POLYNOMIAL GROTHENDIECK PROPERTIES

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Abstract. A Banach space E has the Grothendieck property if every (linear bounded) operator from E into c_0 is weakly compact. It is proved that, for an integer $k > 1$, every k-homogeneous polynomial from E into c_0 is weakly compact if and only if the space $\mathcal{P}(^kE)$ of scalar valued polynomials on E is reflexive. This is equivalent to the symmetric k-fold projective tensor product of E(i.e., the predual of $\mathcal{P}(^kE)$) having the Grothendieck property. The Grothendieck property of the projective tensor product $E \hat{\otimes} F$ is also characterized. Moreover, the Grothendieck property of E is described in terms of sequences of polynomials. Finally, it is shown that if every operator from E into c_0 is completely continuous, then so is every polynomial between these spaces.

1. Introduction. Throughout, E, F will be Banach spaces, and E^* the dual of E. We denote by $\mathcal{L}(E, F)$ the space of all (linear bounded) operators from E to F, and by $\mathcal{Co}(E, F)(\mathcal{WCo}(E, F))$ the subspace of all (weakly) compact operators. We say that $T \in \mathcal{L}(E, F)$ is *completely continuous* if it takes weakly convergent sequences into norm convergent sequences, and we write $T \in \mathcal{CC}(E, F)$.

For an integer k, we shall consider the following classes of polynomials:

(a) $\mathcal{P}(^kE, F)$ is the space of all k-homogeneous (continuous) polynomials from E to F;

(b) $\mathcal{P}_{cc}(^kE, F)$, the subspace of *completely continuous polynomials*, i.e., the polynomials taking weakly convergent sequences into norm convergent ones, equivalently, taking weak Cauchy sequences into convergent ones [**3**, Theorem 2.3];

(c) $\mathcal{P}_{wco}(^kE, F)$, the subspace of weakly compact polynomials;

(d) $\mathcal{P}_{wb}(^kE, F)$, the polynomials whose restrictions to bounded subsets of E are weakly continuous; these are compact polynomials. It is well known that $\mathcal{P}_{wb}(^kE, F) \subseteq \mathcal{P}_{cc}(^kE, F)$, and the equality occurs if and only if E contains no copy of ℓ_1 (see e.g. [**14**]).

The space of k-linear (continuous) mappings from E^k to F is denoted by $\mathcal{L}(^kE, F)$. To each $P \in \mathcal{P}(^kE, F)$ we can associate a unique symmetric $A \in \mathcal{L}(^kE, F)$ such that $P(x) = A(x, \ldots, x)$ for all $x \in E$. Whenever F is omitted, it is understood to be the scalar field. For the general theory of polynomials between Banach spaces, we refer to [**19**].

The projective tensor product of E and F is referred to as $E \hat{\otimes} F$. The closed linear span of the set $\{x \otimes \overset{(k)}{\ldots} \otimes x \colon x \in E\}$ in $\hat{\otimes}^k E := E \hat{\otimes} \overset{(k)}{\ldots} \hat{\otimes} E$ is denoted by $\hat{\Delta}^k E$. Its dual is isomorphic to $\mathcal{P}(^kE)$. The spaces $\mathcal{P}(^kE, F)$ and $\mathcal{L}(\hat{\Delta}^k E, F)$ are linearly isomorphic, and the image of $\mathcal{P}_{wco}(^kE, F)$ under this isomorphism is $\mathcal{WCo}(\hat{\Delta}^k E, F)$ [**22**].

We say that E has the *Grothendieck property*, and write $E \in \mathcal{G}r$, if every sequence in E^* converging to zero in the weak-star (w^*) topology, is also weakly null. Equivalently, if every operator $E \to c_0$ is weakly compact.

In this paper, we investigate conditions on E so that every k-homogeneous

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polynomial $E \to c_0$ be weakly compact, equivalently $\hat{\Delta}^k E \in \mathcal{G}r$, proving that this is the case if and only if $\mathcal{P}(^k E)$, and hence $\hat{\Delta}^k E$, is reflexive. These "polynomially reflexive" Banach spaces have been investigated by various authors [**1, 7, 9, 13**].

