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Dedicated to the memory of Dieter Happel

Abstract

The aim of this paper is to introduce τ-tilting theory, which 'completes' (classical) tilting theory from the viewpoint of mutation. It is well known in tilting theory that an almost complete tilting module for any finite-dimensional algebra over a field k is a direct summand of exactly one or two tilting modules. An important property in cluster-tilting theory is that an almost complete cluster-tilting object in a 2-CY triangulated category is a direct summand of exactly two cluster-tilting objects. Reformulated for path algebras kQ, this says that an almost complete support tilting module has exactly two complements. We generalize (support) tilting modules to what we call (support) τ-tilting modules, and show that an almost complete support τ-tilting module has exactly two complements for any finite-dimensional algebra. For a finite-dimensional k-algebra Λ, we establish bijections between functorially finite torsion classes in $\mathsf{mod}\,\Lambda$, support τ-tilting modules and two-term silting complexes in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$. Moreover, these objects correspond bijectively to cluster-tilting objects in $\mathcal{C}$ if Λ is a 2-CY tilted algebra associated with a 2-CY triangulated category $\mathcal{C}$. As an application, we show that the property of having two complements holds also for two-term silting complexes in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$.

Contents

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Introduction

Let Λ be a finite-dimensional basic algebra over an algebraically closed field k, $\mathsf{mod}\,\Lambda$ the category of finitely generated left Λ-modules, $\mathsf{proj}\,\Lambda$ the category of finitely generated projective left Λ-modules and $\mathsf{inj}\,\Lambda$ the category of finitely generated injective left Λ-modules. For $M \in \mathsf{mod}\,\Lambda$, we denote by $\mathsf{add}\,M$ (respectively, $\mathsf{Fac}\,M$, $\mathsf{Sub}\,M$) the category of all direct summands (respectively, factor modules, submodules) of finite direct sums of copies of M. Tilting theory for Λ, and its predecessors, have been central in the representation theory of finite-dimensional algebras since the early 1970s [APR79, BGP73, Bon81, BB80, HR82]. When T is a (classical) tilting module (which always has the same number of non-isomorphic indecomposable direct summands as Λ), there is an associated torsion pair $(\mathcal{T}, \mathcal{F})$, where $\mathcal{T} = \mathsf{Fac}\,T$, and the interplay between tilting modules and torsion pairs has played a central role. Another important fact is that an almost complete tilting module U can be completed in at most two different ways to a tilting module [RS91, Ung90]. Moreover, there are exactly two ways if and only if U is a faithful Λ-module [HU89].

Even for a finite-dimensional path algebra kQ, where Q is a finite quiver with no oriented cycles, not all almost complete tilting modules U are faithful. However, for the associated cluster category $\mathcal{C}_Q$, where we have cluster-tilting objects induced from tilting modules over path algebras kQ' derived equivalent to kQ, then the almost complete cluster-tilting objects have exactly two complements [BMRRT06]. This fact, and its generalization to 2-Calabi–Yau triangulated categories [IY08], plays an important role in the categorification of cluster algebras. In the case of cluster categories, this can be reformulated in terms of the path algebra $\Lambda = kQ$ as follows [IT09, Rin07]: a Λ-module T is *support tilting* if T is a tilting $(\Lambda/\langle e\rangle)$-module for some idempotent e of Λ. Using the more general class of support tilting modules, it holds for path algebras that almost complete support tilting modules can be completed in exactly two ways to support tilting modules.

The above result for path algebras does not necessarily hold for a finite-dimensional algebra. The reason is that there may be sincere modules that are not faithful. We are looking for a generalization of tilting modules where we have such a result, and where at the same time some of the essential properties of tilting modules still hold. It is then natural to try to find a class of modules satisfying the following properties.

(i) There is a natural connection with torsion pairs in $\mathsf{mod}\,\Lambda$.

(ii) The modules have exactly $|\Lambda|$ non-isomorphic indecomposable direct summands, where $|X|$ denotes the number of non-isomorphic indecomposable direct summands of X.

(iii) The analogues of basic almost complete tilting modules have exactly two complements.

(iv) In the hereditary case, the class of modules should coincide with the classical tilting modules.

For the (classical) tilting modules, we have in addition that when the almost complete ones have two complements, then they are connected in a special short exact sequence. Also, there is a naturally associated quiver, where the isomorphism classes of tilting modules are the vertices.

There is a generalization of classical tilting modules to tilting modules of finite projective dimension [Hap88, Miy86]. But it is easy to see that they do not satisfy the required properties. The category $\mathsf{mod}\,\Lambda$ is naturally embedded in the derived category of Λ. The tilting and silting complexes for Λ [Aih13, AI12, Rin07] are also extensions of the tilting modules. An almost complete silting complex has infinitely many complements. But as we shall see, things work well when we restrict to the two-term silting complexes.

In the module case, it turns out that a natural class of modules to consider is given as follows. As usual, we denote by τ the AR translation (see § 1.2).

DEFINITION 0.1. (a) We call M in $\mathsf{mod}\,\Lambda$ *τ-rigid* if $\mathrm{Hom}_\Lambda(M, \tau M) = 0$.

(b) We call M in $\mathsf{mod}\,\Lambda$ *τ-tilting* (respectively, *almost complete τ-tilting*) if M is τ-rigid and $|M| = |\Lambda|$ (respectively, $|M| = |\Lambda| - 1$).

(c) We call M in $\mathsf{mod}\,\Lambda$ *support τ-tilting* if there exists an idempotent e of Λ such that M is a τ-tilting $(\Lambda/\langle e\rangle)$-module.

Any τ-rigid module is rigid (i.e. $\mathrm{Ext}^1_\Lambda(M, M) = 0$), and the converse holds if the projective dimension is at most one. In particular, any partial tilting module is a τ-rigid module, and any tilting module is a τ-tilting module. Thus we can regard τ-tilting modules as a generalization of tilting modules.

The first main result of this paper is the following analogue of Bongartz completion for tilting modules.

THEOREM 0.2 (Theorem 2.10). *Any τ-rigid Λ-module is a direct summand of some τ-tilting Λ-module.*

As indicated above, in order to get our theory to work nicely, we need to consider support τ-tilting modules. It is often convenient to view them, and the τ-rigid modules, as certain pairs of Λ-modules.

DEFINITION 0.3. Let (M, P) be a pair with $M \in \mathsf{mod}\,\Lambda$ and $P \in \mathsf{proj}\,\Lambda$.

(a) We call (M, P) a *τ-rigid* pair if M is τ-rigid and $\mathrm{Hom}_\Lambda(P, M) = 0$.

(b) We call (M, P) a *support τ-tilting* (respectively, *almost complete support τ-tilting*) pair if (M, P) is τ-rigid and $|M| + |P| = |\Lambda|$ (respectively, $|M| + |P| = |\Lambda| - 1$).

These notions are compatible with those in Definition 0.1 (see Proposition 2.3 for details). As usual, we say that (M, P) is *basic* if M and P are basic. Similarly, we say that (M, P) is a *direct summand* of (M', P') if M is a direct summand of M' and P is a direct summand of P'.

The second main result of this paper is the following.

THEOREM 0.4 (Theorem 2.18). *Let Λ be a finite-dimensional k-algebra. Then any basic almost complete support τ-tilting pair for Λ is a direct summand of exactly two basic support τ-tilting pairs.*

These two support τ-tilting pairs are said to be *mutations* of each other. We will define the support τ-tilting quiver $\mathrm{Q}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ by using mutation (Definition 2.29).

When extending (classical) tilting modules to tilting complexes or silting complexes, we have pointed out that we do not have exactly two complements in the almost complete case. But considering instead only the two-term silting complexes, we prove that this is the case.

The third main result is to obtain a close connection between support τ-tilting modules and other important objects in tilting theory. The corresponding definitions will be given in § 1.

THEOREM 0.5 (Theorems 2.7, 3.2, 4.1 and 4.7). *Let Λ be a finite-dimensional k-algebra. We have bijections between:*

(a) *the set* $\text{f-tors}\,\Lambda$ *of functorially finite torsion classes in* $\mathsf{mod}\,\Lambda$*;*

(b) *the set* $\mathrm{s}\tau\text{-tilt}\,\Lambda$ *of isomorphism classes of basic support τ-tilting modules;*

(c) *the set* $2\text{-silt}\,\Lambda$ *of isomorphism classes of basic two-term silting complexes for Λ;*

(d) *the set* $\text{c-tilt}\,\mathcal{C}$ *of isomorphism classes of basic cluster-tilting objects in a 2-CY triangulated category $\mathcal{C}$ if Λ is an associated 2-CY tilted algebra to $\mathcal{C}$.*

Note that the correspondence between (b) and (d) improves results in [FL09, Smi08].

By Theorem 0.5, we can regard sτ-tilt Λ as a partially ordered set by using the inclusion relation of f-tors Λ (i.e. we write $T \geqslant U$ if $\mathsf{Fac}\, T \supseteq \mathsf{Fac}\, U$). Then we have the following fourth main result, which is an analogue of [HU05, Theorem 2.1] and [AI12, Theorem 2.35].

Theorem 0.6 (Corollary 2.34). *The support τ-tilting quiver* Q(sτ-tilt Λ) *is the Hasse quiver of the partially ordered set* sτ-tilt Λ.

We have the following direct consequences of Theorem 0.5, where the second part is known by [IY08], and the third one by [ZZ11].

Corollary 0.7 (Corollaries 3.8, 4.5). (a) *Two-term almost complete silting complexes have exactly two complements.*

(b) *In a 2-Calabi–Yau triangulated category with cluster-tilting objects, any almost complete cluster-tilting objects have exactly two complements.*

(c) *In a 2-Calabi–Yau triangulated category with cluster-tilting objects, any maximal rigid object is cluster-tilting.*

Part (a) was first proved directly by Derksen–Fei [DF09] without dealing with support τ-tilting modules. Here, we obtain this result by combining a bijection in Theorem 0.5 with Theorem 0.4.

Another important part of our work is to investigate to what extent the main properties of tilting modules mentioned above remain valid in the settings of support τ-tilting modules, two-term silting complexes and cluster-tilting objects in 2-CY triangulated categories.

A motivation for considering the problem of exactly two complements for almost complete support τ-tilting modules was that the condition of a τ-rigid module appears naturally when we express $\mathrm{Ext}^1_{\mathcal{C}}(X, Y)$ for X and Y objects in a 2-CY triangulated category $\mathcal{C}$ in terms of corresponding modules $\overline{X}$ and $\overline{Y}$ over an associated 2-CY tilted algebra (Proposition 4.4).

There is some relationship to the E-invariants of [DWZ10] in the case of finite-dimensional Jacobian algebras, where the expression $\mathrm{Hom}_\Lambda(M, \tau N)$ appears. Here, we introduce E-invariants in § 5 for any finite-dimensional k-algebras, and express them in terms of dimension vectors and g-vectors as defined in [DK08], inspired by [DWZ10].

In the last § 6, we illustrate our results with examples.

There is a curious relationship with interesting independent work by Cerulli Irelli *et al.* [CLS12], where the authors deal with E-invariants in the more general setting of basic algebras that are not necessarily finite-dimensional. We refer to recent work by König and Yang [KY12] for connection with t-structures and co-t-structures. Hoshino *et al.* [HKM02] and Abe [Abe11] studied two-term tilting complexes. Buan and Marsh have considered a direct map from cluster-tilting objects in cluster categories to functorially finite torsion classes for associated cluster-tilted algebras.

Notation

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1. Background and preliminary results

In this section, we give some background material on each of the four topics involved in our main results. This concerns the relationship between tilting modules and functorially finite subcategories and some results on τ-rigid and τ-tilting modules, including new basic results about them that will be useful in the next section. Further, we recall known results on silting complexes, and on cluster-tilting objects in 2-CY triangulated categories.

1.1 Torsion pairs and tilting modules

Let Λ be a finite-dimensional k-algebra. For a subcategory $\mathcal{C}$ of $\mathsf{mod}\,\Lambda$, we let

$$\mathcal{C}^{\perp} := \{X \in \mathsf{mod}\,\Lambda \mid \mathrm{Hom}_\Lambda(\mathcal{C}, X) = 0\},$$
$$\mathcal{C}^{\perp_1} := \{X \in \mathsf{mod}\,\Lambda \mid \mathrm{Ext}^1_\Lambda(\mathcal{C}, X) = 0\}.$$

Dually, we define ${}^{\perp}\mathcal{C}$ and ${}^{\perp_1}\mathcal{C}$. We call T in $\mathsf{mod}\,\Lambda$ a *partial tilting module* if $\mathrm{pd}_\Lambda T \leqslant 1$ and $\mathrm{Ext}^1_\Lambda(T,T)=0$. A partial tilting module is called a *tilting module* if there is an exact sequence $0 \to \Lambda \to T_0 \to T_1 \to 0$ with T_0 and T_1 in $\mathsf{add}\, T$. Then any tilting module satisfies $|T| = |\Lambda|$. Moreover, it is known that for any partial tilting module T, there is a tilting module U such that $T \in \mathsf{add}\, U$ and $\mathsf{Fac}\, U = T^{\perp_1}$, called the *Bongartz completion* of T. Hence a partial tilting module T is a tilting module if and only if $|T| = |\Lambda|$. Dually, T in $\mathsf{mod}\,\Lambda$ is a (*partial*) *cotilting module* if DT is a (partial) tilting Λ^{op}-module.

On the other hand, we say that a full subcategory $\mathcal{T}$ of $\mathsf{mod}\,\Lambda$ is a *torsion class* (respectively, *torsion-free class*) if it is closed under factor modules (respectively, submodules) and extensions. A pair $(\mathcal{T},\mathcal{F})$ is called a *torsion pair* if $\mathcal{T} = {}^{\perp}\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^{\perp}$. In this case, $\mathcal{T}$ is a torsion class and $\mathcal{F}$ is a torsion-free class. Conversely, any torsion class $\mathcal{T}$ (respectively, torsion-free class $\mathcal{F}$) gives rise to a torsion pair $(\mathcal{T},\mathcal{F})$.

We say that $X \in \mathcal{T}$ is Ext-*projective* (respectively, Ext-*injective*) if $\mathrm{Ext}^1_\Lambda(X,\mathcal{T}) = 0$ (respectively, $\mathrm{Ext}^1_\Lambda(\mathcal{T},X)=0$). We denote by $P(\mathcal{T})$ the direct sum of one copy of each of the indecomposable Ext-projective objects in $\mathcal{T}$ up to isomorphism. Similarly, we denote by $I(\mathcal{F})$ the direct sum of one copy of each of the indecomposable Ext-injective objects in $\mathcal{F}$ up to isomorphism.

We first recall the following relevant result on torsion pairs and tilting modules.

