \mathbb{Q}) = 0$. If $n \equiv 0 \ (4)$, $n \equiv 2 \ (k+1)$ and $H^{n/2}(\mathcal{N}) = \{0\}$ the only necssary and sufficient condition is that the characteristic numbers of a_i and b_j with respect to α satisfy the congruences of a cobordism class.

The proof is deferred to Section 3.

REMARK. If $k = n$ we are again in Sullivan's situation ([Su], Theorem 13.2). In this case, we obtain a slightly stronger information by deciding which fundamental classes α (in Sullivan's notation μ) can be realized.

2. **Application to manifolds with formal** $([n/2]+1)$**-skeleton.** A space X is *formal* if its minimal model is determined by the rational cohomology ring. More precisely the condition is that there is a homomorphism from $\mathcal{M}(X)$ to $H^*(X; \mathbb{Q})$ inducing an isomorphism on cohomology. There are large classes of spaces which are formal, e.g., Lie groups, classifying spaces [Su], $(r-1)$-connected manifolds of dimension $\leq 4r-2$ [Mi], 1-connected Kaehler manifolds [DGMS]. An algorithmic method to decide when a space is formal is given in [HS].

In the case of Kaehler manifolds Sullivan showed that the diffeomorphism type (in real dimension > 4) is determined up to finite ambiguity by the integral cohomology ring and the real Pontryagin classes ([Su], Theorem 12.5). In the proof he assumes implicitly that not only the minimal model but also the lattices are determined by the integral cohomology ring. This holds in general for formal spaces.

PROPOSITION 2.1. *Let H be a finitely generated graded commutative ring over $\mathbb{Z}$. Then there are only finitely many homotopy types of simply connected, formal, finite CW-complexes with integral cohomology isomorphic to H.*

REMARK. The result also holds for spaces with finitely generated homotopy $\oplus_i \pi_i(X)$.

We give a proof of Proposition 2.1 in Section 3.

As a consequence of Proposition 2.1 one can generalize Sullivan's result about classification of Kaehler manifolds to general formal manifolds. Similarly one can use it in combination with Theorem 1.1 to classify manifolds with formal $(k+1)$-skeleton for some $k > [n/2]$. As all skeleta of a formal space are again formal it is enough to consider the smallest skeleton, i.e., $k = [n/2]$.

If M has formal $([n/2]+1)$-skeleton it is determined up to finite ambiguity by an invariant of the following algebraic type: isomorphism classes of quadruples

$$(H, p_i, \Delta p_i, \alpha),$$

where H is a finitely generated graded commutative ring over $\mathbb{Z}$ with $H^0 = \mathbb{Z}$, $H^1 = \{0\}$, $H^i = \{0\}$ for $i \geq [n/2]+1$, $p_i \in H^{4i} \otimes \mathbb{R}$ for $1 < 4i \leq [n/2]$, $\Delta p_i \in (H^{n-4i} \otimes R)^*$ for $[n/2] < 4i \leq n$, $\alpha \in \Big(H^n\big(\mathcal{M}(H)[n/2]\big) \otimes \mathbb{R}\Big)^*$.

Two such objects are isomorphic if there is a ring isomorphism on H respecting the p_i, Δp_i and α.

If M has a formal $([n/2]+1)$-skeleton we obtain such a quadruple as follows: $H = \oplus_{i \leq [n/2]+1} H^i(M; \mathbb{Z})$, p_i and Δp_i are the Pontryagin classes and their image under Poincaré duality respectively and $\alpha = \alpha_M \in \Big(H^n\big(\mathcal{M}(H)[n/2]\big) \otimes \mathbb{R}\Big)^*$ is the invariant occurring in Theorem 1.1. Here we use formality to identify $\Big(H^n\big(\mathcal{M}(M)[n/2]\big) \otimes \mathbb{R}\Big)^*$ with $\Big(H^n\big(\mathcal{M}(H)[n/2] \otimes \mathbb{R}\Big)^*$.

THEOREM 2.2. *Let $n \geq 5$. The diffeomorphism type of a 1-connected closed smooth oriented n-manifold M with formal $([n/2]+1)$-skeleton is determined up to finite ambiguity by the isomorphism class*

$$\Big(\bigoplus_{i \leq [n/2]+1} H^i(M; \mathbb{Z}), p_i, \Delta p_i, \alpha_M\Big),$$

PROOF. Let $k = [n/2]$. The statement follows from Theorem 1.1 if $H = \oplus_{i \leq k+1} H^i(M; \mathbb{Z})$ determines $\mathcal{M}[k]$, Z_1 and Z_2 up to finite ambiguity. As $\mathcal{M}[k]$, Z_1 and Z_2 are determined by a $(k+1)$-skeleton of M the proof is completed if there are only finitely many minimal $(k+1)$-skeleta coming from all manifolds M with fixed H. Here X is a minimal $(k+1)$-skeleton if the Betti number $b_{k+1}(X)$ is minimal under all $(k+1)$-skeleta of M.