We also show that the situation is different if we consider the Grothendieck property of $E \hat{\otimes} F$, proving that for $E \in \mathcal{G}r$ and F^* reflexive with the bounded compact approximation property, $E \hat{\otimes} F \in \mathcal{G}r$ if and only if $\mathcal{L}(E, F^*) = \mathcal{C}o(E, F^*)$. In particular, $\ell_\infty \hat{\otimes} \ell_p \in \mathcal{G}r$ for $2 < p < \infty$, and $\ell_\infty \hat{\otimes} T^* = \mathcal{G}r$, where T^* is the original Tsirelson space.

Every $P \in \mathcal{P}(^k E)$ has a standard extension $\tilde{P} \in \mathcal{P}(^k E^{**})$ (see [**6**]). Thus, given $P \in \mathcal{P}(^k E, c_0)$, with $Px = (P_n x)_n$, we can define $\tilde{P} \in \mathcal{P}(^k E^{**}, \ell_\infty)$ by $\tilde{P}z := (\tilde{P}_n z)_n$. It is shown that $E \in \mathcal{G}r$ if and only if, for every sequence $(P_n) \subset \mathcal{P}(^k E)$ such that $P_n x \to 0$ for all $x \in E$, we have $\tilde{P}_n z \to 0$ for every $z \in E^{**}$. Then $E \in \mathcal{G}r$ if and only if for every $P \in \mathcal{P}(^k E, c_0)$, we have $\tilde{P}(E^{**}) \subseteq c_0$.

Several authors [**20, 13, 11**] have studied conditions on E, F so that $\mathcal{P}(^k E, F) = \mathcal{P}_{cc}(^k E, F)$. Here we investigate the equality $\mathcal{P}(^k E, c_0) = \mathcal{P}_{cc}(^k E, c_0)$, proving that it is equivalent to $\mathcal{L}(E, c_0) = \mathcal{C}\mathcal{C}(E, c_0)$. Grothendieck spaces with the Dunford-Pettis property, and Schur spaces satisfy this property. In [**18**], examples are given of Grothendieck spaces with the Dunford-Pettis property, and semigroups of operators on these spaces are studied.

2. We first characterize the spaces E such that $\mathcal{P}(^k E, c_0) = \mathcal{P}_{wco}(^k E, c_0)$. Some previous results are needed.

Proposition 1. *Let E be a nonreflexive space with the Grothendieck property. Then E contains a copy of ℓ_1.*

Proof. Since $E \in \mathcal{G}r$, E has no quotient isomorphic to c_0. Assume E contains no copy of ℓ_1. Then, E^* contains no copy of ℓ_1 [**12**, Corollary 2.3]. Given a bounded sequence $(\phi_n) \subset E^*$, we can find a weak Cauchy subsequence (ϕ_{n_k}). The sequence (ϕ_{n_k}) is w^*-convergent, hence weakly convergent, and we conclude that E is reflexive. □

Proposition 2. *If $\mathcal{P}(^k E, c_0) = \mathcal{P}_{wco}(^k E, c_0)$ for some $k > 1$, then E is reflexive.*

Proof. Suppose there is a nonweakly compact $T \in \mathcal{L}(E, c_0)$. Then we can find a bounded sequence $(x_n) \subset E$ such that (Tx_n) converges in the topology $\sigma(\ell_\infty, \ell_1)$ to some $(a_n) \in \ell_\infty$ with $\limsup a_n = a \neq 0$. Define $P(x) := (Tx)^k$, i.e., take the kth power coordinatewise. We have $P \in \mathcal{P}_{wco}(^k E, c_0)$. However, (Px_n) converges in the topology $\sigma(\ell_\infty, \ell_1)$ to the sequence $(a_n^k)_n \in \ell_\infty$, with $\limsup_n a_n^k = a^k \neq 0$, a contradiction. Hence, $E \in \mathcal{G}r$.

Suppose E is nonreflexive. By Proposition 1, E contains a copy of ℓ_1. Then, there is a quotient map $q: E \to \ell_2$ (see e.g. [**9**, Lemma 12]). Let $P: \ell_2 \to \ell_1$ be the polynomial given by $P((x_n)_n) = (x_n^k)_n$, and let $q': \ell_1 \to c_0$ be a quotient map. Then the product $q'Pq \in \mathcal{P}(^k E, c_0)$ is not weakly compact, a contradiction. □

Lemma 3. *If M is a complemented subspace of E, then $\hat{\Delta}^k M$ is a complemented subspace of $\hat{\Delta}^k E$.*

