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PRODUCT OF TWO COMMUTATORS AS A SQUARE IN A FREE GROUP

BY

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ABSTRACT. We show that, if $[s,t][u,v] = x^2$ in a free group, x need not be a commutator. We arrive at our example by use of a result of D. Piollet which characterizes solutions of such equations using an algebraic interpretation of the mapping class group of the corresponding surface.

1. **Introduction.** In his survey on equations in groups [2], R. C. Lyndon records the following question of C. C. Edmunds: If an element of a free group is a product of two commutators, and is also a square, must it then be the square of a commutator? In this paper we produce an example that shows that the answer to this question is negative.

We are led to our example by a result of D. Piollet [3] which gives the solutions to equations $W = 1$ in a free group with $W = x_1^2 \dots x_g^2$ or $W = [x_1, y_1] \dots [x_g, y_g]$ in terms of the outer automorphisms of

$$G_N = \langle x_1, \dots, x_g; x_1^2 \dots x_g^2 \rangle$$

or

$$G_O = \langle x_1, y_1, \dots, x_g, y_g; [x_1, y_1] \dots [x_g, y_g] \rangle$$

and by an algebraic interpretation of the generators for the mapping class groups given topologically by J. Birman and D. Chillingworth [1] as outer automorphisms of G_O or G_N. These generators for $Out(G)$, in combination with Piollet's results, may be useful in discovering other facts about solutions to quadratic equations in free groups. While we focus on the case of G_N for $g = 5$ in this paper, the appendix lists generators for the general case.

The word $[s,t][u,v]x^{-2}$ is an automorphic image of $s^2t^2u^2v^2x^2$ in the free group $F = \langle s,t,u,v,x \rangle$, so that the group $G = \langle s,t,u,v,x;\ [s,t][u,v]x^{-2} \rangle$ is isomorphic to G_N for $g = 5$. We produce generators for the outer automorphism group of G in Section 2. In Section 3, we state Piollet's characterization of solutions to quadratic equations in a free group, and construct a solution to the equation $[s,t][u,v] = x^2$ in which x is not a commutator.

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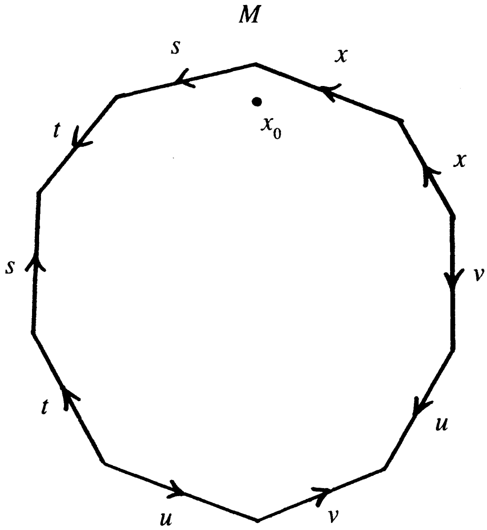

FIGURE 1

2. Let $F = \langle s, t, u, v, x\rangle$ be the free group on five generators and W be the word $st\, s^{-1}t^{-1}uvu^{-1}v^{-1}x^2$ in F. Let N be the normal subgroup of F generated by W and let $G = F/N$.

Let M be the non-orientable surface defined by the word W as in Figure 1, where the edges are identified using the word. Then $G \cong \pi_1(M, x_0)$.

It is well known that each element of $Out(G)$ is represented by an element of the homeotopy group of $M, H(M)$, (or the mapping class group of M) through a canonical homomorphism ψ defined as follows. For any $[f] \in H(M)$ and $\alpha \in \pi_1(M, x_0) \cong G$, $\psi([f])(\alpha) = [\beta_f^{-1} * f\alpha * \beta_f]$, where β_f is an arbitrary path from x_0 to $f(x_0)$.

The group $H(M)$ has been studied by several authors. We use the description of a set of generators of $H(M)$ obtained in [1] to find a set of generators of $Out(G)$. Let T be a torus with 4 holes. There exists an involution J on T (see [1] for the details) such that T/J is homeomorphic to M as in Figure 2, where $P : T \longrightarrow M$ is the projection map induced by the involution. In the figure, M is represented as a torus with 2 holes with the interior of a disk removed. To construct M, one must identify the boundary circle by the antipodal map.