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Proposition 1.1 [AS81, Hos82, Sma84]. *Let $(\mathcal{T},\mathcal{F})$ be a torsion pair in $\mathsf{mod}\,\Lambda$. Then the following conditions are equivalent.*

(a) *$\mathcal{T}$ is functorially finite.*

(b) *$\mathcal{F}$ is functorially finite.*

(c) *$\mathcal{T}=\mathsf{Fac}\,X$ for some X in $\mathsf{mod}\,\Lambda$.*

(d) *$\mathcal{F}=\mathsf{Sub}\,Y$ for some Y in $\mathsf{mod}\,\Lambda$.*

(e) *$P(\mathcal{T})$ is a tilting $(\Lambda/\mathrm{ann}\,\mathcal{T})$-module.*

(f) *$I(\mathcal{F})$ is a cotilting $(\Lambda/\mathrm{ann}\,\mathcal{F})$-module.*

(g) *$\mathcal{T}=\mathsf{Fac}\,P(\mathcal{T})$.*

(h) *$\mathcal{F}=\mathsf{Sub}\,I(\mathcal{F})$.*

Proof. The conditions (a)–(f) are equivalent by [Sma84, Theorem].

(g)⇒(c) is clear.

(e)⇒(g) There exists an exact sequence $0\to\Lambda/\mathrm{ann}\,\mathcal{T}\xrightarrow{a}T^0\to T^1\to 0$ with $T^0,T^1\in\mathsf{add}\,P(\mathcal{T})$. For any $X\in\mathcal{T}$, we take a surjection $f:(\Lambda/\mathrm{ann}\,\mathcal{T})^{\ell}\to X$. It follows from $\mathrm{Ext}^1_{\Lambda}(T^{1\ell},X)=0$ that f factors through $a^{\ell}:(\Lambda/\mathrm{ann}\,\mathcal{T})^{\ell}\to T^{0\ell}$. Thus $X\in\mathsf{Fac}\,P(\mathcal{T})$.

Dually, (h) is also equivalent to the other conditions. □

There is also a tilting quiver associated with the (classical) tilting modules. The vertices are the isomorphism classes of basic tilting modules. Let $X\oplus U$ and $Y\oplus U$ be basic tilting modules, where X and $Y\not\simeq X$ are indecomposable. Then it is known that, after we interchange X and Y if necessary, there is an exact sequence $0\to X\xrightarrow{f}U'\xrightarrow{g}Y\to 0$, where $f:X\to U'$ is a minimal left $(\mathsf{add}\,U)$-approximation and $g:U'\to Y$ is a minimal right $(\mathsf{add}\,U)$-approximation. We say that $Y\oplus U$ is a left mutation of $X\oplus U$. Then we draw an arrow $X\oplus U\to Y\oplus U$, so that we get a quiver for the tilting modules. On the other hand, the set of basic tilting modules has a natural partial order given by $T\geqslant U$ if and only if $\mathsf{Fac}\,T\supseteq\mathsf{Fac}\,U$, and we can consider the associated Hasse quiver. These two quivers coincide [HU05, Theorem 2.1].

1.2 τ-tilting modules

Let Λ be a finite-dimensional k-algebra. We have dualities

$$D:=\mathrm{Hom}_k(-,k):\mathsf{mod}\,\Lambda\leftrightarrow\mathsf{mod}\,\Lambda^{\mathrm{op}}\quad\text{and}\quad(-)^*:=\mathrm{Hom}_{\Lambda}(-,\Lambda):\mathsf{proj}\,\Lambda\leftrightarrow\mathsf{proj}\,\Lambda^{\mathrm{op}},$$

which induce equivalences

$$\nu:=D(-)^*:\mathsf{proj}\,\Lambda\to\mathsf{inj}\,\Lambda\quad\text{and}\quad\nu^{-1}:=(-)^*D:\mathsf{inj}\,\Lambda\to\mathsf{proj}\,\Lambda,$$

called *Nakayama functors*. For X in $\mathsf{mod}\,\Lambda$ with a minimal projective presentation

$$P_1\xrightarrow{d_1}P_0\xrightarrow{d_0}X\longrightarrow 0,$$

we define $\mathrm{Tr}\,X$ in $\mathsf{mod}\,\Lambda^{\mathrm{op}}$ and τX in $\mathsf{mod}\,\Lambda$ by exact sequences

$$P_0^*\xrightarrow{d_1^*}P_1^*\longrightarrow\mathrm{Tr}\,X\longrightarrow 0\quad\text{and}\quad 0\longrightarrow\tau X\longrightarrow\nu P_0\xrightarrow{\nu d_1}\nu P_1.$$

Then Tr and τ give bijections between the isomorphism classes of indecomposable non-projective Λ-modules, the isomorphism classes of indecomposable non-projective Λ^{op}-modules and the isomorphism classes of indecomposable non-injective Λ-modules. We denote by $\underline{\mathsf{mod}}\,\Lambda$ the *stable category* modulo projectives and by $\overline{\mathsf{mod}}\,\Lambda$ the *costable category* modulo injectives. Then Tr gives

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the *Auslander–Bridger transpose duality*

$$\mathrm{Tr} : \underline{\mathsf{mod}}\,\Lambda \leftrightarrow \underline{\mathsf{mod}}\,\Lambda^{\mathrm{op}}$$

and τ gives the *AR translations*

$$\tau = D\,\mathrm{Tr} : \underline{\mathsf{mod}}\,\Lambda \to \overline{\mathsf{mod}}\,\Lambda \quad \text{and} \quad \tau^{-1} = \mathrm{Tr}\,D : \overline{\mathsf{mod}}\,\Lambda \to \underline{\mathsf{mod}}\,\Lambda.$$

We have a functorial isomorphism

$$\underline{\mathrm{Hom}}_\Lambda(X, Y) \simeq D\,\mathrm{Ext}^1_\Lambda(Y, \tau X)$$

for any X and Y in $\mathsf{mod}\,\Lambda$ called *AR duality*. In particular, if M is τ-rigid, then we have $\mathrm{Ext}^1_\Lambda(M, M) = 0$ (i.e. M is rigid) by AR duality. More precisely, we have the following result, which we often use in this paper.

Proposition 1.2. *For X and Y in $\mathsf{mod}\,\Lambda$, we have the following.*

(a) [AS81, Proposition 5.8]. $\mathrm{Hom}_\Lambda(X, \tau Y) = 0$ *if and only if* $\mathrm{Ext}^1_\Lambda(Y, \mathsf{Fac}\,X) = 0$.

(b) [AS81, Theorem 5.10]. *If X is τ-rigid, then $\mathsf{Fac}\,X$ is a functorially finite torsion class and $X \in \mathsf{add}\,P(\mathsf{Fac}\,X)$.*

(c) *If $\mathcal{T}$ is a torsion class in $\mathsf{mod}\,\Lambda$, then $P(\mathcal{T})$ is a τ-rigid Λ-module.*

Proof. (c) Since $T := P(\mathcal{T})$ is Ext-projective in $\mathcal{T}$, we have $\mathrm{Ext}^1_\Lambda(T, \mathsf{Fac}\,T) = 0$. This implies that $\mathrm{Hom}_\Lambda(T, \tau T) = 0$ by (a). □

We have the following direct consequence (see also [ASS06, Sko94]).

Proposition 1.3. *Any τ-rigid Λ-module M satisfies $|M| \leqslant |\Lambda|$.*

Proof. By Proposition 1.2(b), we have $|M| \leqslant |P(\mathsf{Fac}\,M)|$. By Proposition 1.1(e), we have $|P(\mathsf{Fac}\,M)| = |\Lambda/\mathrm{ann}\,M|$. Since $|\Lambda/\mathrm{ann}\,M| \leqslant |\Lambda|$, we have the assertion. □

As an immediate consequence, if τ-rigid Λ-modules M and N satisfy $M \in \mathsf{add}\,N$ and $|M| \geqslant |\Lambda|$, then $\mathsf{add}\,M = \mathsf{add}\,N$.

Finally, we note the following relationship between τ-tilting modules and classical notions.

Proposition 1.4 [ASS06, VIII.5.1]. (a) *Any faithful τ-rigid Λ-module is a partial tilting Λ-module.*

(b) *Any faithful τ-tilting Λ-module is a tilting Λ-module.*

1.3 Silting complexes

Let Λ be a finite-dimensional k-algebra and $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ be the category of bounded complexes of finitely generated projective Λ-modules. We recall the definition of silting complexes and mutations.

Definition 1.5 [Aih13, AI12, BRT11, KV88]. Let $P \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$.

(a) We call P *presilting* if $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(P, P[i]) = 0$ for any $i > 0$.

(b) We call P *silting* if it is presilting and satisfies $\mathsf{thick}\,P = \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$, where $\mathsf{thick}\,P$ is the smallest full subcategory of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ that contains P and is closed under cones, $[\pm 1]$, direct summands and isomorphisms.

We denote by silt Λ the set of isomorphism classes of basic silting complexes for Λ.

The following result is important.

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Proposition 1.6 [AI12, Theorem 2.27, Corollary 2.28]. (a) *For any* $P \in \operatorname{silt}\Lambda$, *we have* $|P| = |\Lambda|$.

(b) *Let* $P = \bigoplus_{i=1}^{n} P_n$ *be a basic silting complex for* Λ *with* P_i *indecomposable. Then* $P_1, \ldots, P_i$ *give a basis of the Grothendieck group* $K_0(\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda))$.

We call a presilting complex P for Λ *almost complete silting* if $|P| = |\Lambda| - 1$. There is a similar type of mutation as for tilting modules.

Definition-Proposition 1.7 [AI12, Theorem 2.31]. Let $P = X \oplus Q$ be a basic silting complex with X indecomposable. We consider a triangle

$$X \xrightarrow{f} Q' \longrightarrow Y \longrightarrow X[1]$$

with a minimal left $(\mathsf{add}\,Q)$-approximation f of X. Then the *left mutation* of P with respect to X is $\mu_X^-(P) := Y \oplus Q$. Dually, we define the *right mutation* $\mu_X^+(P)$ of P with respect to X.[1] Then the left mutation and the right mutation of P are also basic silting complexes.

There is the following partial order on the set $\operatorname{silt}\Lambda$.

Definition-Proposition 1.8 [AI12, Theorem 2.11, Proposition 2.14]. For $P, Q \in \operatorname{silt}\Lambda$, we write

$$P \geqslant Q$$

if $\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(P, Q[i]) = 0$ for any $i > 0$, which is equivalent to $P^{\perp_{>0}} \supseteq Q^{\perp_{>0}}$, where $P^{\perp_{>0}}$ is a subcategory of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ consisting of the X satisfying $\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(P, X[i]) = 0$ for any $i > 0$. Then we have a partial order on $\operatorname{silt}\Lambda$.

We define the *silting quiver* $\mathrm{Q}(\operatorname{silt}\Lambda)$ of Λ as follows.

- The set of vertices is $\operatorname{silt}\Lambda$.
- We draw an arrow from P to Q if Q is a left mutation of P.

Then the silting quiver gives the Hasse quiver of the partially ordered set $\operatorname{silt}\Lambda$ by [AI12, Theorem 2.35], similar to the situation for tilting modules. We shall later restrict to two-term silting complexes to get exactly two complements for almost complete silting complexes.

1.4 Cluster-tilting objects

Let $\mathcal{C}$ be a k-linear Hom-finite Krull–Schmidt triangulated category. Assume that $\mathcal{C}$ is *2-Calabi–Yau* (*2-CY* for short); that is, there exists a functorial isomorphism $D\operatorname{Ext}^1_{\mathcal{C}}(X, Y) \simeq \operatorname{Ext}^1_{\mathcal{C}}(Y, X)$. An important class of objects in these categories are the cluster-tilting objects. We recall the definition of these and related objects.

Definition 1.9. (a) We call T in $\mathcal{C}$ *rigid* if $\operatorname{Hom}_{\mathcal{C}}(T, T[1]) = 0$.

(b) We call T in $\mathcal{C}$ *cluster-tilting* if $\mathsf{add}\,T = \{X \in \mathcal{C} \mid \operatorname{Hom}_{\mathcal{C}}(T, X[1]) = 0\}$.

(c) We call T in $\mathcal{C}$ *maximal rigid* if it is rigid and maximal with respect to this property; that is, $\mathsf{add}\,T = \{X \in \mathcal{C} \mid \operatorname{Hom}_{\mathcal{C}}(T \oplus X, (T \oplus X)[1]) = 0\}$.

We denote by $\operatorname{c-tilt}\mathcal{C}$ the set of isomorphism classes of basic cluster-tilting objects in $\mathcal{C}$. In this setting, there are also mutations of cluster-tilting objects defined via approximations, which we recall [BMRRT06, IY08].

[1] These notations μ^- and μ^+ are the opposite of those in [AI12]. They are easy to remember since they are in the same direction as τ^{-1} and τ, and moreover compatible with the partial order: $\mu_X^-(P) < P < \mu_X^+(P)$.

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DEFINITION-PROPOSITION 1.10 [IY08, Theorem 5.3]. Let $T = X \oplus U$ be a basic cluster-tilting object in $\mathcal{C}$ and X indecomposable in $\mathcal{C}$. We consider the triangle

$$X \xrightarrow{f} U' \longrightarrow Y \longrightarrow X[1]$$

with a minimal left $(\mathsf{add}\, U)$-approximation f of X. Let $\mu_X^-(T) := Y \oplus U$. Dually, we define $\mu_X^+(T)$. A different feature in this case is that we have $\mu_X^-(T) \simeq \mu_X^+(T)$. This is a basic cluster-tilting object, which as before we call the *mutation* of T with respect to X.

In this case, we get just a graph rather than a quiver. We define the *cluster-tilting graph* $\mathrm{G}(\text{c-tilt}\,\mathcal{C})$ of $\mathcal{C}$ as follows.

- The set of vertices is c-tilt $\mathcal{C}$.
- We draw an edge between T and U if U is a mutation of T.

Note that U is a mutation of T if and only if T and U have all but one indecomposable direct summand in common [IY08, Theorem 5.3] (see Corollary 4.5(a)).

2. Support τ-tilting modules

Our aim in this section is to develop a basic theory of support τ-tilting modules over any finite-dimensional k-algebra. We start by discussing some basic properties of τ-rigid modules and connections between τ-rigid modules and functorially finite torsion classes (Theorem 2.7). As an application, we introduce Bongartz completion of τ-rigid modules (Theorem 2.10). Then we give characterizations of τ-tilting modules (Theorem 2.12). We also give left–right duality of τ-rigid modules (Theorem 2.14). Further, we prove our main result, which states that an almost complete support τ-tilting module has exactly two complements (Theorem 2.18). As an application, we introduce mutation of support τ-tilting modules. We show that mutation gives the Hasse quiver of the partially ordered set of support τ-tilting modules (Theorem 2.33).

2.1 Basic properties of τ-rigid modules

When T is a Λ-module with I an ideal contained in ann T, we investigate the relationship between T being τ-rigid as a Λ-module and as a (Λ/I)-module. We have the following.