Proof. If $S \in \mathcal{L}(E, E)$ is a projection with $S(E) = M$, consider the linear mapping defined by

$$x \otimes \overset{(k)}{\dots} \otimes x \mapsto Sx \otimes \overset{(k)}{\dots} \otimes Sx \qquad (x \in E).$$

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Easily, this mapping extends to an operator in $\mathscr{L}(\hat{\Delta}^k E, \hat{\Delta}^k E)$, which is the required projection. □

To simplify notation, we write $x^{(k)} := x \otimes \overset{(k)}{\ldots} \otimes x$, for $x \in E$. Let $P_k : \otimes^k E \to \otimes^k E$ be the projection defined by

$$P_k(x_1 \otimes \ldots \otimes x_k) = \frac{1}{k! 2^k} \sum_{\epsilon_j = \pm 1} \epsilon_1 \ldots \epsilon_k (\epsilon_1 x_1 + \ldots + \epsilon_k x_k)^{(k)}.$$

This mapping extends to an operator on $\hat{\otimes}^k E$, which is a projection of $\hat{\otimes}^k E$ onto $\hat{\Delta}^k E$. The following Lemma is contained in [8]. We include the proof for completeness.

Lemma 4. *Given $u \in \hat{\Delta}^k E$, there exists a sequence $(x_i) \subset E$ such that $\sum_{i=1}^{\infty} \|x_i\|^k < \infty$ and $u = \sum_{i=1}^{\infty} \epsilon_i x_i^{(k)}$, with $\epsilon_i \in \{\pm 1\}$.*

Proof. By the definition of the projective norm, for any $\delta > 0$, we can find sequences $(x_n^1), \ldots, (x_n^k) \subset E$, such that

$$\sum_{n=1}^{\infty} \|x_n^1\| \cdot \ldots \cdot \|x_n^k\| < \|u\| + \delta, \quad \text{and} \quad u = \sum_{n=1}^{\infty} x_n^1 \otimes \ldots \otimes x_n^k.$$

Then,

$$u = P_k u = \sum_{n=1}^{\infty} P_k(x_n^1 \otimes \ldots \otimes x_n^k).$$

We can assume that, for each n, $\|x_n^1\| = \ldots = \|x_n^k\|$. Since

$$P_k(x_n^1 \otimes \ldots \otimes x_n^k) = \frac{1}{k! 2^k} \sum_{\epsilon_j = \pm 1} \epsilon_1 \ldots \epsilon_k (\epsilon_1 x_n^1 + \ldots + \epsilon_k x_n^k)^{(k)},$$

denoting by (x_i) the sequence

$$\left\{ \frac{1}{2\sqrt[k]{k!}} (\epsilon_1 x_n^1 + \ldots + \epsilon_k x_n^k) : \epsilon_1, \ldots, \epsilon_k = \pm 1; n = 1, 2, \ldots \right\},$$

we obtain

$$\sum_{i=1}^{\infty} \|x_i\|^k \leq \frac{k^k}{k!} (\|u\| + \delta), \quad \text{and} \quad u = \sum_{i=1}^{\infty} \epsilon_i x_i^{(k)},$$

completing the proof. □

Before stating the main result of this part, recall that for $P \in \mathscr{P}({}^k E, F)$, its *adjoint* is the operator $P^* : F^* \to \mathscr{P}({}^k E)$ given by $P^*(\psi) = \psi \circ P$ for every $\psi \in F^*$. Then $P \in \mathscr{P}_{wco}({}^k E, F)$ if and only if P^* is weakly compact [23, Proposition 2.1].

Theorem 5. *Given $k > 1$, we have that $\mathscr{P}({}^k E, c_0) = \mathscr{P}_{wco}({}^k E, c_0)$ if and only if the space $\mathscr{P}({}^k E)$ is reflexive.*

Proof. For the "only if" part, if E is separable, then $\hat{\Delta}^k E$ is a Grothendieck

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separable space, hence reflexive. In the general case, suppose $\hat{\Delta}^k E$ is not reflexive. By Lemma 4, we can find

$$w_n = \sum_{i=1}^{\infty} \epsilon_n^i x_n^i \otimes \overset{(k)}{\dots} \otimes x_n^i, \qquad \|w_n\| \leq 1$$

so that $\{w_n\}$ is not relatively weakly compact in $\hat{\Delta}^k E$. Let N be the closed linear span of $\{x_n^i\}_{i,n}$ in E. Since E is reflexive (Proposition 2), there is a separable subspace M complemented in E with $N \subseteq M$. By Lemma 3, $\hat{\Delta}^k M \in \mathcal{G}r$, and is separable, therefore reflexive. However, $(w_n) \subset \hat{\Delta}^k M$, a contradiction, and we conclude that $\hat{\Delta}^k E$ and $\mathcal{P}(^k E)$ are reflexive.