The above will folllow from Proposition 2.1. if we know that:

i) If X and X' are k-skeleta of a simply connected space Y then they are both formal or both non-formal.

ii) The integral cohomology ring of a minimal $(k+1)$-skeleton X of M is determined by $\oplus_{i \leq k+1} H^i(M; \mathbb{Z})$ up to finite ambiguity.

To show i) we first note that X is formal if and only if $X \vee S^k$ is formal. If X and X' are k-skeleta of Y there exists a k-equivalence $Z = X' \vee_r S^k \longrightarrow X$. It follows that $\pi_{k+1}(X, \mathbb{Z}) \cong H_{k+1}(X, \mathbb{Z})$ is free. Let $a_1, \ldots, a_r$ be the generators. Then $\mathcal{M}(Z) = \mathcal{M}(X)[a_1, \ldots, a_r, b_i]$ with $\deg b_i \geq k+1$ and $H^*(Z) = H^*(X) \oplus \mathbb{Z} a_1 \oplus \cdots \oplus \mathbb{Z} a_r$.

Thus if X is formal the map $\mathcal{M}(X) \longrightarrow H^*(X)$ inducing an isomorphism on cohomology has an obvious extension to $\mathcal{M}(Z)$ with the same property. This means that Z is formal.

To show ii) we first note that if X is a $(k+1)$-skeleton of M then $H^* \cong \oplus_{i \leq k+1} H^i(M) \oplus \mathbb{Z}^r$ with integral coefficients.

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Consider the minimal $(k+1)$-skeleta X of all manifolds M with $\oplus_{i\leq k+1}H^i(M) \cong H$. By a simple induction argument over k it can be shown that the Betti numbers $b_{k+1}(X)$ are bounded by some integer N depending on H. This completes the argument. ∎

Now we want to discuss this result and especially the role of the invariant α_M. As mentioned above, all $(r-1)$-connected manifolds of dimension $\leq 4r-2$ are formal [Mi] and thus they are determined up to finite ambiguity by the integral cohomology ring and the Pontryagin classes. This also follows from Theorem 2.2 Indeed the cohomology ring $H^*(M)/$ Tor of M^n is determined by $\oplus_{i\leq k+1}H^i(M;\mathbb{Z})$ and α_M (recall $k=[n/2]$):

As a group $H^i(M;\mathbb{Z})/$ Tor for $i>k$ is by Poincaré duality isomorphic to $H^{n-i}(M;\mathbb{Z})^*$. Also the ring structure is by Poincaré duality equivalent to knowing all triple products $\langle X_1\cup X_2\cup X_3,[M]\rangle$ when $\sum|X_i|=n$ and $|X_i|<k$. But by definition of α_M, $\langle X_1\cup X_2\cup X_3,[M]\rangle$ is equal to $\alpha_M(X_1\cup X_2\cup X_3)$. Here we identify $H^i(M;\mathbb{Z})\otimes\mathbb{Q}$ with $H^i(\mathcal{M}(H)[k])$.

This classification of $(r-1)$-connected manifolds of dimension $\leq 4r-2$ can be generalized by our theorem to manifolds of lower connectivity. Since every $(r-1)$-connected CW-complex of dimension $\leq 3r-2$ is formal, the $([n/2]+1)$-skeleta of $(r-1)$-connected manifolds of dimension $\leq 6r-5$ are formal. Therefore Theorem 2.2 applies.

If M is $(r-1)$-connected and $n>4r-2$, M is in general not formal but it is still possible that the $(k+1)$-skeleton is formal. This is demonstrated in the following example. Consider 1-connected closed smooth n-manifolds with non-vanishing homology only in four dimensions: $0, r, n-r, n$, where $1\leq r\leq[n/2]=k$. Such a manifold has a formal $(k+1)$-skeleton which is a wedge of r-spheres if $r<[n/2]$.

If the Betti number is 1 all such manifolds are formal but if the Betti number is 2, we have two minimal models with this cohomology ring. If r is odd the non-formal model looks as follows (a_1, a_2 and b are the generators of the algebra):

r	$2r-1$	$2r$	$3r-1$	$4r-1$
a_1	$b \stackrel{d}{\longmapsto} a_1a_2$		a_1b	a_1a_2b
a_2			a_2b	

With this example we also want to demonstrate the most important aspect of our results. In the formulation of Theorem 1.1 as well as in Sullivan's original result the most delicate information is contained in the lattices. These are not easy to compute. If M has formal $([n/2]+1)$-skeleton no such information is needed.

As it happens, the algebra described above is also the minimal model $\mathcal{M}$ of $(S^r\vee S^r)[2r-1]$. For simplicity let us restrict ourselves to manifolds with trivial normal bundle. Then these manifolds are up to finite ambiguity classified by $\alpha_M\in H^{4r-1}(\mathcal{M})^*\cong\mathbb{Q}$. On the other hand it follows from [Kr] (compare also the proof of Theorem 1.2 in Section 3) that the set of α which can be realized contains a lattice in $\mathbb{Q}$. In particular, there are infinitely many diffeomorphism types. We can deduce that in Sullivan's theory these diffeomorphism types are distinguished by two minimal models and an infinite number of non-equivalent lattices. This can not be seen directly in that theory since there is no realization result of lattices.