LEMMA 2.1. *Let Λ be a finite-dimensional algebra, and I an ideal in Λ. Let M and N be (Λ/I)-modules. Then we have the following.*

(a) *If $\mathrm{Hom}_\Lambda(N, \tau M) = 0$, then $\mathrm{Hom}_{\Lambda/I}(N, \tau_{\Lambda/I} M) = 0$.*

(b) *Assume that $I = \langle e \rangle$ for an idempotent e in Λ. Then $\mathrm{Hom}_\Lambda(N, \tau M) = 0$ if and only if $\mathrm{Hom}_{\Lambda/I}(N, \tau_{\Lambda/I} M) = 0$.*

Proof. Note that we have a natural inclusion $\mathrm{Ext}^1_{\Lambda/I}(M, N) \to \mathrm{Ext}^1_\Lambda(M, N)$. This is an isomorphism if $I = \langle e \rangle$ for an idempotent e, since $\mathsf{mod}\,(\Lambda/\langle e \rangle)$ is closed under extensions in $\mathsf{mod}\,\Lambda$.

(a) Assume that $\mathrm{Hom}_\Lambda(N, \tau M) = 0$. Then, by Proposition 1.2, we have $\mathrm{Ext}^1_\Lambda(M, \mathsf{Fac}\, N) = 0$. By the above observation, we have $\mathrm{Ext}^1_{\Lambda/I}(M, \mathsf{Fac}\, N) = 0$. By Proposition 1.2 again, we have $\mathrm{Hom}_{\Lambda/I}(N, \tau_{\Lambda/I} M) = 0$.

(b) Assume that $I = \langle e \rangle$ and $\mathrm{Hom}_{\Lambda/I}(N, \tau_{\Lambda/I} M) = 0$. By Proposition 1.2, we have $\mathrm{Ext}^1_{\Lambda/I}(M, \mathsf{Fac}\, N) = 0$. By the above observation, we have $\mathrm{Ext}^1_\Lambda(M, \mathsf{Fac}\, N) = 0$. By Proposition 1.2 again, we have $\mathrm{Hom}_\Lambda(N, \tau M) = 0$. □

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Recall that M in $\mathsf{mod}\,\Lambda$ is *sincere* if every simple Λ-module appears as a composition factor in M. This is equivalent to the fact that there does not exist a non-zero idempotent e of Λ that annihilates M.

Proposition 2.2. (a) *τ-tilting modules are precisely sincere support τ-tilting modules.*

(b) *Tilting modules are precisely faithful support τ-tilting modules.*

(c) *Any τ-tilting (respectively, τ-rigid) Λ-module T is a tilting (respectively, partial tilting) $(\Lambda/\operatorname{ann} T)$-module.*

Proof. (a) Clearly, sincere support τ-tilting modules are τ-tilting. Conversely, if a τ-tilting Λ-module T is not sincere, then there exists a non-zero idempotent e of Λ such that T is a $(\Lambda/\langle e\rangle)$-module. Since T is τ-rigid as a $(\Lambda/\langle e\rangle)$-module by Lemma 2.1(a), we have $|\Lambda/\langle e\rangle| \geqslant |T| = |\Lambda|$ by Proposition 1.3, a contradiction.

(b) Clearly, tilting modules are faithful τ-tilting. Conversely, any faithful support τ-tilting module T is partial tilting by Proposition 1.4 and satisfies $|T| = |\Lambda|$. Thus T is tilting.

(c) By Lemma 2.1(a), we know that T is a faithful τ-tilting (respectively, τ-rigid) $(\Lambda/\operatorname{ann} T)$-module. Thus the assertion follows from (b) (respectively, Proposition 1.4). □

Immediately, we have the following basic observation, which will be used frequently in this paper.

Proposition 2.3. *Let (M, P) be a pair with $M \in \mathsf{mod}\,\Lambda$ and $P \in \mathsf{proj}\,\Lambda$. Let e be an idempotent of Λ such that $\mathsf{add}\,P = \mathsf{add}\,\Lambda e$.*

(a) *(M, P) is a τ-rigid (respectively, support τ-tilting, almost complete support τ-tilting) pair for Λ if and only if M is a τ-rigid (respectively, τ-tilting, almost complete τ-tilting) $(\Lambda/\langle e\rangle)$-module.*

(b) *If (M, P) and (M, Q) are support τ-tilting pairs for Λ, then $\mathsf{add}\,P = \mathsf{add}\,Q$. In other words, M determines P and e uniquely.*

Proof. (a) The assertions follow from Lemma 2.1 and the equation $|\Lambda/\langle e\rangle| = |\Lambda| - |P|$.

(b) This is a consequence of Proposition 2.2(a). □

The following observations are useful.

Proposition 2.4. *Let X be in $\mathsf{mod}\,\Lambda$ with a minimal projective presentation $P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} X \to 0$.*

(a) *For Y in $\mathsf{mod}\,\Lambda$, we have an exact sequence*

$$0 \to \operatorname{Hom}_\Lambda(Y, \tau X) \to D\operatorname{Hom}_\Lambda(P_1, Y) \xrightarrow{D(d_1,Y)} D\operatorname{Hom}_\Lambda(P_0, Y) \xrightarrow{D(d_0,Y)} D\operatorname{Hom}_\Lambda(X, Y) \to 0.$$

(b) *$\operatorname{Hom}_\Lambda(Y, \tau X) = 0$ if and only if the map $\operatorname{Hom}_\Lambda(P_0, Y) \xrightarrow{(d_1,Y)} \operatorname{Hom}_\Lambda(P_1, Y)$ is surjective.*

(c) *X is τ-rigid if and only if the map $\operatorname{Hom}_\Lambda(P_0, X) \xrightarrow{(d_1,X)} \operatorname{Hom}_\Lambda(P_1, X)$ is surjective.*

Proof. (a) We have an exact sequence $0 \to \tau X \to \nu P_1 \xrightarrow{\nu d_1} \nu P_0$. Applying $\operatorname{Hom}_\Lambda(Y, -)$, we have a commutative diagram of exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_\Lambda(Y, \tau X) \arrow[r] & \operatorname{Hom}_\Lambda(Y, \nu P_1) \arrow[r, "(Y,\nu d_1)"] \arrow[d, "\wr"] & \operatorname{Hom}_\Lambda(Y, \nu P_0) \arrow[d, "\wr"] & & \\
& & D\operatorname{Hom}_\Lambda(P_1, Y) \arrow[r, "D(d_1,Y)"] & D\operatorname{Hom}_\Lambda(P_0, Y) \arrow[r, "D(d_0,Y)"] & D\operatorname{Hom}_\Lambda(X, Y) \arrow[r] & 0
\end{tikzcd}$$

Thus the assertion follows.

(b)(c) Immediate from (a). □

We have the following standard observation (cf. [DK08, HU05]).

Proposition 2.5. *Let X be in $\mathsf{mod}\,\Lambda$ with a minimal projective presentation $P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} X \to 0$. If X is τ-rigid, then P_0 and P_1 have no non-zero direct summands in common.*

Proof. We only have to show that any morphism $s : P_1 \to P_0$ is in the radical. By Proposition 2.4(c), there exists $t : P_0 \to X$ such that $d_0 s = t d_1$. Since P_0 is projective, there exists $u : P_0 \to P_0$ such that $t = d_0 u$. Since $d_0(s - u d_1) = 0$, there exists $v : P_1 \to P_1$ such that $s = u d_1 + d_1 v$.

$$\begin{tikzcd}
 & P_1 \arrow[r, "d_1"] \arrow[d, "s"] \arrow[dl, dotted, "v"'] & P_0 \arrow[r, "d_0"] \arrow[d, "t"] \arrow[dl, dotted, "u"] & X \arrow[r] & 0 \\
P_1 \arrow[r, "d_1"'] & P_0 \arrow[r, "d_0"'] & X \arrow[r] & 0 &
\end{tikzcd}$$

Since d_1 is in the radical, so is s. Thus the assertion is shown. □

The following analogue of Wakamatsu's lemma [AR91] will be useful.

Lemma 2.6. *Let $\eta : 0 \to Y \to T' \xrightarrow{f} X$ be an exact sequence in $\mathsf{mod}\,\Lambda$, where T is τ-rigid, and $f : T' \to X$ is a right $(\mathsf{add}\,T)$-approximation. Then we have $Y \in {}^{\perp}(\tau T)$.*

Proof. Replacing X by $\operatorname{Im} f$, we can assume that f is surjective. We apply $\operatorname{Hom}_\Lambda(-, \tau T)$ to η to get the exact sequence

$$0 = \operatorname{Hom}_\Lambda(T', \tau T) \to \operatorname{Hom}_\Lambda(Y, \tau T) \to \operatorname{Ext}^1_\Lambda(X, \tau T) \xrightarrow{\operatorname{Ext}^1(f, \tau T)} \operatorname{Ext}^1_\Lambda(T', \tau T),$$

where we have $\operatorname{Hom}_\Lambda(T', \tau T) = 0$ because T is τ-rigid. Since $f : T' \to X$ is a right $(\mathsf{add}\,T)$-approximation, the induced map $(T, f) : \operatorname{Hom}_\Lambda(T, T') \to \operatorname{Hom}_\Lambda(T, X)$ is surjective. Then also the induced map $\underline{\operatorname{Hom}}_\Lambda(T, T') \to \underline{\operatorname{Hom}}_\Lambda(T, X)$ of the maps modulo projectives is surjective, so by the AR duality the map $\operatorname{Ext}^1(f, \tau T) : \operatorname{Ext}^1_\Lambda(X, \tau T) \to \operatorname{Ext}^1_\Lambda(T', \tau T)$ is injective. It follows that $\operatorname{Hom}_\Lambda(Y, \tau T) = 0$. □

2.2 τ-rigid modules and torsion classes

The following correspondence is basic in our paper, where we denote by $\text{f-tors}\,\Lambda$ the set of functorially finite torsion classes in $\mathsf{mod}\,\Lambda$.

Theorem 2.7. *There is a bijection*

$$\text{s}\tau\text{-tilt}\,\Lambda \longleftrightarrow \text{f-tors}\,\Lambda$$

given by $\text{s}\tau\text{-tilt}\,\Lambda \ni T \mapsto \mathsf{Fac}\,T \in \text{f-tors}\,\Lambda$ and $\text{f-tors}\,\Lambda \ni \mathcal{T} \mapsto P(\mathcal{T}) \in \text{s}\tau\text{-tilt}\,\Lambda$.

Proof. Let first $\mathcal{T}$ be a functorially finite torsion class in $\mathsf{mod}\,\Lambda$. Then we know that $T = P(\mathcal{T})$ is τ-rigid by Proposition 1.2(c). Let $e \in \Lambda$ be a maximal idempotent such that $\mathcal{T} \subseteq \mathsf{mod}\,(\Lambda/\langle e \rangle)$. Then we have $|\Lambda/\langle e \rangle| = |\Lambda/\operatorname{ann} \mathcal{T}|$, and $|\Lambda/\operatorname{ann} \mathcal{T}| = |T|$ by Proposition 1.1(e). Hence $(T, \Lambda e)$ is a support τ-tilting pair for Λ. Moreover, we have $\mathcal{T} = \mathsf{Fac}\,P(\mathcal{T})$ by Proposition 1.1(g).

Assume conversely that T is a support τ-tilting Λ-module. Then T is a τ-tilting $(\Lambda/\langle e \rangle)$-module for an idempotent e of Λ. Thus $\mathsf{Fac}\,T$ is a functorially finite torsion class in $\mathsf{mod}\,(\Lambda/\langle e \rangle)$ such that $T \in \mathsf{add}\,P(\mathsf{Fac}\,T)$ by Proposition 1.2(b). Since $|T| = |\Lambda/\langle e \rangle|$, we have $\mathsf{add}\,T = \mathsf{add}\,P(\mathsf{Fac}\,T)$ by Proposition 1.3. Thus $T \simeq P(\mathsf{Fac}\,T)$. □

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We denote by τ-tilt Λ (respectively, tilt Λ) the set of isomorphism classes of basic τ-tilting Λ-modules (respectively, tilting Λ-modules). On the other hand, we denote by sf-tors Λ (respectively, ff-tors Λ) the set of sincere (respectively, faithful) functorially finite torsion classes in $\mathsf{mod}\,\Lambda$.

Corollary 2.8. *The bijection in Theorem 2.7 induces bijections*

$$\tau\text{-tilt}\,\Lambda \longleftrightarrow \text{sf-tors}\,\Lambda \quad \textit{and} \quad \text{tilt}\,\Lambda \longleftrightarrow \text{ff-tors}\,\Lambda.$$

Proof. Let T be a support τ-tilting Λ-module. By Proposition 2.2, it follows that T is a τ-tilting Λ-module (respectively, tilting Λ-module) if and only if T is sincere (respectively, faithful) if and only if $\mathsf{Fac}\,T$ is sincere (respectively, faithful). □

We are interested in the torsion classes where our original module U is a direct summand of $T = P(\mathcal{T})$, since we would like to complete U to a (support) τ-tilting module. The conditions for this to be the case are the following.

Proposition 2.9. *Let $\mathcal{T}$ be a functorially finite torsion class and U a τ-rigid Λ-module. Then $U \in \mathsf{add}\,P(\mathcal{T})$ if and only if $\mathsf{Fac}\,U \subseteq \mathcal{T} \subseteq {}^{\perp}(\tau U)$.*

Proof. We have $\mathcal{T} = \mathsf{Fac}\,P(\mathcal{T})$ by Proposition 1.1(g).

Assume that $\mathsf{Fac}\,U \subseteq \mathcal{T} \subseteq {}^{\perp}(\tau U)$. Then U is in $\mathcal{T}$. We want to show that U is Ext-projective in $\mathcal{T}$; that is, $\mathrm{Ext}^1_\Lambda(U, \mathcal{T}) = 0$ or, equivalently, $\mathrm{Hom}_\Lambda(P(\mathcal{T}), \tau U) = 0$, by Proposition 1.2(a). This follows since $P(\mathcal{T}) \in \mathcal{T} \subseteq {}^{\perp}(\tau U)$. Hence U is a direct summand of $P(\mathcal{T})$.

Conversely, assume that $U \in \mathsf{add}\,P(\mathcal{T})$. Then we must have $U \in \mathcal{T}$, and hence $\mathsf{Fac}\,U \subseteq \mathcal{T}$. Since U is Ext-projective in $\mathcal{T}$, we have $\mathrm{Ext}^1_\Lambda(U, \mathcal{T}) = 0$. Since $\mathcal{T} = \mathsf{Fac}\,\mathcal{T}$, we have $\mathrm{Hom}_\Lambda(\mathcal{T}, \tau U) = 0$ by Proposition 1.2(a). Hence we have $\mathcal{T} \subseteq {}^{\perp}(\tau U)$. □

We now prove the analogue, for τ-tilting modules, of the Bongartz completion of classical tilting modules.

Theorem 2.10. *Let U be a τ-rigid Λ-module. Then $\mathcal{T} := {}^{\perp}(\tau U)$ is a sincere functorially finite torsion class and $T := P(\mathcal{T})$ is a τ-tilting Λ-module satisfying $U \in \mathsf{add}\,T$ and ${}^{\perp}(\tau T) = \mathsf{Fac}\,T$.*

We call $P({}^{\perp}(\tau U))$ the *Bongartz completion* of U.