Let a_i, b_i, c_i and e be the homeomorphisms of T induced by a right hand full twist along the oriented circles α_i, β_i, γ_i and ϵ in Figure 2, respectively. A set of generators of $H(M)$ is represented by the homeomorphisms $\bar{a}_1 = Pa_1a_4^{-1}P^{-1}$, $\bar{a}_2 = Pa_2a_3^{-1}P^{-1}$, $\bar{b}_1 = Pb_1b_4^{-1}P^{-1}$, $\bar{b}_2 = Pb_2b_3^{-1}P^{-1}$, $\bar{c}_1 = Pc_1c_3^{-1}P^{-1}$, and $y_4 = P(c_2a_2a_3)^2e^{-1}P^{-1}$, where the maps are composed from the right. By abusing notation, we will regard $\bar{a}_1$, $\bar{a}_2$, $\bar{b}_1$, $\bar{b}_2$, $\bar{c}_1$ and y_4 as generators of $H(M)$.

The loops, s, t, u, v and x on M in Figure 1, representing generators of $\pi_1(M, x_0) \cong$

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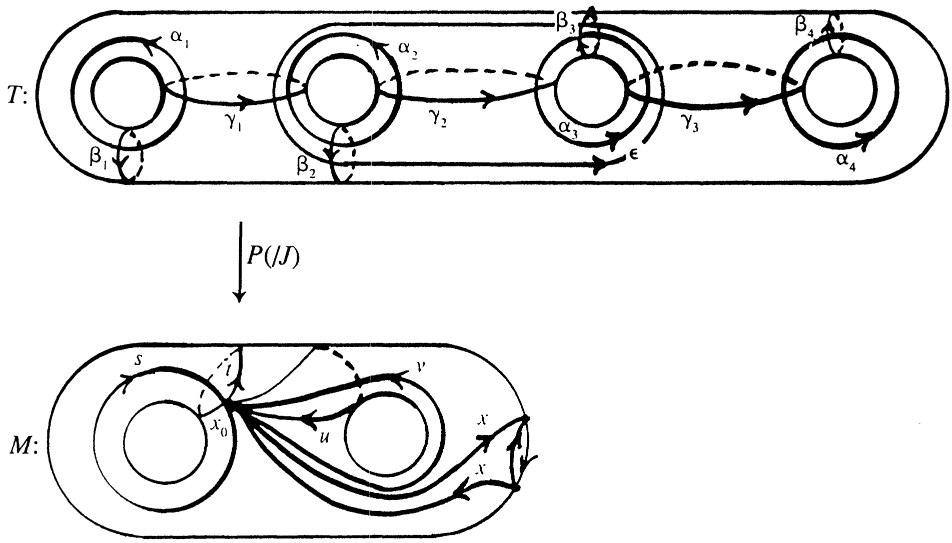

FIGURE 2

G, appear in M as in Figure 2. We may assume that the base point x_0 does not intersect the image under P of any circles drawn in T. Therefore, we may assume that each generator of $H(M)$ fixes x_0. Thus each generator induces an automorphism of $\pi_1(M, x_0)$; we denote the automorphisms induced by $\bar{a}_1$, $\bar{a}_2$, $\bar{b}_1$, $\bar{b}_2$, $\bar{c}_1$, y_4 by σ_1, σ_2, σ_3, σ_4, σ_5 and σ_6, respectively. Then σ_i, $1 \leqq i \leqq 6$, are generators of $Out(G)$.

We give a description of σ_i, $1 \leqq i \leqq 6$, in the following list. Each homomorphism is described on the generators which are not fixed under the homomorphism.

(2.1) $$\sigma_1(t) = ts$$

(2.2) $$\sigma_2(u) = uv^{-1}$$

(2.3) $$\sigma_3(s) = st^{-1}$$

(2.4) $$\sigma_4(v) = x^{-2}vu$$
$$\sigma_4(x) = x^{-2}vuv^{-1}xvu^{-1}v^{-1}x^2$$

(2.5) $$\sigma_5(s) = t^{-1}us$$
$$\sigma_5(v) = t^{-1}uv$$
$$\sigma_5(x) = t^{-1}uxu^{-1}t$$

(2.6) $$\sigma_6(u) = v^{-1}xvu^{-1}v^{-1}xv$$
$$\sigma_6(v) = v^{-1}x^{-1}vtst^{-1}s^{-1}x^{-1}v$$
$$\sigma_6(x) = v^{-1}x^{-1}vtst^{-1}s^{-1}$$

We demonstrate in two cases how we arrive at the above description.

(2.4) $$\sigma_4(x) = x^{-2}vuv^{-1}xvu^{-1}v^{-1}x^2$$

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We need to read $\bar{b}_2(x) = Pb_2b_3^{-1}P^{-1}(x)$ as an element of $\pi_1(M, x_0)$. The following diagram gives $\bar{b}_2(x)$ step by step.