In this example we have seen that at least part of the information carried by the lattices in Sullivan's result is contained in the invariant α_M. We finish this chapter with a brief discussion of the role of α_M. As noted above, for $(r-1)$-connected manifolds M^n with $n \leq 4r-2$, α_M is equivalent to the ring structure of $H^*(M;\mathbb{Z})/\operatorname{Tor}$. In general for $n > 4r-2$ there is further structure on the cohomology ring which one should be able to deduce from α, namely higher order cohomology operations, for instance the Massey products. In principle one could specify the relation between the higher order cohomology operations and α but the formulation becomes very technical.

Instead we show this relation in the situation of the manifolds $M_k^{e,p}$ defined in the introduction. As mentioned above, all 1-connected 4-complexes are formal and thus 2.2 applies. We suppose $e \neq 0$. The Gysin sequence shows that the only non-trivial cohomology groups of $H^*(M_k^{e,p};\mathbb{Z})$ in dimension ≤ 4 are $H^0 \cong \mathbb{Z}$, $H^2 \cong \mathbb{Z}^{2k}$, $H^4 \cong \mathbb{Z}_e$.

From this information one can read off $\mathcal{M}(H)[3]$, where $H = \oplus_{i\leq 4} H^i(M;\mathbb{Z})$. $\mathcal{M}(H)[3]$ is generated by elements $a_1,\ldots,a_{2k}$ in dimension 2 and b_{ij}, $1 \leq i \leq j \leq 2k$, in dimension 3. The differential is given by: $d(b_{ij}) = a_i a_j$. From these data one can derive that $H^5\big(\mathcal{M}(H)[3]\big)$ is generated by the 3-fold Massey products $[a_1,a_2,a_3]$, and $H^7\big(\mathcal{M}(H)[3]\big)$ by $[a_1,a_2,a_3]\cdot a_4, a_i \in H^2\big(\mathcal{M}(H)[3]\big)$. Here $[a_1,a_2,a_3] := a_{12}a_3 - a_1 a_{23}$, where a_{ij} are cochains with $da_{ij} = a_i a_j$, and a_i denotes the cocycle and its cohomology class indiscriminately.

Thus by naturality α_M is determined by knowing $\langle [a_1,a_2,a_3]\cdot a_4, [M]\rangle$ for all $a_i \in H^2(M;\mathbb{R})$. By construction $M = M_k^{e,p}$ is the boundary of a disk-bundle $N = N_k^{e,p}$. We have an isomorphism $d\colon H^5(M) \longrightarrow H^6(N,M)$. In this situation a standard argument shows that $d[a_1,a_2,a_3] = \alpha_{12}\bar{a}_3 - \alpha_{23}\bar{a}_1$, where $\bar{a}_i \in H^2(N) \xrightarrow{\cong} H^2(M)$ are the corresponding elements to a_i and $\alpha_{ij} \in H^4(N,M)$ map to $\bar{a}_i\bar{a}_j$ in $H^4(N)$ (with $\mathbb{R}$-coefficients). If we identify $H^6(N,M)$ with $H^2(\#_k S^2\times S^2)$ by the Thom-isomorphism and $H^2(M)$ with $H^2(\#_k S^2 \times S^2)$ by the natural map, the Massey product $[a_1,a_2,a_3]$ is $(a_1\cdot a_2/e)\cdot a_3 - (a_2\cdot a_3/e)\cdot a_1$, where $a_i\cdot a_j/e \in H^0(M;\mathbb{R}) = \mathbb{R}$ is the unique element which under the cup-product with e maps to $a_i\cdot a_j \in H^4(M;\mathbb{R})$. Thus we see that $\langle [a_1,a_2,a_3]\cdot a_4, [M]\rangle = \langle (a_1a_2/e)\cdot a_3a_4 - (a_2a_3/e)\cdot a_1\cdot a_4, [\#_k S^2\times S^2]\rangle$ and so α_M depends only on e and not on p.

Another consequence is that, for $e \neq 0$ and $k > 0$, $M_k^{e,p}$ is not formal. For, otherwise we have a morphism $\mathcal{M} = \mathcal{M}(M_k^{e,p}) \longrightarrow H^*(M_k^{e,p})$ inducing an isomorphism on cohomology groups. Restricting to $\mathcal{M}[3] = \mathcal{M}(H)[3]$ and using the fact that by construction of $\mathcal{M}[3]$ the induced map $H^7(\mathcal{M}[3]) \longrightarrow H^7(M_k^{e,p})$ is trivial, we obtain $\alpha_M = 0$. On the other hand if $k > 0$ we can find $a_1,a_2,a_3,a_4 \in H^2(M) = H^2(\mathcal{M}[3])$ such that by the formula above $\langle [a_1,a_2,a_3]\cdot a_4, [M]\rangle = \langle [a_1,a_2,a_3]\cdot a_4, \alpha_M\rangle$ is non-trivial contradicting the assumption.