Proof. The first part follows from the following observation.

Lemma 2.11. *For any τ-rigid Λ-module U, we have a sincere functorially finite torsion class ${}^{\perp}(\tau U)$.*

Proof. When U is τ-rigid, then $\mathsf{Sub}\,\tau U$ is a torsion-free class by the dual of Proposition 1.2(b). Then $({}^{\perp}(\tau U), \mathsf{Sub}\,\tau U)$ is a torsion pair, and $\mathsf{Sub}\,\tau U$ and ${}^{\perp}(\tau U)$ are functorially finite by Proposition 1.1.

Assume that ${}^{\perp}(\tau U)$ is not sincere. Then we have ${}^{\perp}(\tau U) \subseteq \mathsf{mod}\,(\Lambda/\langle e\rangle)$ for some primitive idempotent e in Λ. The corresponding simple Λ-module S is not a composition factor of any module in ${}^{\perp}(\tau U)$; in particular, $\mathrm{Hom}({}^{\perp}(\tau U), D(e\Lambda)) = 0$. Then $D(e\Lambda)$ is in $\mathsf{Sub}\,\tau U$. But this is a contradiction, since τU, and hence also any module in $\mathsf{Sub}\,\tau U$ has no non-zero injective direct summands. □

By Corollary 2.8, it follows that T is a τ-tilting Λ-module such that ${}^{\perp}(\tau U) = \mathsf{Fac}\,T$. By Proposition 2.9, we have $U \in \mathsf{add}\,T$. Clearly, ${}^{\perp}(\tau U) \supseteq {}^{\perp}(\tau T)$, since U is in $\mathsf{add}\,T$. Hence we get $\mathsf{Fac}\,T = {}^{\perp}(\tau U) \supseteq {}^{\perp}(\tau T) \supseteq \mathsf{Fac}\,T$, and consequently ${}^{\perp}(\tau T) = \mathsf{Fac}\,T$. □

We have the following characterizations of a τ-rigid module being τ-tilting.

THEOREM 2.12. *The following are equivalent for a τ-rigid Λ-module T.*

(a) *T is τ-tilting.*

(b) *T is maximal τ-rigid; that is, if $T \oplus X$ is τ-rigid for some Λ-module X, then $X \in \mathsf{add}\, T$.*

(c) ${}^{\perp}(\tau T) = \mathsf{Fac}\, T$.

(d) *If $\mathrm{Hom}_\Lambda(T, \tau X) = 0$ and $\mathrm{Hom}_\Lambda(X, \tau T) = 0$, then $X \in \mathsf{add}\, T$.*

Proof. (a)⇒(b). Immediate from Proposition 1.3.

(b)⇒(c). Let U be the Bongartz completion of T. Since T is maximal τ-rigid, we have $T \simeq U$, and hence ${}^{\perp}(\tau T) = {}^{\perp}(\tau U) = \mathsf{Fac}\, U = \mathsf{Fac}\, T$, using Theorem 2.10.

(c)⇒(a). Let T be τ-rigid with ${}^{\perp}(\tau T) = \mathsf{Fac}\, T$. Let U be the Bongartz completion of T. Then we have

$$\mathsf{Fac}\, T = {}^{\perp}(\tau T) \supseteq {}^{\perp}(\tau U) \supseteq \mathsf{Fac}\, U \supseteq \mathsf{Fac}\, T,$$

and hence all inclusions are equalities. Since $\mathsf{Fac}\, U = \mathsf{Fac}\, T$, there exists an exact sequence

$$0 \longrightarrow Y \longrightarrow T' \xrightarrow{f} U \longrightarrow 0. \tag{1}$$

where $f : T' \to U$ is a right $(\mathsf{add}\, T)$-approximation. By the Wakamatsu-type Lemma 2.6, we have $\mathrm{Hom}_\Lambda(Y, \tau T) = 0$, and hence $\mathrm{Hom}_\Lambda(Y, \tau U) = 0$, since ${}^{\perp}(\tau T) = {}^{\perp}(\tau U)$. By the AR duality we have $\mathrm{Ext}^1_\Lambda(U, Y) \simeq D\overline{\mathrm{Hom}}_\Lambda(Y, \tau U) = 0$, and hence the sequence (1) splits. Then it follows that U is in $\mathsf{add}\, T$. Thus T is a τ-tilting Λ-module.

(a) + (c) ⇒ (d). Assume that (a) and (c) hold, and $\mathrm{Hom}_\Lambda(T, \tau X) = 0$ and $\mathrm{Hom}_\Lambda(X, \tau T) = 0$. Then $\mathrm{Ext}^1_\Lambda(X, \mathsf{Fac}\, T) = 0$ by Proposition 1.2(a) and X is in ${}^{\perp}(\tau T) = \mathsf{Fac}\, T$. Thus X is in $\mathsf{add}\, P(\mathsf{Fac}\, T) = \mathsf{add}\, T$ by Theorem 2.7.

(d)⇒(b). This is clear. ◻

We note the following generalization.

COROLLARY 2.13. *The following are equivalent for a τ-rigid pair (T, P) for Λ.*

(a) *(T, P) is a support τ-tilting pair for Λ.*

(b) *If $(T \oplus X, P)$ is τ-rigid for some Λ-module X, then $X \in \mathsf{add}\, T$.*

(c) ${}^{\perp}(\tau T) \cap P^{\perp} = \mathsf{Fac}\, T$.

(d) *If $\mathrm{Hom}_\Lambda(T, \tau X) = 0$, $\mathrm{Hom}_\Lambda(X, \tau T) = 0$ and $\mathrm{Hom}_\Lambda(P, X) = 0$, then $X \in \mathsf{add}\, T$.*

Proof. In view of Lemma 2.1(b), the assertion follows immediately from Theorem 2.12 by replacing Λ by $\Lambda/\langle e\rangle$ for an idempotent e of Λ satisfying $\mathsf{add}\, P = \mathsf{add}\, \Lambda e$. ◻

In the rest of this subsection, we discuss the left–right symmetry of τ-rigid modules. It is somehow surprising that there exists a bijection between support τ-tilting Λ-modules and support τ-tilting Λ^{op}-modules. We decompose M in $\mathsf{mod}\, \Lambda$ as $M = M_{\mathrm{pr}} \oplus M_{\mathrm{np}}$, where M_{pr} is a maximal projective direct summand of M. For a τ-rigid pair (M, P) for Λ, let

$$(M, P)^{\dagger} := (\mathrm{Tr}\, M_{\mathrm{np}} \oplus P^*, M^*_{\mathrm{pr}}) = (\mathrm{Tr}\, M \oplus P^*, M^*_{\mathrm{pr}}).$$

We denote by $\tau\text{-rigid}\, \Lambda$ the set of isomorphism classes of basic τ-rigid pairs of Λ.

THEOREM 2.14. *$(-)^{\dagger}$ gives bijections*

$$\tau\text{-rigid}\, \Lambda \longleftrightarrow \tau\text{-rigid}\, \Lambda^{\mathrm{op}} \quad \textit{and} \quad \mathrm{s}\tau\text{-tilt}\, \Lambda \longleftrightarrow \mathrm{s}\tau\text{-tilt}\, \Lambda^{\mathrm{op}}$$

such that $(-)^{\dagger\dagger} = \mathrm{id}$.

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For a support τ-tilting Λ-module M, we simply write $M^\dagger := \operatorname{Tr} M_{\rm np} \oplus P^*$, where (M, P) is a support τ-tilting pair for Λ.

Proof. We only have to show that $(M, P)^\dagger$ is a τ-rigid pair for $\Lambda^{\rm op}$, since the correspondence $(M, P) \mapsto (M, P)^\dagger$ is clearly an involution. We have

$$0 = \operatorname{Hom}_\Lambda(M_{\rm np}, \tau M) = \operatorname{Hom}_{\Lambda^{\rm op}}(\operatorname{Tr} M, DM_{\rm np}) = \operatorname{Hom}_{\Lambda^{\rm op}}(\operatorname{Tr} M, \tau \operatorname{Tr} M). \tag{2}$$

Moreover, we have

$$0 = \operatorname{Hom}_\Lambda(M_{\rm pr}, \tau M) = \operatorname{Hom}_{\Lambda^{\rm op}}(\operatorname{Tr} M, DM_{\rm pr}) = D \operatorname{Hom}_{\Lambda^{\rm op}}(M_{\rm pr}^*, \operatorname{Tr} M). \tag{3}$$

On the other hand, we have

$$0 = \operatorname{Hom}_\Lambda(P, M) = \operatorname{Hom}_\Lambda(P, M_{\rm pr}) \oplus \operatorname{Hom}_\Lambda(P, M_{\rm np}). \tag{4}$$

Thus we have

$$0 = D(P^* \otimes_\Lambda M_{\rm np}) = \operatorname{Hom}_{\Lambda^{\rm op}}(P^*, DM_{\rm np}) = \operatorname{Hom}_{\Lambda^{\rm op}}(P^*, \tau \operatorname{Tr} M).$$

This, together with (2), shows that $\operatorname{Tr} M \oplus P^*$ is a τ-rigid $\Lambda^{\rm op}$-module. We have $\operatorname{Hom}_{\Lambda^{\rm op}}(M_{\rm pr}^*, P^*) = 0$ by (4). This, together with (3), shows that $(M, P)^\dagger$ is a τ-rigid pair for $\Lambda^{\rm op}$. $\square$

Now, we discuss dual notions of τ-rigid and τ-tilting modules, even though we do not use them in this paper.

- We call M in $\mathsf{mod}\,\Lambda$ *τ^--rigid* if $\operatorname{Hom}_\Lambda(\tau^- M, M) = 0$.
- We call M in $\mathsf{mod}\,\Lambda$ *τ^--tilting* if M is τ^--rigid and $|M| = |\Lambda|$.
- We call M in $\mathsf{mod}\,\Lambda$ *support τ^--tilting* if M is a τ^--tilting $(\Lambda/\langle e\rangle)$-module for some idempotent e of Λ.

Clearly, M is τ^--rigid (respectively, τ^--tilting, support τ^--tilting) Λ-module if and only if DM is τ-rigid (respectively, τ-tilting, support τ-tilting) $\Lambda^{\rm op}$-module.

We denote by $\operatorname{cotilt} \Lambda$ (respectively, $\tau^-\text{-tilt}\,\Lambda$, $\mathrm{s}\tau^-\text{-tilt}\,\Lambda$) the set of isomorphism classes of basic cotilting (respectively, τ^--tilting, support τ^--tilting) Λ-modules. On the other hand, we denote by $\text{f-torf}\,\Lambda$ the set of functorially finite torsion-free classes in $\mathsf{mod}\,\Lambda$, and by $\text{sf-torf}\,\Lambda$ (respectively, $\text{ff-torf}\,\Lambda$) the set of sincere (respectively, faithful) functorially finite torsion-free classes in $\mathsf{mod}\,\Lambda$. We have the following results immediately from Theorem 2.7 and Corollary 2.8.

Theorem 2.15. *We have bijections*

$$\mathrm{s}\tau^-\text{-tilt}\,\Lambda \longleftrightarrow \text{f-torf}\,\Lambda, \quad \tau^-\text{-tilt}\,\Lambda \longleftrightarrow \text{sf-torf}\,\Lambda \quad \textit{and} \quad \operatorname{cotilt}\Lambda \longleftrightarrow \text{ff-torf}\,\Lambda$$

given by $\mathrm{s}\tau^-\text{-tilt}\,\Lambda \ni T \mapsto \mathsf{Sub}\,T \in \text{f-torf}\,\Lambda$ *and* $\text{f-torf}\,\Lambda \ni \mathcal{F} \mapsto I(\mathcal{F}) \in \mathrm{s}\tau^-\text{-tilt}\,\Lambda$.

On the other hand, we have a bijection

$$\mathrm{s}\tau\text{-tilt}\,\Lambda \longleftrightarrow \mathrm{s}\tau^-\text{-tilt}\,\Lambda$$

given by $(M, P) \mapsto D((M, P)^\dagger) = (\tau M \oplus \nu P, \nu M_{\rm pr})$. Thus we have bijections

$$\text{f-tors}\,\Lambda \longleftrightarrow \mathrm{s}\tau\text{-tilt}\,\Lambda \longleftrightarrow \mathrm{s}\tau^-\text{-tilt}\,\Lambda \longleftrightarrow \text{f-torf}\,\Lambda$$

by Theorems 2.7 and 2.15. We end this subsection with the following observation.

Proposition 2.16. (a) *The above bijections send $\mathcal{T} \in \text{f-tors}\,\Lambda$ to $\mathcal{T}^\perp \in \text{f-torf}\,\Lambda$.*

(b) *For any support τ-tilting pair (M, P) for Λ, the torsion pairs $(\mathsf{Fac}\,M, M^\perp)$ and $({}^\perp(\tau M \oplus \nu P), \mathsf{Sub}\,(\tau M \oplus \nu P))$ in $\mathsf{mod}\,\Lambda$ coincide.*

Proof. (b) We only have to show that $\mathsf{Fac}\, M = {}^{\perp}(\tau M \oplus \nu P)$. It follows from Proposition 1.2(b) and its dual that $(\mathsf{Fac}\, M, M^{\perp})$ and $({}^{\perp}(\tau M \oplus \nu P), \mathsf{Sub}\,(\tau M \oplus \nu P))$ are torsion pairs in $\mathsf{mod}\, \Lambda$. They coincide since $\mathsf{Fac}\, M = {}^{\perp}(\tau M) \cap P^{\perp} = {}^{\perp}(\tau M \oplus \nu P)$ holds by Corollary 2.13(c).

(a) Let $\mathcal{T} \in \text{f-tors}\, \Lambda$ and (M, P) be the corresponding support τ-tilting pair for Λ. Since $\mathcal{T}^{\perp} = M^{\perp}$ and $D(M^{\dagger}) = \tau M \oplus \nu P$, the assertion follows from (b). $\square$

2.3 Mutation of support τ-tilting modules

In this section, we prove our main result on complements for almost complete support τ-tilting pairs. Let us start with the following result.

Proposition 2.17. *Let T be a basic τ-rigid module that is not τ-tilting. Then there are at least two basic support τ-tilting modules that have T as a direct summand.*

Proof. By Theorem 2.12, $\mathcal{T}_1 = \mathsf{Fac}\, T$ is properly contained in $\mathcal{T}_2 = {}^{\perp}(\tau T)$. By Theorem 2.7 and Lemma 2.11, we have two different support τ-tilting modules $P(\mathcal{T}_1)$ and $P(\mathcal{T}_2)$ up to isomorphism. By Proposition 2.9, they are extensions of T. $\square$

Our aim is to prove the following result.