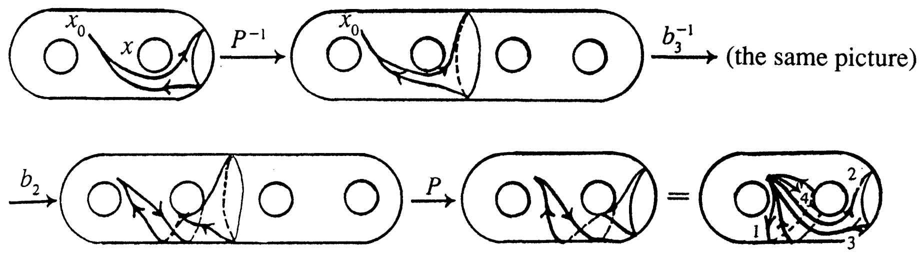

FIGURE 3

From the last picture in the above diagram, we see easily that $\bar{b}_2(x) = (x^{-2}vuv^{-1})x(vu^{-1}v^{-1}x^2)$.

(2.6)
$$\sigma_6(u) = v^{-1}xvu^{-1}v^{-1}xv.$$
$$y_4(u) = P(c_2a_2a_3)^2e^{-1}P^{-1}(u)$$

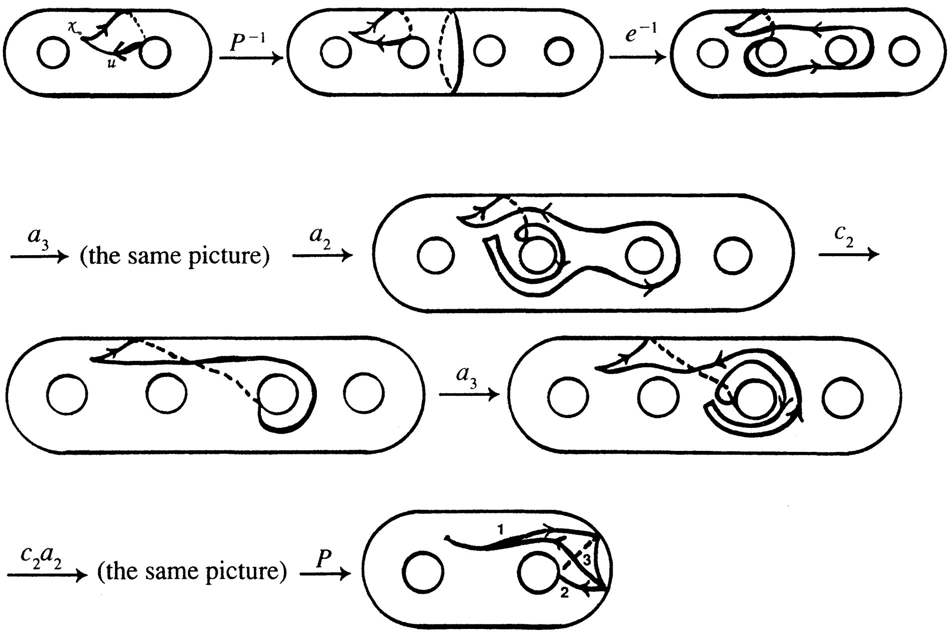

FIGURE 4

Finally, $y_4(u) = (v^{-1}xv)(u^{-1}v^{-1}xv)$.

3. Recall that $G = F/N$ where $F = \langle s, t, u, v, x\rangle$ is a free group and N is the normal subgroup of F generated by $W = sts^{-1}t^{-1}uvu^{-1}v^{-1}x^2$. Let $X = \{s, t, u, v, x\}$, $X^{-1} = \{s^{-1}, t^{-1}, u^{-1}, v^{-1}, x^{-1}\}$ and $X^{\pm} = X \cup X^{-}$. We define a literal solution as one which maps X into $X^{\pm} \cup \{1\}$.

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Piollet provides a characterization of the solutions of $W = 1$ in [3]. The set of all solutions to the equation $W = 1$ in F is comprised of all the homomorphisms of the form hgf where h is an arbitrary homomorphism of F in G, g is a literal solution, and f is an outer automorphism of G. Since there are only finitely many literal solutions, a description of the group of outer automorphisms of G, $Out(G) \cong Aut(G)/Inn(G)$, completes the information we need to find the solutions to the equation.