The "if" part is clear by the previous comment. □

It is well known that the space $\mathcal{P}(^k \ell_p)$ is reflexive if and only if $k < p < \infty$. If $E = T^*$, then $\mathcal{P}(^k E)$ is reflexive for all k [**1**].

In the last Theorem, c_0 can be replaced by any superspace F. However, for F containing no copy of c_0, the result is not true since, for instance, every polynomial from $E = c_0$ into $F \not\supset c_0$ is weakly continuous on bounded subsets (see e.g. [**11**]).

3. In this part, we show that the situation is different for general projective tensor products. We refine a result of [**17**], proving that for $E \in \mathcal{G}r$ and F^* reflexive with the bounded compact approximation property, $E \hat{\otimes} F \in \mathcal{G}r$ if and only if $\mathcal{L}(E, F^*) = \mathcal{C}o(E, F^*)$. As a consequence, $\ell_\infty \hat{\otimes} \ell_p$ for $2 < p < \infty$ and $\ell_\infty \hat{\otimes} T^*$ have the Grothendieck property.

PROPOSITION 6. *Suppose* $E \hat{\otimes} F \in \mathcal{G}r$. *Then* $E, F \in \mathcal{G}r$ *and at least one of them is reflexive.*

Proof. Since E and F are complemented in $E \hat{\otimes} F$, the first assertion is clear. Suppose E and F are nonreflexive. Then each of them contains a copy of ℓ_1 (Proposition 1). Hence, there are quotient maps (see e.g. [**9**, Lemma 12])

$$q_1 : E \to \ell_2 \quad \text{and} \quad q_2 : F \to \ell_2.$$

Consider the quotient maps

$$E \hat{\otimes} F \xrightarrow{q_1 \hat{\otimes} \mathrm{id}} \ell_2 \hat{\otimes} F \xrightarrow{\mathrm{id} \hat{\otimes} q_2} \ell_2 \hat{\otimes} \ell_2.$$

If is well known that $\ell_2 \hat{\otimes} \ell_2 \notin \mathcal{G}r$ (separable Grothendieck spaces are reflexive). Hence, $E \hat{\otimes} F \notin \mathcal{G}r$, a contradiction. □

REMARK 7. It follows from Proposition 6 that whenever $E_1 \hat{\otimes} \dots \hat{\otimes} E_k \in \mathcal{G}r$ and, for example, E_1 is not reflexive, $E_2 \hat{\otimes} \dots \hat{\otimes} E_k$ is reflexive. In particular, $E_2, \dots, E_k$ are reflexive.

Before stating the next result, recall that the dual of $E \hat{\otimes} F$ may be identified with $\mathcal{L}(E, F^*)$.

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PROPOSITION 8. *Assume* $E \in \mathscr{G}r$ *and* F *is reflexive. If* $\mathscr{L}(E,F^*) = \mathscr{C}o(E,F^*)$, *then* $E\hat{\otimes}F \in \mathscr{G}r$.

Proof. Let $(A_n) \subset \mathscr{L}(E,F^*)$ be a w^*-null sequence. Then for every $x \in E$ and $y \in F$,

$$\langle y, A_n(x)\rangle = \langle x \otimes y, A_n\rangle \to 0.$$

Applying Kalton's test for the weak convergence of sequences in spaces of compact operators (see Theorem 3 in [16]), we have that (A_n) is weakly null. ☐

We say that E has the *bounded compact approximation property* (BCAP) [4] if there exists $\lambda \geq 1$ so that for each compact subset $K \subset E$ and for each $\epsilon > 0$ there is $S \in \mathscr{C}o(E,E)$ such that

$$\sup\{\|Sx - x\| : x \in K\} \leq \epsilon, \qquad \|S - \mathrm{id}\| \leq \lambda.$$

Every space with the bounded approximation property has the BCAP. The converse is not true [24].