Theorem 2.18. *Let Λ be a finite-dimensional k-algebra. Then any basic almost complete support τ-tilting pair (U, Q) for Λ is a direct summand of exactly two basic support τ-tilting pairs (T, P) and (T', P') for Λ. Moreover, we have $\{\mathsf{Fac}\, T, \mathsf{Fac}\, T'\} = \{\mathsf{Fac}\, U, {}^{\perp}(\tau U) \cap Q^{\perp}\}$.*

Before proving Theorem 2.18, we introduce a notion of mutation.

Definition 2.19. Two basic support τ-tilting pairs (T, P) and (T', P') for Λ are said to be *mutations* of each other if there exists a basic almost complete support τ-tilting pair (U, Q) that is a direct summand of (T, P) and (T', P'). In this case, we write $(T', P') = \mu_X(T, P)$ or simply $T' = \mu_X(T)$ if X is an indecomposable Λ-module satisfying either $T = U \oplus X$ or $P = Q \oplus X$.

We can also describe mutation as follows: let (T, P) be a basic support τ-tilting pair for Λ, and X an indecomposable direct summand of either T or P.

(a) If X is a direct summand of T, precisely one of the following holds.

- There exists an indecomposable Λ-module Y such that $X \not\simeq Y$ and $\mu_X(T, P) := ((T/X) \oplus Y, P)$ is a basic support τ-tilting pair for Λ.
- There exists an indecomposable projective Λ-module Y such that $\mu_X(T, P) := (T/X, P \oplus Y)$ is a basic support τ-tilting pair for Λ.

(b) If X is a direct summand of P, there exists an indecomposable Λ-module Y such that $\mu_X(T, P) := (T \oplus Y, P/X)$ is a basic support τ-tilting pair for Λ.

Moreover, such a module Y in each case is unique up to isomorphism.

In the rest of this subsection, we give a proof of Theorem 2.18. The following is the first step.

Lemma 2.20. *Let (T, P) be a τ-rigid pair for Λ. If U is a τ-rigid Λ-module satisfying ${}^{\perp}(\tau T) \cap P^{\perp} \subseteq {}^{\perp}(\tau U)$, then there is an exact sequence $U \xrightarrow{f} T' \to C \to 0$ satisfying the following conditions.*

- *f is a minimal left $(\mathsf{Fac}\, T)$-approximation.*
- *T' is in $\mathsf{add}\, T$, C is in $\mathsf{add}\, P(\mathsf{Fac}\, T)$ and $\mathsf{add}\, T' \cap \mathsf{add}\, C = 0$.*

Proof. Consider the exact sequence $U \xrightarrow{f} T' \xrightarrow{g} C \to 0$, where f is a minimal left $(\mathsf{add}\, T)$-approximation. Then $g \in \mathrm{rad}(T', C)$.

(i) We show that f is a minimal left $(\mathsf{Fac}\, T)$-approximation. Take any $X \in \mathsf{Fac}\, T$ and $s: U \to X$. By the Wakamatsu-type Lemma 2.6, there exists an exact sequence

$$0 \to Y \to T'' \xrightarrow{h} X \to 0,$$

where h is a right $(\mathsf{add}\, T)$-approximation and $Y \in {}^{\perp}(\tau T)$. Moreover, we have $Y \in P^{\perp}$, since $T'' \in P^{\perp}$. By the assumption that ${}^{\perp}(\tau T) \cap P^{\perp} \subseteq {}^{\perp}(\tau U)$, we have $\mathrm{Hom}_{\Lambda}(Y, \tau U) = 0$, hence $\mathrm{Ext}^1_{\Lambda}(U, Y) = 0$. Then we have an exact sequence

$$\mathrm{Hom}_{\Lambda}(U, T'') \to \mathrm{Hom}_{\Lambda}(U, X) \to \mathrm{Ext}^1_{\Lambda}(U, Y) = 0.$$

Thus there is some $t: U \to T''$ such that $s = ht$.

$$\begin{tikzcd} & & & U \arrow[r, "f"] \arrow[dl, dotted, "t"'] \arrow[d, "s"] & T' \\ 0 \arrow[r] & Y \arrow[r] & T'' \arrow[r, "h"] & X \arrow[r] & 0 \end{tikzcd}$$

Since $T'' \in \mathsf{add}\, T$ and f is a left $(\mathsf{add}\, T)$-approximation, there is some $u: T' \to T''$ such that $t = uf$. Hence we have $hu: T' \to X$ such that $(hu)f = ht = s$, and the claim follows.

(ii) We show that $C \in \mathsf{add}\, P(\mathsf{Fac}\, T)$. We have an exact sequence $0 \to \mathrm{Im}\, f \xrightarrow{i} T' \to C \to 0$, which gives rise to an exact sequence

$$\mathrm{Hom}_{\Lambda}(T', \mathsf{Fac}\, T) \xrightarrow{(i, \mathsf{Fac}\, T)} \mathrm{Hom}_{\Lambda}(\mathrm{Im}\, f, \mathsf{Fac}\, T) \to \mathrm{Ext}^1_{\Lambda}(C, \mathsf{Fac}\, T) \to \mathrm{Ext}^1_{\Lambda}(T', \mathsf{Fac}\, T).$$

We know from (i) that $(f, \mathsf{Fac}\, T): \mathrm{Hom}_{\Lambda}(T', \mathsf{Fac}\, T) \to \mathrm{Hom}_{\Lambda}(U, \mathsf{Fac}\, T)$ is surjective, and hence $(i, \mathsf{Fac}\, T)$ is surjective. Further, $\mathrm{Ext}^1_{\Lambda}(T', \mathsf{Fac}\, T) = 0$ by Proposition 1.2 since T' is in $\mathsf{add}\, T$ and T is τ-rigid. Then it follows that $\mathrm{Ext}^1_{\Lambda}(C, \mathsf{Fac}\, T) = 0$. Since $C \in \mathsf{Fac}\, T$, this means that C is Ext-projective in $\mathsf{Fac}\, T$.

(iii) We show that $\mathsf{add}\, T' \cap \mathsf{add}\, C = 0$. To show this, it is clearly sufficient to show that $\mathrm{Hom}_{\Lambda}(T', C) \subseteq \mathrm{rad}(T', C)$.

Let $s: T' \to C$ be an arbitrary map. We have an exact sequence $\mathrm{Hom}_{\Lambda}(U, T') \to \mathrm{Hom}_{\Lambda}(U, C) \to \mathrm{Ext}^1_{\Lambda}(U, \mathrm{Im}\, f)$. Since $\mathrm{Ext}^1_{\Lambda}(U, \mathrm{Im}\, f) = 0$ because $\mathrm{Im}\, f$ is in $\mathsf{Fac}\, U$, and U is τ-tilting, there is a map $t: U \to T'$ such that $sf = gt$. Since f is a left $(\mathsf{add}\, T)$-approximation, and T' is in $\mathsf{add}\, T$, there is a map $u: T' \to T'$ such that $t = uf$. Then $(s - gu)f = sf - gt = 0$; hence there is some $v: C \to C$ such that $s - gu = vg$, and hence $s = gu + vg$.

$$\begin{tikzcd} & U \arrow[r, "f"] \arrow[d, "t"] & T' \arrow[r, "g"] \arrow[dl, dotted, "u"'] \arrow[d, "s"] & C \arrow[r] \arrow[dl, dotted, "v"'] & 0 \\ U \arrow[r, "f"] \arrow[dr] & T' \arrow[r, "g"] & C \arrow[r] & 0 & \\ & \mathrm{Im}\, f \arrow[u] & & & \end{tikzcd}$$

Since $g \in \mathrm{rad}(T', C)$, it follows that $s \in \mathrm{rad}(T', C)$. Hence $\mathrm{Hom}_{\Lambda}(T', C) \subseteq \mathrm{rad}(T', C)$, and consequently $\mathsf{add}\, T' \cap \mathsf{add}\, C = 0$. $\square$

The following information on the previous lemma is useful.

Lemma 2.21. *In Lemma 2.20, assume $C = 0$. Then $f: U \to T'$ induces an isomorphism $U/\langle e \rangle U \simeq T'$ for a maximal idempotent e of Λ satisfying $eT = 0$. In particular, if T is sincere, then $U \simeq T'$.*

Proof. By our assumption, we have an exact sequence:

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow U \xrightarrow{f} T' \longrightarrow 0. \tag{5}$$

Applying $\operatorname{Hom}_\Lambda(-, \mathsf{Fac}\, T)$, we have an exact sequence:

$$\operatorname{Hom}_\Lambda(T', \mathsf{Fac}\, T) \xrightarrow{(f, \mathsf{Fac}\, T)} \operatorname{Hom}_\Lambda(U, \mathsf{Fac}\, T) \to \operatorname{Hom}_\Lambda(\operatorname{Ker} f, \mathsf{Fac}\, T) \to \operatorname{Ext}^1_\Lambda(T', \mathsf{Fac}\, T).$$

We have $\operatorname{Ext}^1_\Lambda(T', \mathsf{Fac}\, T) = 0$ because T' is in $\mathsf{add}\, T$ and T is τ-rigid. Since $(f, \mathsf{Fac}\, T)$ is surjective, it follows that $\operatorname{Hom}_\Lambda(\operatorname{Ker} f, \mathsf{Fac}\, T) = 0$ and so $\operatorname{Ker} f \in {}^\perp(\mathsf{Fac}\, T)$. On the other hand, since T is a sincere $(\Lambda/\langle e\rangle)$-module, $\mathsf{mod}\,(\Lambda/\langle e\rangle)$ is the smallest torsion-free class of $\mathsf{mod}\,\Lambda$ containing $\mathsf{Fac}\, T$. Thus we have a torsion pair $({}^\perp(\mathsf{Fac}\, T), \mathsf{mod}\,(\Lambda/\langle e\rangle))$, and the canonical sequence for X associated with this torsion pair is given by

$$0 \longrightarrow \langle e\rangle X \longrightarrow X \longrightarrow X/\langle e\rangle X \longrightarrow 0.$$

Since $\operatorname{Ker} f \in {}^\perp(\mathsf{Fac}\, T)$ and $T' \in \mathsf{Fac}\, T \subseteq \mathsf{mod}\,(\Lambda/\langle e\rangle)$, the canonical sequence of U is given by (5). Thus we have $U/\langle e\rangle U \simeq T'$. □

In the next result, we prove a useful restriction on X when $T = X \oplus U$ is τ-tilting and X is indecomposable.

PROPOSITION 2.22. *Let $T = X \oplus U$ be a basic τ-tilting Λ-module, with X indecomposable. Then exactly one of ${}^\perp(\tau U) \subseteq {}^\perp(\tau X)$ and $X \in \mathsf{Fac}\, U$ holds.*

Proof. First, we assume that ${}^\perp(\tau U) \subseteq {}^\perp(\tau X)$ and $X \in \mathsf{Fac}\, U$ both hold. Then we have

$$\mathsf{Fac}\, U = \mathsf{Fac}\, T = {}^\perp(\tau T) = {}^\perp(\tau U),$$

which implies that U is τ-tilting by Theorem 2.12, a contradiction.

Let $Y \oplus U$ be the Bongartz completion of U. Then we have ${}^\perp\tau(Y \oplus U) = {}^\perp(\tau U) \supseteq {}^\perp(\tau T)$. Using the triple $(T, 0, Y \oplus U)$ instead of (T, P, U) in Lemma 2.20, there is an exact sequence

$$Y \oplus U \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & 1 \end{smallmatrix}\right)} T' \oplus U \longrightarrow T'' \longrightarrow 0\,,$$

where $f : Y \to T'$ and $\left(\begin{smallmatrix} f & 0 \\ 0 & 1 \end{smallmatrix}\right) : Y \oplus U \to T' \oplus U$ are minimal left $(\mathsf{Fac}\, T)$-approximations, T' and T'' are in $\mathsf{add}\, T$ and $\mathsf{add}\,(T' \oplus U) \cap \mathsf{add}\, T'' = 0$. Then we have $T'' \in \mathsf{add}\, X$.

Assume first $T'' \neq 0$. Then $T'' \simeq X^\ell$ for some $\ell \geqslant 1$, so we have $T' \in \mathsf{add}\, U$. Since we have a surjective map $T' \to T''$, we have $X \in \mathsf{Fac}\, T' \subseteq \mathsf{Fac}\, U$.

Assume now that $T'' = 0$. Applying Lemma 2.21, we have that $\left(\begin{smallmatrix} f & 0 \\ 0 & 1 \end{smallmatrix}\right) : Y \oplus U \to T' \oplus U$ is an isomorphism, since T is sincere. Thus $Y \in \mathsf{add}\, T$, and we must have $Y \simeq X$. Thus ${}^\perp(\tau X) = {}^\perp(\tau Y) \supseteq {}^\perp(\tau U)$. □

Now we are ready to prove Theorem 2.18.

(i) First, we assume that $Q = 0$ (i.e. U is an almost complete τ-tilting module).

In view of Proposition 2.17, it only remains to show that there are at most two extensions of U to a support τ-tilting module. Using the bijection in Theorem 2.7, we only have to show that for any support τ-tilting module $X \oplus U$, the torsion class $\mathsf{Fac}\,(X \oplus U)$ is either $\mathsf{Fac}\, U$ or ${}^\perp(\tau U)$. If $X = 0$ (i.e. U is a support τ-tilting module), then this is clear. If $X \neq 0$, then $X \oplus U$ is a τ-tilting Λ-module. Moreover, by Proposition 2.22 either $X \in \mathsf{Fac}\, U$ or ${}^\perp(\tau U) \subseteq {}^\perp(\tau X)$ holds. If $X \in \mathsf{Fac}\, U$, then we have $\mathsf{Fac}\,(X \oplus U) = \mathsf{Fac}\, U$. If ${}^\perp(\tau U) \subseteq {}^\perp(\tau X)$, then we have $\mathsf{Fac}\,(X \oplus U) = {}^\perp(\tau(X \oplus U)) = {}^\perp(\tau U)$. Thus the assertion follows.

(ii) Let (U, Q) be a basic almost complete support τ-tilting pair for Λ and e be an idempotent of Λ such that $\mathsf{add}\, Q = \mathsf{add}\, \Lambda e$. Then U is an almost complete τ-tilting $(\Lambda/\langle e\rangle)$-module by Proposition 2.3(a). It follows from (i) that U is a direct summand of exactly two basic support τ-tilting $(\Lambda/\langle e\rangle)$-modules. Thus the assertion follows, since basic support τ-tilting $(\Lambda/\langle e\rangle)$-modules that have U as a direct summand correspond bijectively to basic support τ-tilting pairs for Λ that have (U, Q) as a direct summand. □

The following special case of Lemma 2.20 is useful.

Proposition 2.23. *Let T be a support τ-tilting Λ-module. Assume that one of the following conditions is satisfied.*

(i) *U is a τ-rigid Λ-module such that $\mathsf{Fac}\, T \subseteq {}^{\perp}(\tau U)$.*

(ii) *U is a support τ-tilting Λ-module such that $U \geqslant T$.*

Then there exists an exact sequence $U \xrightarrow{f} T^0 \to T^1 \to 0$ such that f is a minimal left $(\mathsf{Fac}\, T)$-approximation of U and T^0 and T^1 are in $\mathsf{add}\, T$ and satisfy $\mathsf{add}\, T^0 \cap \mathsf{add}\, T^1 = 0$.