Our example has been found through the use of computer. Let $f = \sigma_1\sigma_5\sigma_6\sigma_5\sigma_6$ and let g fix $u^{\pm 1}$ and $s^{\pm 1}$ and send the remaining variables to 1. We find that

$$gf(s) = s^{-1}u^{-1}s^{-1}us, \tag{3.1}$$

$$gf(t) = s, \tag{3.2}$$

$$gf(u) = u^{-1}suus^{-1}, \tag{3.3}$$

$$gf(v) = su^{-1}s^{-1}s^{-1}us^{-1}, \tag{3.4}$$

$$gf(x) = su^{-1}s^{-1}s^{-1}us^{-1}u^{-1}sus. \tag{3.5}$$

M. J. Wicks [4] proved that an element in a free group is a commutator if and only if cyclically it can be represented as either $a^{-1}b^{-1}ab$ or $a^{-1}b^{-1}c^{-1}abc$. This result easily demonstrates that the image of x, (3.5), is not a commutator.

Appendix. Let $N_{g+1}(g \geqq 2)$ be the non-orientable closed surface obtained by taking a connected sum of $(g + 1)$ copies of projective 2-space. A set of generators of the mapping class group $H(N_{g+1})$ of N_{g+1} is given in [1]. We describe how these generators act on the fundamental group of N_{g+1}.

Using the notation of [1], we describe the generators. Let O_g be the closed orientable surface of genus g. Then there exists a free involution J on O_g such that the projection map $P : O_g \longrightarrow O_g/J = N_{g+1}$ is a double covering projection (see Figure 5).

Define homeomorphisms a_i, b_i, c_j, d_j, e, f on O_g as the right hand full twist of O_g along the oriented circles α_i, β_i, γ_j, δ_j, ϵ and ϕ given in Figure 5, respectively.

Let

$$\bar{a}_i = Pa_ia_{g+1-i}^{-1}P^{-1},\ \bar{b}_i = Pb_ib_{g+1-i}^{-1}P^{-1},\ \bar{c}_j = Pc_jc_{g-j}^{-1}P^{-1},$$
$$\bar{b}_{r+1} = Pd_1d_2^{-1}P,\ y_{2r+1} = P(a_{r+1}c_rc_{r+1})^2f^{-1}P^{-1}$$

and $y_{2r} = P(c_ra_ra_{r+1})^2e^{-1}P^{-1}$. These are homeomorphisms of N_{g+1} and the homeotopy classes will be denoted by the same notation. According to [1] $H(N_{g+1}) = \langle \bar{a}_i, \bar{b}_i, \bar{c}_j, \bar{b}_{r+1}\, y_{2r+1}\colon 1 \leqq i, j \leqq r\rangle$ if $g = 2r+1$ and $H(N_{g+1}) = \langle \bar{a}_i, \bar{b}_i, \bar{c}_j\, y_{2r}\colon 1 \leqq i \leqq r, 1 \leqq j \leqq r-1\rangle$ if $g = 2r$. (J. Birman has informed us that the generator $\bar{b}_{r+1}$ was overlooked in [1].)

Choose the generators of the fundamental group $\pi_1(N_{g+1}, \acute{x})$ as the elements represented by the oriented loops in N_{g+1} (Figure 5). Then the group is isomorphic to $\langle s_i, t_i, 1 \leqq i \leqq r+1;\ \prod_{1\leqq i\leqq r}[s_i, t_i]s_{r+1}t_{r+1}s_{r+1}^{-1}t_{r+1} = 1\rangle$ if $g = 2r+1$ and $\langle s_i, t_i, v, 1 \leqq i \leqq r;\ \prod_{1\leqq i\leqq r}[s_i, t_i]v^2 = 1\rangle$ if $g = 2r$, where $[s_i, t_i] = s_it_is_i^{-1}t_i^{-1}$.