PROPOSITION 9. *Suppose* F^* *is reflexive and has the BCAP, and* $E\hat{\otimes}F \in \mathscr{G}r$. *Then we have* $\mathscr{L}(E,F^*) = \mathscr{C}o(E,F^*)$.

Proof. Suppose first that F^* is separable. Then there is a bounded sequence $(T_n) \subset \mathscr{C}o(F^*,F^*)$ such that $T_n\psi \to \psi$ for all $\psi \in F^*$. Assume $T \in \mathscr{L}(E,F^*)$ is not compact. For $x \in E$, $y \in F$ we have

$$\langle x \otimes y, T_nT\rangle = \langle y, T_n(Tx)\rangle \to \langle y, Tx\rangle = \langle x \otimes y, T\rangle.$$

Since (T_nT) is bounded and $\{x \otimes y : x \in E, y \in F\}$ generates a dense subset of $E\hat{\otimes}F$, we have that (T_nT) is w^*-convergent to T. Since $(T_nT) \subset \mathscr{C}o(E,F^*)$, (T_nT) is not weakly convergent to T, a contradiction.

For F^* nonseparable, suppose T as above. There is a bounded sequence $(x_n) \subset E$ such that (Tx_n) has no Cauchy subsequence. The closed linear span of $\{Tx_n\}$ is contained in a separable space M^* complemented in F^*. If $q : F^* \to M^*$ is the identity on M^*, then $qT \in \mathscr{L}(E,M^*)$ is noncompact. By the above, $E\hat{\otimes}M \notin \mathscr{G}r$, a contradiction since $E\hat{\otimes}M$ is a quotient of $E\hat{\otimes}F$. ☐

COROLLARY 10. *For* $1 \leq p \leq \infty$, *the space* $\ell_\infty\hat{\otimes}\ell_p$ *has the Grothendieck property if and only if* $2 < p < \infty$.

Proof. If $2 < p < \infty$, we have $\ell_p^* = \ell_q$ with $1 < q < 2$, and it is known that every operator $\ell_\infty \to \ell_q$ factors through ℓ_2 [21, Corollary 4.4] and is therefore compact. The converse is easy. ☐

REMARK 11. (a) We note that the space $\ell_\infty\hat{\otimes}\ell_p\hat{\otimes}\ell_p$ does not have the Grothendieck property, for $2 \leq p \leq 3$. Indeed, there is a noncompact operator $T : \ell_p \to (\ell_p\hat{\otimes}\ell_p)^*$, for instance, the operator T associated to the polynomial $Px := \sum_{i=1}^{\infty} x_i^3$, for $x = (x_i) \in \ell_p$, given by $(Tx)(y \otimes z) = \hat{P}(x,y,z)$, where $\hat{P}$ is the symmetric 3-linear form associated to P. Then

T is not completely continuous, so we can find a weakly null sequence $(x_n) \subset \ell_p$ such that $\{Tx_n\}$ is not relatively compact. Passing to a subsequence, we can assume that (x_n) is equivalent to a block basis and hence equivalent to the basis of ℓ_p. Let $q: \ell_\infty \to \ell_2$ be a quotient, and $j: \ell_2 \to \ell_p$ the operator taking the ℓ_2-basis into (x_n). Then $Tjq: \ell_\infty \to (\ell_p \hat{\otimes} \ell_p)^*$ is not compact, and it is enough to apply Proposition 9.

(b) It is proved in [**2**, Corollary 8] that the space $(\hat{\otimes}^k T^*) \hat{\otimes} \ell_p$ is reflexive, for $1 < p < \infty$.

Corollary 12. *The space $\ell_\infty \hat{\otimes} T^*$ has the Grothendieck property.*

The proof relies on the following Lemma.

Lemma 13. *Let T be the dual of T^*. Then $\mathscr{L}(\ell_\infty, T) = \mathscr{C}o(\ell_\infty, T)$.*

Proof. Assume $S \in \mathscr{L}(\ell_\infty, T)$ is not compact. Let $(y_k) \subset \ell_\infty$ be a bounded sequence such that (Sy_k) has no convergent subsequence. Choose a weakly convergent subsequence (Sy_{k_n}), and take $x_n := y_{k_{2n}} - y_{k_{2n-1}}$. Then (Sx_n) is weakly null, and we can assume that it is equivalent to a block basis in T.