Proof. Let (T, P) be a support τ-tilting pair for Λ. Then ${}^{\perp}(\tau T) \cap P^{\perp} = \mathsf{Fac}\, T$ holds by Corollary 2.13(c). Thus ${}^{\perp}(\tau T) \cap P^{\perp} \subseteq {}^{\perp}(\tau U)$ holds for both cases. Hence the assertion is immediate from Lemma 2.20, since C is in $\mathsf{add}\, P(\mathsf{Fac}\, T) = \mathsf{add}\, T$ by Theorem 2.7. □

The following well-known result [HU89] can be shown as an application of our results.

Corollary 2.24. *Let Λ be a finite-dimensional k-algebra and U a basic almost complete tilting Λ-module. Then U is faithful if and only if U is a direct summand of precisely two basic tilting Λ-modules.*

Proof. It follows from Theorem 2.18 that U is a direct summand of exactly two basic support τ-tilting Λ-modules T and T' such that $\mathsf{Fac}\, T = \mathsf{Fac}\, U$. If U is faithful, then T and T' are tilting Λ-modules by Proposition 2.2(b). Thus the 'only if' part follows. If U is not faithful, then T is not a tilting Λ-module, since it is not faithful because $\mathsf{Fac}\, T = \mathsf{Fac}\, U$. Thus the 'if' part follows. □

2.4 Partial order, exchange sequences and Hasse quiver

In this section, we investigate two quivers. One is defined by partial order, and the other one by mutation. We show that they coincide.

Since we have a bijection $T \mapsto \mathsf{Fac}\, T$ between sτ-tilt Λ and f-tors Λ, then inclusion in f-tors Λ gives rise to a partial order on sτ-tilt Λ, and we have an associated Hasse quiver. Note that sτ-tilt Λ has a unique maximal element Λ and a unique minimal element 0.

The following description of when $T \geqslant U$ holds will be useful.

Lemma 2.25. *Let (T, P) and (U, Q) be support τ-tilting pairs for Λ. Then the following conditions are equivalent.*

(a) *$T \geqslant U$.*

(b) *$\mathrm{Hom}_\Lambda(U, \tau T) = 0$ and $\mathsf{add}\, P \subseteq \mathsf{add}\, Q$.*

(c) *$\mathrm{Hom}_\Lambda(U_{\mathrm{np}}, \tau T_{\mathrm{np}}) = 0$, $\mathsf{add}\, T_{\mathrm{pr}} \supseteq \mathsf{add}\, U_{\mathrm{pr}}$ and $\mathsf{add}\, P \subseteq \mathsf{add}\, Q$.*

Proof. (a)⇒(c). Since $\mathsf{Fac}\, T \supseteq \mathsf{Fac}\, U$, we have $\mathsf{add}\, T_{\mathrm{pr}} \supseteq \mathsf{add}\, U_{\mathrm{pr}}$ and $\mathrm{Hom}_\Lambda(U, \tau T) = 0$. Moreover, $\mathsf{add}\, P \subseteq \mathsf{add}\, Q$ holds by Proposition 2.2(a).

(b)⇒(a). We have $\mathsf{Fac}\,T = {}^{\perp}(\tau T) \cap P^{\perp}$ by Corollary 2.13(c). Since $\mathsf{add}\,P \subseteq \mathsf{add}\,Q$, we have $U \in Q^{\perp} \subseteq P^{\perp}$. Since $\mathrm{Hom}_{\Lambda}(U, \tau T) = 0$, we have $U \in {}^{\perp}(\tau T) \cap P^{\perp} = \mathsf{Fac}\,T$, which implies $\mathsf{Fac}\,T \supseteq \mathsf{Fac}\,U$.

(c)⇒(b). This is clear. □

Also we shall need the following.

Proposition 2.26. *Let $T, U, V \in \mathrm{s}\tau\text{-tilt}\,\Lambda$ such that $T \geqslant U \geqslant V$. Then $\mathsf{add}\,T \cap \mathsf{add}\,V \subseteq \mathsf{add}\,U$.*

Proof. Clearly, we have $P(\mathsf{Fac}\,T) \cap \mathsf{Fac}\,U \subseteq P(\mathsf{Fac}\,U) = \mathsf{add}\,U$. Thus we have $\mathsf{add}\,T \cap \mathsf{add}\,V \subseteq P(\mathsf{Fac}\,T) \cap \mathsf{Fac}\,U \subseteq \mathsf{add}\,U$. □

The following observation is immediate.

Proposition 2.27. (a) *For any idempotent e of Λ, the inclusion $\mathrm{s}\tau\text{-tilt}\,(\Lambda/\langle e\rangle) \to \mathrm{s}\tau\text{-tilt}\,\Lambda$ preserves the partial order.*

(b) *The bijection $(-)^{\dagger} : \mathrm{s}\tau\text{-tilt}\,\Lambda \to \mathrm{s}\tau\text{-tilt}\,\Lambda^{\mathrm{op}}$ in Theorem 2.14 reverses the partial order.*

Proof. (a) This is clear.

(b) Let (T, P) and (U, Q) be support τ-tilting pairs of Λ. By Lemma 2.25, $T \geqslant U$ if and only if $\mathrm{Hom}_{\Lambda}(U_{\mathrm{np}}, \tau T_{\mathrm{np}}) = 0$, $\mathsf{add}\,T_{\mathrm{pr}} \supseteq \mathsf{add}\,U_{\mathrm{pr}}$ and $\mathsf{add}\,P \subseteq \mathsf{add}\,Q$. This is equivalent to $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\,T_{\mathrm{np}}, \tau\,\mathrm{Tr}\,U_{\mathrm{np}}) = 0$, $\mathsf{add}\,T^{*}_{\mathrm{pr}} \supseteq \mathsf{add}\,U^{*}_{\mathrm{pr}}$ and $\mathsf{add}\,P^{*} \subseteq \mathsf{add}\,Q^{*}$. By Lemma 2.25 again, this is equivalent to $(\mathrm{Tr}\,T_{\mathrm{np}} \oplus P^{*}, T^{*}_{\mathrm{pr}}) \leqslant (\mathrm{Tr}\,U_{\mathrm{np}} \oplus Q^{*}, U^{*}_{\mathrm{pr}})$. □

In the rest of this section, we study a relationship between partial order and mutation.

Definition-Proposition 2.28. Let $T = X \oplus U$ and T' be support τ-tilting Λ-modules such that $T' = \mu_X(T)$ for some indecomposable Λ-module X. Then either $T > T'$ or $T < T'$ holds by Theorem 2.18. We say that T' is a *left mutation* (respectively, *right mutation*) of T and we write $T' = \mu_X^{-}(T)$ (respectively, $T' = \mu_X^{+}(T)$) if the following equivalent conditions are satisfied.

(a) $T > T'$ (respectively, $T < T'$).

(b) $X \notin \mathsf{Fac}\,U$ (respectively, $X \in \mathsf{Fac}\,U$).

(c) ${}^{\perp}(\tau X) \supseteq {}^{\perp}(\tau U)$ (respectively, ${}^{\perp}(\tau X) \not\supseteq {}^{\perp}(\tau U)$).

If T is a τ-tilting Λ-module, then the following condition is also equivalent to the above conditions.

(d) T is a Bongartz completion of U (respectively, T is a non-Bongartz completion of U).

Proof. This follows immediately from Theorem 2.18 and Proposition 2.22. □

Definition 2.29. We define the *support τ-tilting quiver* $\mathrm{Q}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ of Λ as follows.

- The set of vertices is $\mathrm{s}\tau$-tilt Λ.
- We draw an arrow from T to U if U is a left mutation of T.

Next, we show that one can calculate left mutation of support τ-tilting Λ-modules by exchange sequences that are constructed from left approximations.

Theorem 2.30. *Let $T = X \oplus U$ be a basic τ-tilting module that is the Bongartz completion of U, where X is indecomposable. Let $X \xrightarrow{f} U' \xrightarrow{g} Y \to 0$ be an exact sequence, where f is a minimal left $(\mathsf{add}\,U)$-approximation. Then we have the following.*

(a) *If U is not sincere, then $Y = 0$. In this case, $U = \mu_X^{-}(T)$ holds and this is a basic support τ-tilting Λ-module that is not τ-tilting.*

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(b) *If U is sincere, then Y is a direct sum of copies of an indecomposable Λ-module Y_1 and is not in* $\mathsf{add}\, T$. *In this case, $Y_1 \oplus U = \mu_X^-(T)$ holds and this is a basic τ-tilting Λ-module.*

Proof. We first make some preliminary observations. We have ${}^\perp(\tau U) \subseteq {}^\perp(\tau X)$ because T is a Bongartz completion of U. By Lemma 2.20, we have an exact sequence

$$X \xrightarrow{f} U' \xrightarrow{g} Y \to 0$$

such that U' is in $\mathsf{add}\, U$, Y is in $\mathsf{add}\, P(\mathsf{Fac}\, U)$, $\mathsf{add}\, U' \cap \mathsf{add}\, Y = 0$ and f is a left $(\mathsf{Fac}\, U)$-approximation. We have $\mathrm{Ext}^1_\Lambda(Y, \mathsf{Fac}\, U) = 0$ since $Y \in \mathsf{add}\, P(\mathsf{Fac}\, U)$, and hence $\mathrm{Hom}_\Lambda(U, \tau Y) = 0$ by Proposition 1.2. We have an injective map $\mathrm{Hom}_\Lambda(Y, \tau(Y \oplus U)) \to \mathrm{Hom}_\Lambda(U', \tau(Y \oplus U))$. Since U is τ-rigid, we have that $\mathrm{Hom}_\Lambda(U', \tau(Y \oplus U)) = 0$, and consequently $\mathrm{Hom}_\Lambda(Y, \tau(Y \oplus U)) = 0$. It follows that $Y \oplus U$ is τ-rigid.

We show that $g : U' \to Y$ is a right $(\mathsf{add}\, T)$-approximation. To see this, consider the exact sequence

$$\mathrm{Hom}_\Lambda(T, U') \to \mathrm{Hom}_\Lambda(T, Y) \to \mathrm{Ext}^1_\Lambda(T, \mathrm{Im}\, f).$$

Since $\mathrm{Im}\, f \in \mathsf{Fac}\, T$, we have $\mathrm{Ext}^1_\Lambda(T, \mathrm{Im}\, f) = 0$, which proves the claim.

We have that Y does not have any indecomposable direct summand from $\mathsf{add}\, T$. For if T' in $\mathsf{add}\, T$ is an indecomposable direct summand of Y, then the natural inclusion $T' \to Y$ factors through $g : U' \to Y$. This contradicts the fact that $f : X \to U'$ is left minimal.

Now we are ready to prove the claims (a) and (b).

(a) Assume first that U is not sincere. Let e be a primitive idempotent with $eU = 0$. Then U is a τ-rigid $(\Lambda/\langle e\rangle)$-module. Since $|U| = |\Lambda| - 1 = |\Lambda/\langle e\rangle|$, we have that U is a τ-tilting $(\Lambda/\langle e\rangle)$-module, and hence a support τ-tilting Λ-module that is not τ-tilting.

(b) Next, assume that U is sincere. Since we have already shown that $Y \oplus U$ is τ-rigid and Y does not have any indecomposable direct summand from $\mathsf{add}\, T$, it is enough to show $Y \neq 0$. Otherwise, we have $X \simeq U'$ by Lemma 2.21, since U is sincere. This is not possible since U' is in $\mathsf{add}\, U$, but X is not. Hence it follows that $Y \neq 0$. □

We do not know the answer to the following.

Question 2.31. Is Y always indecomposable in Theorem 2.30(b)?

Note that right mutation cannot be calculated as directly as left mutation:

Remark 2.32. Let T and T' be support τ-tilting Λ-modules such that $T' = \mu_X(T)$ for some indecomposable Λ-module X.

(a) If $T' = \mu_X^-(T)$, then we can calculate T' by applying Theorem 2.30.

(b) If $T' = \mu_X^+(T)$, then we can calculate T' using the following three steps. First, calculate $T^\dagger$. Then, calculate $T'^\dagger$ by applying Theorem 2.30 to $T^\dagger$. Finally, calculate T' by applying $(-)^\dagger$ to $T'^\dagger$.

Our next main result is the following.

Theorem 2.33. *For $T, U \in \mathrm{s}\tau\text{-tilt}\,\Lambda$, the following conditions are equivalent.*

(a) *U is a left mutation of T.*

(b) *T is a right mutation of U.*

(c) *$T > U$ and there is no $V \in \mathrm{s}\tau\text{-tilt}\,\Lambda$ such that $T > V > U$.*

Before proving Theorem 2.33, we give the following result as a direct consequence.

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COROLLARY 2.34. *The support τ-tilting quiver* $\mathrm{Q}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ *is the Hasse quiver of the partially ordered set* $\mathrm{s}\tau\text{-tilt}\,\Lambda$.

The following analogue of [AI12, Proposition 2.36] is a main step to prove Theorem 2.33.

THEOREM 2.35. *Let U and T be basic support τ-tilting Λ-modules such that $U > T$. Then:*

(a) *there exists a right mutation V of T such that $U \geqslant V$;*

(b) *there exists a left mutation V' of U such that $V' \geqslant T$.*

Before proving Theorem 2.35, we finish the proof of Theorem 2.33 by using Theorem 2.35.

(a)$\Leftrightarrow$(b). Immediate from the definitions.

(a)$\Rightarrow$(c). Assume that $V \in \mathrm{s}\tau\text{-tilt}\,\Lambda$ satisfies $T > V \geqslant U$. Then we have $\mathsf{add}\,T \cap \mathsf{add}\,U \subseteq \mathsf{add}\,V$ by Proposition 2.26. Thus T and V have an almost complete support τ-tilting pair for Λ as a common direct summand. Hence we have $V \simeq U$ by Theorem 2.18.

(c)$\Rightarrow$(a). By Theorem 2.35, there exists a left mutation V of T such that $T > V \geqslant U$. Then $V \simeq U$ by our assumption. Thus U is a left mutation of T. $\square$

To prove Theorem 2.35, we shall need the following results.

LEMMA 2.36. *Let U and T be basic support τ-tilting Λ-modules such that $U > T$. Let $U \xrightarrow{f} T^0 \to T^1 \to 0$ be an exact sequence as given in Proposition 2.23. If X is an indecomposable direct summand of T that does not belong to $\mathsf{add}\,T^0$, then we have $U \geqslant \mu_X(T) > T$.*

Proof. First, we show $\mu_X(T) > T$. Since X is in $\mathsf{Fac}\,T \subseteq \mathsf{Fac}\,U$, there exists a surjective map $a : U^\ell \to X$ for some $\ell > 0$. Since $f^\ell : U^\ell \to (T^0)^\ell$ is a left $(\mathsf{add}\,T)$-approximation, a factors through f^ℓ and we have $X \in \mathsf{Fac}\,T^0$. It follows from $X \notin \mathsf{add}\,T^0$ that $X \in \mathsf{Fac}\,T^0 \subseteq \mathsf{Fac}\,\mu_X(T)$. Thus $\mathsf{Fac}\,T \subseteq \mathsf{Fac}\,\mu_X(T)$ and we have $\mu_X(T) > T$.