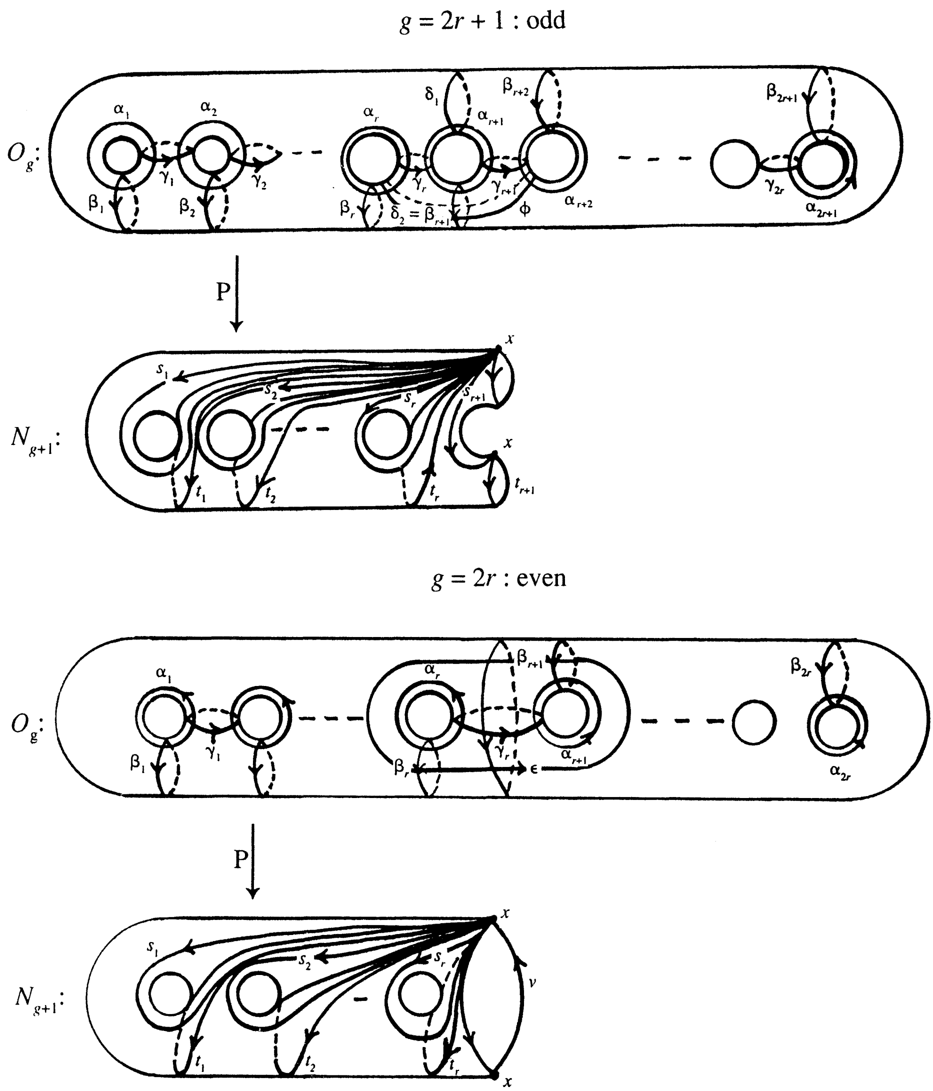

FIGURE 5

The generators of $H(N_{g+1})$ induce the following homomorphisms on the fundamental group, where the homomorphisms are described only on the elements which are not fixed under the homomorphisms. The indices vary over the positive integers as long as the identities make sense.

(1) $$\bar{a}_{i*}(t_i) = t_i s_i^{-1}$$

(2) $$\bar{b}_{i*}(s_i) = s_i t_i$$

$$
\begin{aligned}
(3) \qquad \bar{c}_{j^*}(s_j) &= s_j s_{j+1} t_{j+1}^{-1} s_{j+1}^{-1} t_j \\
\bar{c}_{j^*}(s_{j+1}) &= t_j^{-1} s_{j+1} t_{j+1} \\
\bar{c}_{j^*}(t_j) &= t_j^{-1} s_{j+1} t_{j+1} s_{j+1}^{-1} t_j s_{j+1} t_{j+1}^{-1} s_{j+1}^{-1} t_j \\
(4) \qquad y_{(2r+1)^*}(s_r) &= s_r s_{r+1}^2 t_{r+1}^{-1} s_{r+1} t_{r+1}^{-1} s_{r+1}^{-1} t_r \\
y_{(2r+1)^*}(s_{s+1}) &= t_r^{-1} s_{r+1} t_{r+1} s_{r+1}^{-1} t_{r+1} s_{r+1}^{-2} t_r s_{r+1} \\
y_{(2r+1)^*}(t_r) &= t_r^{-1} s_{r+1} t_{r+1} s_{r+1}^{-1} t_{r+1} s_{r+1}^{-2} t_r s_{r+1}^2 t_{r+1}^{-1} s_{r+1} t_{r+1}^{-1} s_{r+1}^{-1} t_r \\
y_{(2r+1)^*}(t_{r+1}) &= t_r^{-1} s_{r+1} t_{r+1} s_{r+1}^{-2} t_r s_{r+1} \\
(5) \qquad y_{(2r)^*}(s_r) &= v^{-1} t_r s_r t_r^{-1} v \\
y_{(2r)^*}(t_r) &= v^{-1} t_r s_r^{-1} t_r^{-1} v s_r t_r s_r^{-1} t_r^{-1} v s_r t_r^{-1} v \\
y_{(2r)^*}(v) &= v^{-1} t_r s_r^{-1} t_r^{-1} v s_r v
\end{aligned}
$$

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