These considerations prove

COROLLARY 2.3. *For $p = 2e \bmod 4$ let $M_k^{e,p}$ be the total space of the 3-sphere bundle over $\#_k S^2\times S^2$ with Euler and Pontryagin class e and p respectively. Suppose $e \neq 0$.*

i) For $k > 0$ these manifolds are not formal.

ii) Their diffeomorphism type is up to finite ambiguity determined by $k = \frac{1}{2}\operatorname{rank} H^2(M)$ and $e = |H^4(M)|$ alone.

REMARK 2.4. If $k = 0$ and $e = 1$, $M_0^{1,p}$ are Milnor's exotic 7-spheres implying that the actual diffeomorphism classification is rather complicated.

3. **Proofs.** The proofs of Theorems 1.1 and 1.2 follow the same pattern as Sullivan's proofs [Su]. He uses his homotopy theory up to finite ambiguity and Browder's and Novikov's surgery theory for 1-connected manifolds. The first named author has extended this surgery theory to attack the problem of classification of n-manifolds with prescribed $(k+1)$-skeleton [Kr]. A combination of this theory and Sullivan's homotopy theory leads to the proof of Theorem 1.1 and 1.2.

We shortly summarize this surgery theory. For details see [Kr]. Let $p: B \longrightarrow BO$ be a fibration. An *n-dimensional normal k-smoothing in* (B, p) is a pair $(M^n, \bar{\nu}_M)$, where M is a closed smooth n-manifold and $\bar{\nu}_M: M \longrightarrow B$ a $(k+1)$-equivalence such that $\nu_M = p\bar{\nu}_M$; here ν_M is the normal Gauss map of M. These conditions imply that M and B have homotopy equivalent $(k+1)$-skeleton and the restriction of the normal bundle of M to this skeleton is isomorphic to the restriction of $p^*\gamma$ to it, where γ is the universal bundle over BO.

If we omit the condition that $\bar{\nu}_M$ is a $(k+1)$-equivalence, $(M, \bar{\nu}_M)$ is called a (B,p)-manifold. The bordism group of n-dimensional (B,p)-manifolds is denoted by $\Omega_n^{(B,p)}$.

The most interesting case occurs when B is connected and homotopy equivalent to a CW-complex and $p: B \longrightarrow BO$ is $(k+1)$-coconnected (i.e., the homotopy groups of the fibre vanish in dimension $\geq k+1$). Such a fibration is obtained if one considers the k-th stage of a Postnikov decomposition of the normal Gauss map of some manifold M which is characterized by a commutative diagram.

$$\begin{tikzcd} & B \arrow[d, "p"] \\ M \arrow[ur] \arrow[r, "\nu_M"'] & BO \end{tikzcd};$$

here $M \longrightarrow B$ is a $(k+1)$-equivalence and $B \longrightarrow BO$ is a $(k+1)$-coconnected fibration. This fibration is determined by M and k up to fibre homotopy equivalence and is called the *normal k-type of M*.

LEMMA 4. *Let $p: B \longrightarrow BO$ be a $(k+1)$-coconnected fibration such that B is 1-connected and homotopy equivalent to a CW-complex with finite $([n/2]+1)$-skeleton. Let $n \geq 5$.*

i) If $k \geq [n/2]$ then the set of diffeomorphism classes of n-dimensional normal k-smoothings in (B,p) representing a fixed bordism class in $\Omega_n^{(B,p)}$ is finite.

ii) If $n = 2(k+1)$ then the same holds if we restrict ourselves to manifolds with $H_{k+1}(M; \mathbb{Q}) = \{0\}$.

PROOF. i) is an immediate consequence of [Kr], Theorem 7.3, B, C, D as the obstructions to transforming a B-bordism between two normal k-smoothings into an h-cobordism are contained in the ordinary surgery obstruction group $L_{n+1}\{e\}$ or rather a quotient of it.

Thus we can obtain a B-bordism with vanishing obstruction if we add to one of the normal k-smoothings an appropriate homotopy sphere. Here we make use of the additivity of surgery obstructions ([Kr], Proposition 7.1).

ii) follows from [Kr], Theorem 2.1 and Theorem 3.1.

By the Pontryagin-Thom construction, $\Omega_n^{(B,p)} \cong \pi_n(Mp^*\gamma)$, where $\pi_n(Mp^*\gamma)$ is the stable homotopy group of the Thom spectrum of the pull back of γ. On the other hand by Serre [Se], $\pi_n(Mp^*\gamma) \otimes \mathbb{Q} \cong H_n(B;\mathbb{Q})$. Thus, if B is 1-connected and has finite $(n+1)$-skeleton this implies that the map $\Omega_n^{(B,p)} \longrightarrow H_n(B;\mathbb{Q})$ given by $[M,\bar{\nu}_M] \longmapsto (\bar{\nu}_M)_*[M]$ has finite kernel (if we orient $p^*\gamma$; any B-manifold can be oriented by pulling back the orientation of $p^*\gamma$).