Since $\{Sx_n\}$ spans a complemented subspace $[Sx_n]$ [**5**, Proposition II.6], there is an operator $V: T \to [Sx_n]$ which is the identity on $[Sx_n]$. For $1 < q < 2$, T has lower q-estimates [**5**, Proposition V.10], so there is an operator $U: [Sx_n] \to \ell_q$ given by $U(Sx_n) = e_n$, where (e_n) is the unit vector basis of ℓ_q. Then $UVS: \ell_\infty \to \ell_q$ is not compact, a contradiction [**21**, Corollary 4.4]. □

In [**17**], the following result was obtained (see *Zentralblatt Math.* **599** #46017 (1987)):

"Let E be a Banach space with the Grothendieck property, and F a reflexive space with the metric approximation property. For $E \hat{\otimes} F$ to have the Grothendieck property it is necessary and sufficient, that each operator $E \to F^*$ be compact".

Another related result is the following of [**15**]:

"If E and F are reflexive and both have the approximation property, then $\mathscr{L}(E, F)$ is reflexive if and only if $\mathscr{L}(E, F) = \mathscr{C}o(E, F)$".

4. Next we describe the Grothendieck property in terms of polynomials. Recall that each $P \in \mathscr{P}(^kE)$ has a Davie-Gamelin extension $\tilde{P} \in \mathscr{P}(^kE^{**})$ (see the Introduction). The authors are indebted to Professor Richard M. Aron, who suggested this study. Namely, he asked if, given $E \in \mathscr{G}r$ and a sequence $(P_n) \subset \mathscr{P}(^kE)$ with $P_n x \to 0$ for all $x \in E$, it is true that $\tilde{P}_n z \to 0$ for all $z \in E^{**}$. The following theorem shows that the answer is affirmative.

Theorem 14. *The following assertions are equivalent*:

(a) *E has the Grothendieck property*;

(b) *for every integer k, given a sequence $(P_n) \subset \mathscr{P}(^kE)$ with $P_n x \to 0$ for all $x \in E$, then $\tilde{P}_n z \to 0$ for all $z \in E^{**}$*;

(c) *the same statement as* (b) *is true for some k.*

Proof. (a)$\Rightarrow$(b). By induction on k. For $k = 1$, the result is nothing but the definition of the Grothendieck property. Suppose it holds for $k-1$, and let $(P_n) \subset \mathscr{P}(^kE)$ be a sequence such that $P_n x \to 0$ for all $x \in E$. Denote by $F_n \in \mathscr{L}(^kE)$ the associated symmetric k-linear form, and by $G_n \in \mathscr{L}(E \times E^{**} \times \overset{(k-1)}{\ldots} \times E^{**})$ an extension obtained by the Davie-Gamelin method.

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Thanks to the polarization formula [**19**, Theorem 1.10], we have that

$$F_n(x_1, x_2, \ldots, x_k) \to 0 \quad \text{for every} \quad x_1, \ldots, x_k \in E.$$

Fixing $x_1 \in E$, we define $Q_n \in \mathcal{P}(^{k-1}E)$ by $Q_n(x) = F_n(x_1, x, \ldots, x)$. Then $Q_n x \to 0$ for all $x \in E$.

By the induction hypothesis and polarization,

$$G_n(x_1, z_2, \ldots, z_k) \to 0, \quad \text{for} \quad z_2, \ldots, z_k \in E^{**}.$$

Then, for $z_2, \ldots, z_k \in E^{**}$ fixed, the sequence $(\phi_n) \subset E^*$, given by $\phi_n(x) = G_n(x, z_2, \ldots, z_k)$, is w^*-null, hence weakly null, and so

$$\tilde{F}_n(z_1, z_2, \ldots, z_k) \to 0 \quad \text{for every} \quad z_1, \ldots, z_k \in E^{**},$$

where $\tilde{F}_n \in \mathcal{L}(^k E^{**})$ is the Davie-Gamelin extension of F_n.

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (a). Given a w^*-null sequence $(\phi_n) \subset E^*$, apply (c) to $P_n x := (\phi_n(x))^k$. □

Given a polynomial $P \in \mathcal{P}(^k E, c_0)$, with $Px = (P_n x)_n$, we define $\tilde{P} \in \mathcal{P}(^k E^{**}, \ell_\infty)$ by $\tilde{P}z := (\tilde{P}_n z)_n$. Then we have the following corollary.