Next, we show that $U \geqslant \mu_X(T)$. Let $(U, \Lambda e)$ and $(T, \Lambda e')$ be support τ-tilting pairs for Λ. By Proposition 2.27(b), we know that $U^\dagger = \mathrm{Tr}\,U \oplus e\Lambda$ and $T^\dagger = \mathrm{Tr}\,T \oplus e'\Lambda$ are support τ-tilting Λ^{op}-modules such that $U^\dagger < T^\dagger$. In particular, any minimal right $(\mathsf{add}\,T^\dagger)$-approximation

$$\mathrm{Tr}\,T_0 \oplus P \to U^\dagger \tag{6}$$

of $U^\dagger$ with $T_0 \in \mathsf{add}\,T_{\mathrm{np}}$ and $P \in \mathsf{add}\,e'\Lambda$ is surjective. The following observation shows that $T_0 \in \mathsf{add}\,T^0$.

LEMMA 2.37. *Let X and Y be in $\mathsf{mod}\,\Lambda$ and P in $\mathsf{proj}\,\Lambda^{\mathrm{op}}$. Let $f : Y \to X^0$ be a left $(\mathsf{add}\,X)$-approximation of Y and $g : \mathrm{Tr}\,X_0 \oplus P_0 \to \mathrm{Tr}\,Y$ be a minimal right $(\mathsf{add}\ \mathrm{Tr}\,X \oplus P)$-approximation of $\mathrm{Tr}\,Y$ with $X_0 \in \mathsf{add}\,X_{\mathrm{np}}$ and $P_0 \in \mathsf{add}\,P$. If g is surjective, then X_0 is a direct summand of X^0.*

Proof. Assume that g is surjective and consider the exact sequence

$$0 \longrightarrow K \xrightarrow{h} \mathrm{Tr}\,X_0 \oplus P_0 \xrightarrow{g} \mathrm{Tr}\,Y \longrightarrow 0.$$

Then h is in $\mathrm{rad}(K, \mathrm{Tr}\,X_0 \oplus P_0)$, since g is right minimal. It is easy to see that in the stable category $\underline{\mathsf{mod}}\,\Lambda^{\mathrm{op}}$, a pseudokernel of g is given by h, which is in the radical of $\underline{\mathsf{mod}}\,\Lambda^{\mathrm{op}}$. In particular, g is a minimal right $(\mathsf{add}\ \mathrm{Tr}\,X)$-approximation in $\underline{\mathsf{mod}}\,\Lambda^{\mathrm{op}}$. Since $\mathrm{Tr} : \underline{\mathsf{mod}}\,\Lambda \to \underline{\mathsf{mod}}\,\Lambda^{\mathrm{op}}$ is a duality, we have that $\mathrm{Tr}\,g : \mathrm{Tr}\,\mathrm{Tr}\,Y \to \mathrm{Tr}(\mathrm{Tr}\,X_0 \oplus P_0) = X_0$ is a minimal left $(\mathsf{add}\ X)$-approximation of $\mathrm{Tr}\,\mathrm{Tr}\,Y$ in $\underline{\mathsf{mod}}\,\Lambda$. On the other hand, $f : Y \to X^0$ is clearly a left $(\mathsf{add}\ X)$-approximation of Y in $\underline{\mathsf{mod}}\,\Lambda$. Since $\mathrm{Tr}\,\mathrm{Tr}\,Y$ is a direct summand of Y, we have that X_0 is a direct summand of X^0 in $\underline{\mathsf{mod}}\,\Lambda$. Thus the assertion follows. $\square$

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We now finish the proof of Lemma 2.36.

Since $T_0 \in \mathsf{add}\, T^0$ and $X \notin \mathsf{add}\, T^0$, we have $X \notin \mathsf{add}\, T_0$ and hence $U^\dagger \in \mathsf{Fac}\,(\mathrm{Tr}(T/X) \oplus e'\Lambda)$ by (6). Hence we have $U^\dagger \leqslant \mu_X(T)^\dagger$, which implies $U \geqslant \mu_X(T)$ by Proposition 2.27(b). □

Now we are ready to prove Theorem 2.35.

We only prove (a), since (b) follows from (a) and Proposition 2.27(b).

(i) Let $(U, \Lambda e)$ and $(T, \Lambda e')$ be support τ-tilting pairs for Λ. Let

$$U \longrightarrow T^0 \longrightarrow T^1 \longrightarrow 0 \tag{7}$$

be an exact sequence given by Proposition 2.23. If $T \notin \mathsf{add}\, T^0$, then any indecomposable direct summand X of T that is not in $\mathsf{add}\, T^0$ satisfies $U \geqslant \mu_X(T) > T$ by Lemma 2.36. Thus we assume that $T \in \mathsf{add}\, T^0$ in the rest of proof. Since $\mathsf{add}\, T^0 \cap \mathsf{add}\, T^1 = 0$, we have $T^1 = 0$, which implies $T^0 = U/\langle e'\rangle U$ by Lemma 2.21.

(ii) By Proposition 2.27(b), we know that $U^\dagger = \mathrm{Tr}\, U \oplus e\Lambda$ and $T^\dagger = \mathrm{Tr}\, T \oplus e'\Lambda$ are support τ-tilting Λ^{op}-modules such that $U^\dagger < T^\dagger$. Let

$$T_0^\dagger \xrightarrow{f} U^\dagger \longrightarrow 0$$

be a minimal right $(\mathsf{add}\, T^\dagger)$-approximation of $U^\dagger$. If $e'\Lambda \notin \mathsf{add}\, T_0^\dagger$, then any indecomposable direct summand Q of $e'\Lambda$ that is not in $\mathsf{add}\, T_0^\dagger$ satisfies $U^\dagger \in \mathsf{Fac}\,(T^\dagger/Q)$. Thus we have $U^\dagger \leqslant \mu_Q(T^\dagger)$ and $U \geqslant \mu_{Q^*}(T) > T$ by Proposition 2.27. In the rest of the proof, we assume that $e'\Lambda \in \mathsf{add}\, T_0^\dagger$.

(iii) We show that there exists an exact sequence

$$P_1 \xrightarrow{a} \mathrm{Tr}\, T^0 \oplus P_0 \longrightarrow \mathrm{Tr}\, U \longrightarrow 0 \tag{8}$$

in $\mathsf{mod}\, \Lambda^{\mathrm{op}}$ such that $P_0 \in \mathsf{proj}\, \Lambda^{\mathrm{op}}$, $P_1 \in \mathsf{add}\, e'\Lambda$, $a \in \mathrm{rad}(P_1, \mathrm{Tr}\, T^0 \oplus P_0)$ and the map

$$(a, U^\dagger) : \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, T^0 \oplus P_0, U^\dagger) \longrightarrow \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(P_1, U^\dagger) \tag{9}$$

is surjective.

Let $Q_1 \xrightarrow{d} Q_0 \to U \to 0$ be a minimal projective presentation of U. Let $d' : Q_1' \to Q_0$ be a right $(\mathsf{add}\, \Lambda e')$-approximation of Q_0. Since $T^0 = U/\langle e'\rangle U$ by (i), we have a projective presentation $Q_1' \oplus Q_1 \xrightarrow{\binom{d'}{d}} Q_0 \to T^0 \to 0$ of T^0. Thus we have an exact sequence

$$Q_0^* \xrightarrow{(d'^* \ d^*)} Q_1'^* \oplus Q_1^* \xrightarrow{\binom{c'}{c}} \mathrm{Tr}\, T^0 \oplus Q \longrightarrow 0$$

for some projective Λ^{op}-module Q. We have a commutative diagram

$$\begin{tikzcd}
Q_0^* \arrow[r, "d^*"] \arrow[d, "d'^*"] & Q_1^* \arrow[r] \arrow[d, "-c"] & \mathrm{Tr}\, U \arrow[r] \arrow[d, equal] & 0 \\
Q_1'^* \arrow[r, "c'"] & \mathrm{Tr}\, T^0 \oplus Q \arrow[r] & \mathrm{Tr}\, U \arrow[r] & 0
\end{tikzcd}$$

of exact sequences. Now we decompose the morphism c' as

$$c' = \begin{pmatrix} a & 0 \\ 0 & 1_{Q''} \end{pmatrix} : Q_1'^* = P_1 \oplus Q'' \longrightarrow \mathrm{Tr}\, T^0 \oplus Q = \mathrm{Tr}\, T^0 \oplus P_0 \oplus Q'',$$

where a is in the radical. Then we naturally have an exact sequence (8), and clearly we have $P_0 \in \mathsf{proj}\, \Lambda^{\mathrm{op}}$ and $P_1 \in \mathsf{add}\, e'\Lambda$ by our construction. It remains to show that (9) is surjective.

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We only have to show that the map

$$(c', U^\dagger) : \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, T^0 \oplus Q, U^\dagger) \longrightarrow \mathrm{Hom}_{\Lambda^{\mathrm{op}}}({Q'_1}^*, U^\dagger)$$

is surjective. Take any map $s : {Q'_1}^* \to U^\dagger$. By Proposition 2.4(c), there exists $t : Q_1^* \to U^\dagger$ such that $sd'^* = td^*$. Thus there exists $u : \mathrm{Tr}\, T^0 \oplus Q \to U^\dagger$ such that $s = uc'$ and $t = -uc$, which shows the assertion.

(iv) First, we assume that P_1 in (iii) is non-zero. Since $e'\Lambda \in \mathsf{add}\, T_0^\dagger$ by (ii) and $P_1 \in \mathsf{add}\, e'\Lambda$, we have $P_1 \in \mathsf{add}\, T_0^\dagger$. Thus there exists a morphism $s : P_1 \to T_0^\dagger$ that is not in the radical. Since (9) is surjective, there exists $t : \mathrm{Tr}\, T^0 \oplus P_0 \to U^\dagger$ such that $ta = fs$. Since f is a surjective right $(\mathsf{add}\, T^\dagger)$-approximation and P_0 is projective, there exists $u : \mathrm{Tr}\, T^0 \oplus P_0 \to T_0^\dagger$ such that $t = fu$.

$$\begin{tikzcd} P_1 \arrow[r, "a"] \arrow[d, "s"] & \mathrm{Tr}\, T^0 \oplus P_0 \arrow[r] \arrow[d, "t"] \arrow[dl, dotted, "u"'] & \mathrm{Tr}\, U \arrow[r] & 0 \\ T_0^\dagger \arrow[r, "f"'] & U^\dagger \arrow[rr] & & 0 \end{tikzcd}$$

Since $f(s - ua) = 0$ and f is right minimal, we have that $s - ua$ is in the radical. Since a is in the radical, so is s, a contradiction.

Consequently, we have $P_1 = 0$. Thus $\mathrm{Tr}\, T^0 \oplus P_0 \simeq \mathrm{Tr}\, U$ and $\mathrm{Tr}\, T^0 \simeq \mathrm{Tr}\, U$. Since $T \in \mathsf{add}\, T^0$ by our assumption, we have $\mathsf{add}\, T_{\mathrm{np}} = \mathsf{add}\, U_{\mathrm{np}}$. Since $U > T$, we have $T_{\mathrm{pr}} \in \mathsf{add}\, U_{\mathrm{pr}}$. Thus $U \simeq T \oplus P$ for some projective Λ-module P.

(v) It remains to consider the case $U \simeq T \oplus P$ for some projective Λ-module P.

Since $U > T$, we have $\mathsf{add}\, \Lambda e \subsetneq \mathsf{add}\, \Lambda e'$. Take any indecomposable summand $\Lambda e''$ of $\Lambda(e' - e)$ and let $V := \mu_{\Lambda e''}(T, \Lambda e')$, which has a form $(T \oplus X, \Lambda(e' - e''))$, with X indecomposable. Clearly, $V > T$ holds. Since $\tau U \in \mathsf{add}\, \tau(T \oplus X)$ by our assumption and $\Lambda e \in \mathsf{add}\, \Lambda(e' - e'')$ by our choice of e'', we have

$$\mathsf{Fac}\, U = {}^\perp(\tau U) \cap (\Lambda e)^\perp \supseteq {}^\perp(\tau(T \oplus X)) \cap (\Lambda(e' - e''))^\perp = \mathsf{Fac}\, V$$

by Corollary 2.13(c). Thus $U \geqslant V$ holds. □

We end this section with the following application, which is an analogue of [HU05, Corollary 2.2].

Corollary 2.38. *If* $\mathrm{Q}(\mathrm{s}\tau\text{-tilt}\, \Lambda)$ *has a finite connected component* C*, then* $\mathrm{Q}(\mathrm{s}\tau\text{-tilt}\, \Lambda) = C$.

Proof. Fix T in C. Applying Theorem 2.35(a) to $\Lambda \geqslant T$, we have a sequence $T = T_0 < T_1 < T_2 < \cdots$ of right mutations of support τ-tilting modules such that $\Lambda \geqslant T_i$ for any i. Since C is finite, this sequence must be finite. Thus $\Lambda = T_i$ for some i, and Λ belongs to C. Now we fix any $U \in \mathrm{s}\tau\text{-tilt}\, \Lambda$. Applying Theorem 2.35(b) to $\Lambda \geqslant U$, we have a sequence $\Lambda = V_0 > V_1 > V_2 > \cdots$ of left mutations of support τ-tilting modules such that $V_i \geqslant U$ for any i. Since C is finite, this sequence must be finite. Thus $U = V_j$ for some j, and U belongs to C. □

3. Connection with silting theory

Throughout this section, let Λ be a finite-dimensional algebra over a field k. Any almost complete silting complex has infinitely many complements. But if we restrict to two-term silting complexes, we get another class of objects extending the (classical) tilting modules and satisfying the two-complement property (Corollary 3.8). Moreover, we will show that there is a bijection between

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support τ-tilting Λ-modules and two-term silting complexes for Λ, which is of independent interest (Theorem 3.2). The two-term silting complexes are defined as follows.

Definition 3.1. We call a complex $P = (P^i, d^i)$ in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ *two-term* if $P^i = 0$ for all $i \neq 0, -1$. Clearly, $P \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ is two-term if and only if $\Lambda \geqslant P \geqslant \Lambda[1]$.

We denote by 2-silt Λ (respectively, 2-presilt Λ) the set of isomorphism classes of basic two-term silting (respectively, presilting) complexes for Λ.

Clearly, any two-term complex is isomorphic to a two-term complex $P = (P^i, d^i)$ satisfying $d^{-1} \in \mathrm{rad}(P^{-1}, P^0)$ in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$. Moreover, for any two-term complexes P and Q, we have $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(P, Q[i]) = 0$ for any $i \neq -1, 0, 1$.

The aim of this section is to prove the following result.