We compute $H_n(B;\mathbb{Q})$ by a rationalization B_0 of B such that $H_n(B;\mathbb{Q}) = H_n(B_0)$. $BO_0 \cong \prod K(\mathbb{Q},4i)$ is a product of Eilenberg-Mac Lane spaces. Thus if $p\colon B \longrightarrow BO$ is $(k+1)$-coconnected, we have a homotopy commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & B_0[k] \times BO_0\langle k+\varepsilon\rangle \\ \big\downarrow p & & \big\downarrow \\ BO & \longrightarrow & BO_0 \end{array},$$

where the upper horizontal map induces isomorphisms on the rational homotopy groups and thus induces a homotopy equivalence $B_0 = B_0[k] \times BO_0\langle k+\varepsilon\rangle$. As before $M[k]$ stands for the k-th stage of a Postnikov tower. $B\langle r\rangle$ stands for the $(r-1)$-connected cover of B. In the diagram $\varepsilon = 1$, if $k+1 \not\equiv 0 \ (4)$ or $k+1 = 4m$ and $p^*(p_m) = 0$ in $H^{4m}(B;\mathbb{Q})$. Otherwise $\varepsilon = 2$. The map $B \longrightarrow B_0[k]$ is the rational k-th Postnikov map whereas the map $B \longrightarrow BO_0\langle k+\varepsilon\rangle$ is given by the higher Pontryagin classes of $p^*\gamma$. The map $B_0[k] \times BO_0\langle k+\varepsilon\rangle \longrightarrow BO_0$ is the sum of the lower Pontryagin classes $p^*\gamma$ and the inclusion $BO_0\langle k+\varepsilon\rangle \longrightarrow BO_0$

The homotopy equivalence $B_0 \longrightarrow B_0[k] \times BO_0\langle k+\varepsilon\rangle$ described above and the Kuenneth formula imply, $(\bar{\nu}_M)_*[M] = (\bar{\nu}_N)_*[N]$ in $H_n(B;\mathbb{Q}) = H_n(B_0)$. This is equivalent to $(\rho_M)_*[M] = (\rho_N)_*[N]$ and $(\rho_M)_*\Delta p_r(M) = (\rho_N)_*\Delta p_r(N)$ for $4r > k$ in $H_*(B_0[k])$; here ρ_M and ρ_N are the compositions of the maps ν_M and ν_N respectively with the map $B \longrightarrow B_0[k]$ in the diagram.

If we combine this information with Lemma 4 we obtain

LEMMA 5. *Consider the same assumptions as in Lemma 4. The set of diffeomorphism classes of n-dimensional normal k-smoothings* $(M,\bar{\nu}_M)$ *with the same homology classes* $(\rho_M)_*[M]$ *and* $(\rho_M)_*\Delta p_r(M)$ *in* $H_*(B_0[k])$ *for* $4r > k$ *is finite.*

PROOF OF THEOREM 1.1. Let $\mathcal{N}$ be a real minimal d.g.a. generated by elements of degree $\leq k$, Z_1, Z_2 lattices, $a_r \in H^{4r}(\mathcal{N})$ for $4r \leq k$ and $b_r \in H^{n-4r}(\mathcal{N})^*$ for $4r > k$ and $\alpha \in H_n(\mathcal{N})$. By ([Su], p. 324, step 1) we can assume that $\mathcal{N}$ is a rational minimal d.g.a. We want to show that there are only finitely many diffeomorphism classes of manifolds of dimension n with k-th rational invariant $(\mathcal{N}, Z_1, Z_2, a_r, b_r, \alpha)$ and with torsion bounded by N.

By ([Su], Theorem 10.4) there exist only finitely many 1-connected homotopy types with rational minimal model and integer lattice isomorphic to $(\mathcal{N}, Z_1)$ and with torsion

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bounded by N. Consider the subset of these homotopy types which has a representative X with finite $(n+1)$-skeleton and $\pi_r(X) = \{0\}$ for $r > k$. Fix for each homotopy type a representative X and denote its $(k+1)$-skeleton by Y. We translate our data a_r, b_s and α into the rational cohomology and homology of X. They are only well defined up to composition with the action of $\operatorname{Aut} X$ on the corresponding cohomology and homology groups of X. On the other hand our invariant is the orbit of a_r, b_s and α under the action of the automorphisms of the minimal model of X which preserve the lattices. One has to control the difference between the orbits under these two actions. Fortunately the group of automorphisms induced by $\operatorname{Aut} X$ and the one induced by the automorphisms of $\mathcal{N}$ which preserve the lattices are commensurable ([Su], p. 325).