Corollary 15. *The space E has the Grothendieck property if and only if for every $P \in \mathcal{P}(^k E, c_0)$, we have that $\tilde{P}(E^{**}) \subseteq c_0$.*

For polynomials whose restrictions to bounded sets are weakly continuous, we can deduce a result on weak convergence.

Corollary 16. *The following assertions are equivalent*:

(a) *E has the Grothendieck property*;

(b) *for every integer k and every F, if for a sequence $(P_n) \subset P_{wb}(^k E, F)$ we have that $\langle P_n x, \psi \rangle \to 0$ for all $x \in E$ and $\psi \in F^*$, then (P_n) is weakly null*;

(c) *the same statement as* (b) *is true for some k and some $F \neq \{0\}$*;

(d) *for some k, if for a sequence $(P_n) \subset \mathcal{P}_{wb}(^k E)$ we have that $P_n x \to 0$ for all $x \in E$, then (P_n) is weakly null.*

Proof. (a) $\Rightarrow$ (b). It is proved in Theorem 4 of [**10**] that a sequence $(P_n) \subset \mathcal{P}_{wb}(^k E, F)$ is weakly null if and only if, for every $z \in E^{**}$ and $\psi \in F^*$, we have $\langle \tilde{P}_n z, \psi \rangle \to 0$. Therefore, it is enough to apply Theorem 14(b).

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (d). Take $0 \neq y \in F$ and define $Q_n x := (P_n x) y$.

(d) $\Rightarrow$ (a). Given a w^*-null sequence $(\phi_n) \subset E^*$, apply (d) to $P_n x := (\phi_n(x))^k$. □

5. Several authors [**20, 13, 11**] have studied conditions on E, F so that $\mathcal{P}(^k E, F) = \mathcal{P}_{cc}(^k E, F)$. Here we investigated the equality $\mathcal{P}(^k E, c_0) = \mathcal{P}_{cc}(^k E, c_0)$, proving that it is equivalent to $\mathcal{L}(E, c_0) = \mathcal{CC}(E, c_0)$. Therefore, the Grothendieck spaces with the Dunford-Pettis property, and the Schur spaces satisfy this property.

Theorem 17. *The following assertions are equivalent*:

(a) $\mathcal{L}(E, c_0) = \mathcal{CC}(E, c_0)$;

(b) $\mathcal{P}(^k E, c_0) = \mathcal{P}_{cc}(^k E, c_0)$ *for all integers k.*

(c) $\mathcal{P}(^k E, c_0) = \mathcal{P}_{cc}(^k E, c_0)$ *for some integer k.*

Proof. (a)$\Rightarrow$(b). By induction on k. For $k=1$ there is nothing to prove. Assume the result is true for $k-1$, and consider $P \in \mathcal{P}({}^kE, c_0)$ with associated k-linear mapping A. We only sketch the proof, since it follows the lines of that in [**11**, Theorem 6]. In fact, we can prove that every k-linear mapping from E^k into c_0 takes weak Cauchy sequences into convergent ones. Let $(x_n^1), \ldots, (x_n^k) \subset E$ be weak Cauchy sequences. Suppose first that one of them, say (x_n^1), is weakly null.

Define the operator $T: E \to c_0(c_0)$ by

$$y \mapsto (A(x_n^1, \ldots, x_n^{k-1}, y))_n.$$

Using the induction hypothesis, it is not difficult to see that T is well-defined. Since $c_0(c_0)$ is isomorphic to c_0, T is completely continuous. From this, we have

$$\lim_m \|A(x_m^1, \ldots, x_m^k)\| \leq \lim_m \sup_n \|A(x_n^1, \ldots, x_n^{k-1}, x_m^k)\| = 0.$$

In the general case, the proof follows that of Theorem 6 in [**11**].

(b)$\Rightarrow$(c) is obvious.

(c)$\Rightarrow$(a) is clear. $\square$

The condition $\mathcal{L}(E, c_0) = \mathcal{CC}(E, c_0)$ implies that E has the Dunford-Pettis property. However, there are spaces with the Dunford-Pettis property that admit non-completely continuous operators into c_0 (e.g. $E = c_0$, $E = L_1[0,1]$).

Added in Proof. While this paper was in press, G. Emmanuele pointed out that Propositions 6 and 8 and Corollary 10 are contained in his note About certain isomorphic properties of Banach spaces in projective tensor products, *Extracta Math.* **5** (1990), 23–25.

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