Thus for studying the classification up to finite ambiguity we can equally well work with the orbit under $\operatorname{Aut} X$ instead of the automorphisms of $\mathcal{N}$ which preserve the lattices.

Now, we want to construct a $(k+1)$-coconnected fibration $p: B \longrightarrow BO$ from the classes $a_r \in H^{4r}(X; \mathbb{Q})$ and, if $k+1 = 4m$, $b_{n-m} \in H^{n-m}(X; \mathbb{Q})^*$. We require that the fibre homotopy equivalence class of this fibration is determined by the classes above up to finite ambiguity. We will later show that the normal k-type of a manifold M with the given data as its k-th rational invariant is contained in this finite set. Furthermore we know that if $k+1 = 4m$, $\Delta p_{4m}(M) = b_{n-M}$ is in the image of the map $\cap\, \alpha : H^m(X; \mathbb{Q}) \longrightarrow H^{n-m}(X; \mathbb{Q})^*$. This map is injective and we can assume for our proof that if $k+1 = 4m$, $b_{n-m} = a_{4m} \cap \alpha$ with $a_{4m} \in H^{4m}(X; \mathbb{Q})$ completely determined by b_{n-m}.

The classes a_r, $4r \leq k+1$, determine a finite set of isomorphism classes of vector bundles over Y, the $(k+1)$-skeleton of X, given by a map $p: Y \longrightarrow BO$ with $p^*(p_r) = a_r$. We obtain the $(k+1)$-coconnected fibrations $p: B \longrightarrow BO$ if we pass to the k-th stage of a Postnikov decomposition of such a map.

By Lemma 5 for each such fibration $p: B \longrightarrow BO$ there are only finitely many diffeomorphism classes of manifolds with this normal k-type whose k-th rational invariant is isomorphic to $(\mathcal{N}, Z_1, Z_2, a_r, b_s, \alpha)$. Thus we are done if, as announced, we can show that for all these manifolds the normal k-type is contained in the finite set constructed above. But this is obvious as the normal k-type of a manifold can be constructed from the k-th stage of a Postnikov tower $M[k]$ (which is homotopy equivalent to one of the X described above). We restrict the normal bundle of M to the $(k+1)$-skeleton of M which is equal to the $(k+1)$-skeleton of $M[k]$ and we take the k-th stage of a Postnikov decomposition of it. ∎

PROOF OF THEOREM 1.2. Again we follow the same pattern as Sullivan. Given $\mathcal{N}$ let X be a rational space representing $\mathcal{N}$ ([Su], § 8). Consider the map $X \times BO_0\langle k+\varepsilon\rangle \longrightarrow BO_0$ given by the classes a_r and the inclusion $BO_0\langle k+\varepsilon\rangle \longrightarrow BO_0$, where $\varepsilon = 1$ or 2 as described in the proof of Theorem 1.1; $\varepsilon = 1$ if $k+1 \not\equiv 0$ (4) or $k+1 = 4m$ and $b_{n-m} = 0$ and $\varepsilon = 2$ otherwise. Consider the fibre product B of this map and the map $BSO \longrightarrow BO_0$:

$$\begin{array}{ccc} B & \xrightarrow{p} & BSO \\ \downarrow & & \downarrow \\ X \times BO_0\langle k+\varepsilon\rangle & \longrightarrow & BO_0 \end{array} .$$

Now, we assume that we have a class $\alpha \in \Omega_n^{(B,p)}$ whose image under the Hurewicz map in $H_n(B;\mathbb{Q}) \cong H_n(X\times BO_0\langle k+\varepsilon\rangle)$ is the class given by α and the b_{n-4r} (including b_0 if $n \equiv 0\ (4)$ as determined by the other classes and α by means of the L-polynomial). We represent α by a B-manifold $(N, \bar{\nu}_N)$. By rational surgery we can assume that $\bar{\nu}_N\colon N \longrightarrow B$ is a rational $[n/2]$-equivalence (this follows from an obvious modification of [Kr], Lemma 2.3; compare [Su], p. 326).

If a is a k-partial Poincaré duality class we can transform the map $\bar{\nu}_N\colon N \longrightarrow B$ into a rational $(k+1)$-equivalence (into one with $H^{n/2}(M;\mathbb{Q}) = \{0\}$, if $n = 2(k+1)$) by killing the kernel of $\pi_{[n/2]}(N)\otimes\mathbb{Q} \longrightarrow \pi_{[n/2]}(B)\otimes\mathbb{Q}$ (compare with the arguments in [Kr], §§5 and 6). The only obstruction for doing this is the Witt class of the quadratic form on this kernel which for $n \equiv 0\ (4)$ is classified by the signature. This signature vanishes by our construction of b_0. This proves that we can realize the given data if our data represent a homology class in $H_n(B;\mathbb{Q})$ which is in the image of the Hurewicz homomorphism.

In order to show that this is true if the characteristic numbers of a_i and b_j with repsect to α satisfy the congruences of a cobordism class we first note that if G is a finite group or $\mathbb{Q}/\mathbb{Z}$ the map $p\colon B \longrightarrow BSO$ induces as isomorphism of homology groups $H_*(B;G) \longrightarrow H_*(BSO;G)$. This follows from the Serre spectral sequence and the fact that the homotopy fibre F of $p\colon B \longrightarrow BSO$ is by construction a rational space.

Next we note that we can find a representative in $H_n(B;\mathbb{Z})$ of the class given by our data whose image in $H_n(BSO;\mathbb{Z})$ is contained in the image of the Hurewicz map $\Omega_n \longrightarrow H_n(B\Sigma 0;\mathbb{Z})$. This follows from our assumptions and diagram chasing in

$$\begin{array}{ccccccc}
H_{n+1}(B;\mathbb{Q}/\mathbb{Z}) & \longrightarrow & H_n(B;\mathbb{Z}) & \longrightarrow & H_n(B;\mathbb{Q}) & \longrightarrow & H_n(B;\mathbb{Q}/\mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow\cong \\
H_{n+1}(BSO;\mathbb{Q}/\mathbb{Z}) & \longrightarrow & H_n(BSO;\mathbb{Z}) & \longrightarrow & H_n(BSO;\mathbb{Q}) & \longrightarrow & H_n(BSO;\mathbb{Q}/\mathbb{Z})
\end{array}.$$

Finally we use the Atiyah-Hirzebruch spectral sequence. This determines the image of the Hurewicz map $\Omega_n^{(B,p)} \longrightarrow H_n(B;\mathbb{Z})$ as $\bigcap_{j\geq 2}\ker d_j\colon H_n(B;\mathbb{Z}) \longrightarrow H_{n-j}(B;\pi_{j-1}^s)$ where d_j are the differentials in this spectral sequence. We compare this situation with the Atiyah-Hirzebruch spectral sequence for BSO. As the stable stems π_{j-1}^s are finite groups the commutative diagram

$$\begin{array}{ccc}
H_n(B;\mathbb{Z}) & \xrightarrow[d_j]{} & H_{n-j}(B;\pi_{j-1}^s) \\
\downarrow & & \downarrow\cong \\
H_n(BSO;\mathbb{Z}) & \xrightarrow[d_j]{} & H_{n-j}(BSO;\pi_{j-1}^s)
\end{array}$$

implies that an element in $H_n(B;\mathbb{Z})$ is in the image of the Hurewicz map if and only if its image in $H_n(BSO;\mathbb{Z})$ is contained in the image of the corresponding Hurewicz map.∎

PROOF OF PROPOSITION 2.1. We proceed inductively over the Postnikov tower. Let X_n be the nth stage of the Postnikov tower of X where X fulfills the properties of the propsition. Assume that there are only finitely many homotopy types of such X_n's. We fix one such $[X_n]$ and we consider all the spaces X above, the nth Postnikov stage of which is homotopy equivalent to X_n. We will show that their $(n+1)$st stages X_{n+1} fall into finitely many homotopy types.

Let $\mathcal{M}$ be a minimal model of $H_{\mathbb{Q}} = H \otimes \mathbb{Q}$. Let $\mathcal{M}[n] \xrightarrow{\varphi_n} E_{X_n}$ and $\varphi \colon \mathcal{M} \longrightarrow E_X$ be cohomology isomorphisms such that the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\varphi} & E_X \\ \cup & & \uparrow \\ \mathcal{M}[n] & \xrightarrow{\varphi_n} & E_{X_n} \end{array}$$

commutes up to homotopy.

These maps induce lattices Z_1 and Z_2 such that

$$Z_1 \otimes \mathbb{Q} \cong \bigoplus_{i \leq n+1} \pi_i(\mathcal{M}[i])$$

and

$$Z_2 \otimes \mathbb{Q} \cong \bigoplus_{i \leq n} H^{i+2}(\mathcal{M}[i]).$$

By Sullivan [Su] the model $\mathcal{M}[n+1]$ with these lattices and a torsion bound determine $[X_{n+1}]$ up to finite ambiguity.

We will show that for fixed $[X_n]$ various choices of X, φ and φ_n together with the torsion of H induce finitely many equivalence classes of lattices. We recall that the lattices (Z_1, Z_2) and (Z_1', Z_2') are equivalent if there is an isomorphism of the model $\mathcal{M}[n+1]$ which induces an isomorphism between (Z_1, Z_2) and (Z_1', Z_2').

Since $[X_n]$ is fixed, the sublattices in $\pi_i(\mathcal{M}[i])$ for $i \leq n$ and $H^{i+2}(\mathcal{M}[i])$ for $i \leq n$ are also fixed up to equivalence.

The only lattice that can vary is the lattice in $\pi_{n+1}(\mathcal{M}[n+1])$.

We will show that there are only finitely many such lattices in $\pi_{n+1}(\mathcal{M}[n+1])$ up to equivalence. This will complete the inductive step since the torsion of H determines a torsion bound on the homotopy of X_{n+1} as well. To complete the proof we only need to recall that two finite complexes X and Y are homotopy equivalent if X_n and Y_n are homotopy equivalent for n sufficiently large. We consider the long exact sequence

$$\begin{array}{ccccccccc} \cdots \to H^{n+1}(\mathcal{M}[n]) & \to & H^{n+1}(\mathcal{M}) & \xrightarrow{i} & H^{n+1}(\mathcal{M}, \mathcal{M}[n]) & \xrightarrow{p} & H^{n+2}(\mathcal{M}[n]) & \to & \cdots \\ & & \cong \downarrow \varphi^* & & \cong \downarrow & & \varphi_n^* \downarrow \cong & & \\ \cdots \to & & H^{n+1}(X; \mathbb{Q}) & \to & H^{n+2}(X_n, X; \mathbb{Q}) & \to & H^{n+2}(X_n; \mathbb{Q}) & \to & \cdots. \\ & & \cup & & \cup & & \cup & & \\ & & L' & & L & & L'' & & \end{array}$$

Let L', L and L'' be the lattices corresponding to the integral cohomology. We consider L', L, and L'' as lattices of $H^{n+1}(\mathcal{M})$, $H^{n+1}(\mathcal{M}, \mathcal{M}[n])$ and $H^{n+2}(\mathcal{M}[n])$ respectively. Recall that

$$H^{n+1}(\mathcal{M}, \mathcal{M}[n]) \cong \mathcal{M}[n+1]/\, \mathcal{M}[n] = \pi_{n+1}(\mathcal{M}[n+1]),$$

where the first isomorphism is canonical and the second is the definition.

We observe that L can be written as $L = L_1 \oplus L_2$, where L_1 is determined up to finite ambiguity by $i(L) \subset L_1$ and the torsion of $H^{n+2}(X_n, \mathbb{Z})$ and $L_2 \cong p(L)$ is determined up to finite ambiguity by the torsion of H^{n+2} and a splitting $\sigma \colon p(L) \longrightarrow L$.

If another splitting $\sigma' : p(L) \longrightarrow L_2' \subset L$ is chosen one can construct an isomorphism ρ of $\mathcal{M}[n+1]$ with the following properties:

$$\begin{aligned} \rho|_{\mathcal{M}[n]} &= \mathrm{id}_{\mathcal{M}[n]} \\ \rho|_{L_1} &= \mathrm{id}_{L_1} \\ \rho|_{L_2} &= \sigma' \cdot p|_{L_2} \end{aligned}$$

Here we used the fact that

$$\mathcal{M}[n+1] = \mathcal{M}[n]\big(H^{n+1}(\mathcal{M}, \mathcal{M}[n])\big),$$

where the differential is the composition

$$H^{n+1}(\mathcal{M}, \mathcal{M}[n]) \xrightarrow{p} H^{n+2}(\mathcal{M}[n]) \xrightarrow{r} Z^{n+2}(\mathcal{M}[n])$$

and r is a splitting of cohomology classes into cocycles. The isomorphism ρ induces an isomorphism between $L_1 \oplus L_2$ and $L_1 \oplus L_2'$.

The only statement that remains to be shown is that another choice X', φ' and φ_n' with $[X_n]$ fixed induces lattices M' and M'' equivalent to L' and L'' respectively.

Let $f : X_n \longrightarrow X_n'$ be a homotopy equivalence and let $\bar{f} : \mathcal{M}[n] \longrightarrow \mathcal{M}[n]$ be an isomorphism induced by f. The map $\tilde{f}$ induces an isomorphism of the lattices M'' and L'' (and of the lattices in $H^{i+2}(\mathcal{M}[n])$ for $i \leq n$ and $\pi_i(\mathcal{M})$ for $i \leq n$).

Consider the diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "a"] \arrow[dd, dashed, "g"] & H^*(\mathcal{M}) \arrow[r, "\varphi^*"] & H^*(X; \mathbb{Q}) \arrow[dr, "a"] & \\
 & & & H_{\mathbb{Q}} . \\
\mathcal{M} \arrow[r, "a"] & H^*(\mathcal{M}) \arrow[r, "\varphi'^*"] & H^*(X'; \mathbb{Q}) \arrow[ur, "a'"'] &
\end{tikzcd}$$

Here a and a' are isomorphisms induced by integral isomorphisms $H^*(X; \mathbb{Z}) \cong H$ and $H^*(X'; \mathbb{Z}) \cong H$ and the map a can be chosen such that $a^* = \mathrm{id}$ on cohomology. The map g is an isomorphism which completes the diagram of solid arrows up to homotopy. Moreover g can be constructed as an extension of $\tilde{f} : \mathcal{M}[n] \longrightarrow \mathcal{M}[n]$. Therefore g is the required equivalence between the lattices M' and L' and betwen M'' and L''. ∎

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