Theorem 3.2. *Let Λ be a finite-dimensional k-algebra. Then there exists a bijection*

$$\text{2-silt}\,\Lambda \longleftrightarrow \text{s}\tau\text{-tilt}\,\Lambda$$

given by $\text{2-silt}\,\Lambda \ni P \mapsto H^0(P) \in \text{s}\tau\text{-tilt}\,\Lambda$ *and* $\text{s}\tau\text{-tilt}\,\Lambda \ni (M, P) \mapsto (P_1 \oplus P \xrightarrow{(f\ 0)} P_0) \in \text{2-silt}\,\Lambda$, *where $f : P_1 \to P_0$ is a minimal projective presentation of M.*

The following result is quite useful.

Proposition 3.3. *Let P be a two-term presilting complex for Λ.*

(a) *P is a direct summand of a two-term silting complex for Λ.*

(b) *P is a silting complex for Λ if and only if $|P| = |\Lambda|$.*

Proof. (a) This is shown in [Aih13, Proposition 2.16].

(b) The 'only if' part follows from Proposition 1.6(a). We will show the 'if' part. Let P be a two-term presilting complex for Λ with $|P| = |\Lambda|$. By (a), there exists a complex X such that $P \oplus X$ is silting. Then we have $|P \oplus X| = |\Lambda| = |P|$ by Proposition 1.6(a), so X is in $\mathsf{add}\,P$. Thus P is silting. □

The following lemma is important.

Lemma 3.4. *Let $M, N \in \mathsf{mod}\,\Lambda$. Let $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$ and $Q_1 \xrightarrow{q_1} Q_0 \xrightarrow{q_0} N \to 0$ be minimal projective presentations of M and N, respectively. Let $P = (P_1 \xrightarrow{p_1} P_0)$ and $Q = (Q_1 \xrightarrow{q_1} Q_0)$ be two-term complexes for Λ. Then the following conditions are equivalent.*

(a) $\mathrm{Hom}_\Lambda(N, \tau M) = 0$.

(b) $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(P, Q[1]) = 0$.

In particular, M is a τ-rigid Λ-module if and only if P is a presilting complex for Λ.

Proof. The condition (a) is equivalent to the fact that $(p_1, N) : \mathrm{Hom}_\Lambda(P_0, N) \to \mathrm{Hom}_\Lambda(P_1, N)$ is surjective by Proposition 2.4(b).

(a)⇒(b). Any morphism $f \in \mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(P, Q[1])$ is given by some $f \in \mathrm{Hom}_\Lambda(P_1, Q_0)$. Since (p_1, N) is surjective, there exists $g : P_0 \to N$ such that $q_0 f = g p_1$. Moreover, since P_0 is projective, there exists $h_0 : P_0 \to Q_0$ such that $q_0 h_0 = g$. Since $q_0(f - h_0 p_1) = 0$, we have $h_1 : P_1 \to Q_1$ with $f = q_1 h_1 + h_0 p_1$.

$$\begin{tikzcd}
 & 0 \arrow[r] & P_1 \arrow[r, "p_1"] \arrow[d, "f"] \arrow[dl, dotted, "h_1"'] & P_0 \arrow[r, "p_0"] \arrow[d, "g"] \arrow[dl, dotted, "h_0"'] & M \arrow[r] & 0 \\
0 \arrow[r] & Q_1 \arrow[r, "q_1"'] & Q_0 \arrow[r, "q_0"'] & N \arrow[r] & 0 &
\end{tikzcd}$$

Hence we have $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(P, Q[1]) = 0$.

(b)$\Rightarrow$(a). Take any $f \in \mathrm{Hom}_\Lambda(P_1, N)$. Since P_1 is projective, there exists $g : P_1 \to Q_0$ such that $q_0 g = f$.

$$\begin{tikzcd} & P_1 \arrow[r, "p_1"] \arrow[d, dotted, "g"'] \arrow[dr, "f"] & P_0 & \\ Q_1 \arrow[r, "q_1"'] & Q_0 \arrow[r, "q_0"'] & N \arrow[r] & 0 \end{tikzcd}$$

Then g gives a morphism $P \to Q[1]$ in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$. Since $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(P, Q[1]) = 0$, there exist $h_0 : P_0 \to Q_0$ and $h_1 : P_1 \to Q_1$ such that $g = q_1 h_1 + h_0 p_1$. Hence we have $f = q_0(q_1 h_1 + h_0 p_1) = q_0 h_0 p_1$. Therefore (p_1, N) is surjective. $\square$

We also need the following observation.

Lemma 3.5. *Let $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$ be a minimal projective presentation of M in $\mathsf{mod}\,\Lambda$ and $P := (P_1 \xrightarrow{p_1} P_0)$ be a two-term complex for Λ. Then, for any Q in $\mathsf{proj}\,\Lambda$, the following conditions are equivalent.*

(a) $\mathrm{Hom}_\Lambda(Q, M) = 0$.

(b) $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(Q, P) = 0$.

Proof. The proof is left to the reader, since it is straightforward. $\square$

The following result shows that silting complexes for Λ give support τ-tilting modules.

Proposition 3.6. *Let $P = (P_1 \xrightarrow{d} P_0)$ be a two-term complex for Λ and $M := \mathrm{Cok}\, d$.*

(a) *If P is a silting complex for Λ and d is right minimal, then M is a τ-tilting Λ-module.*

(b) *If P is a silting complex for Λ, then M is a support τ-tilting Λ-module.*

Proof. (b) We write $d = (d'\ 0) : P_1 = P_1' \oplus P_1'' \to P_0$, where d' is right minimal. Then the sequence $P_1' \xrightarrow{d'} P_0 \to M \to 0$ is a minimal projective presentation of M. We show that (M, P_1'') is a support τ-tilting pair for Λ. Since P is silting, M is a τ-rigid Λ-module by Lemma 3.4. On the other hand, since P is silting, we have $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(P_1'', P) = 0$. By Lemma 3.5, we have $\mathrm{Hom}_\Lambda(P_1'', M) = 0$. Thus (M, P_1'') is a τ-rigid pair for Λ. Since d' is a minimal projective presentation of M, we have $|M| = |P_1' \xrightarrow{d'} P_0|$. Thus we have

$$|M| + |P_1''| = |P_1' \xrightarrow{d'} P_0| + |P_1''| = |P|,$$

which is equal to $|\Lambda|$ by Proposition 1.6(a). Hence (M, P_1'') is a support τ-tilting pair for Λ.

(a) This is the case $P_1'' = 0$ in (b). $\square$

The following result shows that support τ-tilting Λ-modules give silting complexes for Λ.

Proposition 3.7. *Let $P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ be a minimal projective presentation of M in $\mathsf{mod}\,\Lambda$.*

(a) *If M is a τ-tilting Λ-module, then $(P_1 \xrightarrow{d_1} P_0)$ is a silting complex for Λ.*

(b) *If (M, Q) is a support τ-tilting pair for Λ, then $P_1 \oplus Q \xrightarrow{(d_1\ 0)} P_0$ is a silting complex for Λ.*

Proof. (b) We know that $(P_1 \xrightarrow{d_1} P_0)$ is a presilting complex for Λ by Lemma 3.4. Let $P := (P_1 \oplus Q \xrightarrow{(d_1\ 0)} P_0)$. By Lemma 3.5, we have that P is a presilting complex for Λ. Since d_1 is a

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minimal projective presentation, we have $|P_1 \xrightarrow{d_1} P_0| = |M|$. Moreover, since (M, Q) is a support τ-tilting pair for Λ, we have $|M| + |Q| = |\Lambda|$. Thus we have

$$|P| = |P_1 \xrightarrow{d_1} P_0| + |Q| = |M| + |Q| = |\Lambda|.$$

Hence P is a silting complex for Λ by Proposition 3.3(b).

(a) This is the case $Q = 0$ in (b). □

Now Theorem 3.2 follows from Propositions 3.6 and 3.7. □

We give some applications of Theorem 3.2.

Corollary 3.8. *Let Λ be a finite-dimensional k-algebra.*

(a) *Any basic two-term presilting complex P for Λ with $|P| = |\Lambda| - 1$ is a direct summand of exactly two basic two-term silting complexes for Λ.*

(b) *Let $P, Q \in$ 2-silt Λ. Then P and Q have all but one indecomposable direct summand in common if and only if P is a left or right mutation of Q.*

Proof. (a) This follows from Theorems 2.18 and 3.2.

(b) This is immediate from (a). □

Now we define $\mathrm{Q}(\text{2-silt}\,\Lambda)$ as the full subquiver of $\mathrm{Q}(\text{silt}\,\Lambda)$, with vertices corresponding to two-term silting complexes for Λ.

Corollary 3.9. *The bijection in Theorem 3.2 is an isomorphism of the partially ordered sets. In particular, it induces an isomorphism between the two-term silting quiver* $\mathrm{Q}(\text{2-silt}\,\Lambda)$ *and the support τ-tilting quiver* $\mathrm{Q}(\text{s}\tau\text{-tilt}\,\Lambda)$.

Proof. Let $(M, \Lambda e)$ and $(N, \Lambda f)$ be support τ-tilting pairs for Λ. Let $P := (P_1 \to P_0)$ and $Q := (Q_1 \to Q_0)$ be minimal projective presentations of M and N respectively. We only have to show that $M \geqslant N$ if and only if $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(P \oplus \Lambda e[1], (Q \oplus \Lambda f[1])[1]) = 0$.

We know that $M \geqslant N$ if and only if $\mathrm{Hom}_\Lambda(N, \tau M) = 0$ and $\Lambda e \in \mathsf{add}\,\Lambda f$ by Lemma 2.25. Moreover, $\mathrm{Hom}_\Lambda(N, \tau M) = 0$ if and only if $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(P, Q[1]) = 0$ by Lemma 3.4. On the other hand, $\Lambda e \in \mathsf{add}\,\Lambda f$ if and only if $\mathrm{Hom}_\Lambda(\Lambda e, N) = 0$, since N is a sincere $(\Lambda/\langle f\rangle)$-module. Thus $\Lambda e \in \mathsf{add}\,\Lambda f$ is equivalent to $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(\Lambda e, Q) = 0$ by Lemma 3.5. Consequently, $M \geqslant N$ if and only if $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(P \oplus \Lambda e[1], Q[1]) = 0$, and this is equivalent to $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(P \oplus \Lambda e[1], (Q \oplus \Lambda f[1])[1]) = 0$, since $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(P \oplus \Lambda e[1], \Lambda f[2]) = 0$ is automatic. Thus the assertion follows. □

Immediately, we have the following application.

Corollary 3.10. *If* $\mathrm{Q}(\text{2-silt}\,\Lambda)$ *has a finite connected component C, then* $\mathrm{Q}(\text{2-silt}\,\Lambda) = C$.

Proof. This is immediate from Corollaries 2.38 and 3.9. □

Note also that Theorem 3.2 and Corollary 3.9 give an alternative proof of Theorem 2.35, since the corresponding property for two-term silting complexes holds by [AI12, Proposition 2.36].

4. Connection with cluster-tilting theory

Let $\mathcal{C}$ be a Hom-finite Krull–Schmidt 2-Calabi–Yau (2-CY for short) triangulated category (for example, the cluster category $\mathcal{C}_Q$ associated with a finite acyclic quiver Q [BMRRT06]). We shall assume that our category $\mathcal{C}$ has a cluster-tilting object T. Associated with T, we have by definition

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the 2-CY tilted algebra $\Lambda = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$, whose module category is closely connected with the 2-CY triangulated category $\mathcal{C}$. In particular, there is an equivalence of categories [BMR07, KR07]:

$$\overline{(-)} := \mathrm{Hom}_{\mathcal{C}}(T, -) : \mathcal{C}/[T[1]] \to \mathsf{mod}\,\Lambda. \tag{10}$$

In this section, we investigate this relationship more closely by giving a bijection between cluster-tilting objects in $\mathcal{C}$ and support τ-tilting Λ-modules (Theorem 4.1). This was the starting point for the theory of τ-rigid and τ-tilting modules. As an application, we give a proof of some known results for cluster-tilting objects (Corollary 4.5). Also, we give a direct connection between cluster-tilting objects in $\mathcal{C}$ and two-term silting complexes for Λ (Theorem 4.7). There is an induced isomorphism between the associated graphs (Corollary 4.8).

4.1 Support τ-tilting modules and cluster-tilting objects

In this subsection, we show that there is a close relationship between the cluster-tilting objects in $\mathcal{C}$ and support τ-tilting Λ-modules. We use this to apply our main Theorem 0.4 to get a new proof of the fact that almost complete cluster-tilting objects have exactly two complements, and of the fact that all maximal rigid objects are cluster-tilting, as first proved in [IY08, ZZ11], respectively.

We denote by $\mathrm{iso}\,\mathcal{C}$ the set of isomorphism classes of objects in a category $\mathcal{C}$. From our equivalence (10), we have a bijection

$$\widetilde{(-)} : \mathrm{iso}\,\mathcal{C} \longleftrightarrow \mathrm{iso}\,(\mathsf{mod}\,\Lambda) \times \mathrm{iso}\,(\mathsf{proj}\,\Lambda)$$

given by $X = X' \oplus X'' \mapsto \widetilde{X} := (\overline{X'}, \overline{X''[-1]})$, where X'' is a maximal direct summand of X that belongs to $\mathsf{add}\,T[1]$. We denote by $\mathrm{rigid}\,\mathcal{C}$ (respectively, $\mathrm{m\text{-}rigid}\,\mathcal{C}$) the set of isomorphism classes of basic rigid (respectively, maximal rigid) objects in $\mathcal{C}$, and by $\mathrm{c\text{-}tilt}_{\,T}\mathcal{C}$ the set of isomorphism classes of basic cluster-tilting objects in $\mathcal{C}$ that do not have non-zero direct summands in $\mathsf{add}\,T[1]$.

Our main result in this section is the following.

THEOREM 4.1. *The bijection* $\widetilde{(-)}$ *induces bijections*

$$\mathrm{rigid}\,\mathcal{C} \longleftrightarrow \tau\text{-rigid}\,\Lambda, \quad \mathrm{c\text{-}tilt}\,\mathcal{C} \longleftrightarrow \mathrm{s}\tau\text{-tilt}\,\Lambda \quad \textit{and} \quad \mathrm{c\text{-}tilt}_{\,T}\mathcal{C} \longleftrightarrow \tau\text{-tilt}\,\Lambda.$$

Moreover, we have $\mathrm{c\text{-}tilt}\,\mathcal{C} = \mathrm{m\text{-}rigid}\,\mathcal{C} = \{U \in \mathrm{rigid}\,\mathcal{C} \mid |U| = |T|\}$.

We start with the following easy observation (see [KR07]).

LEMMA 4.2. *The functor* $\overline{(-)}$ *induces an equivalence of categories between* $\mathsf{add}\,T$ *(respectively,* $\mathsf{add}\,T[2]$*) and* $\mathsf{proj}\,\Lambda$ *(respectively,* $\mathsf{inj}\,\Lambda$*). Moreover, we have an isomorphism* $\overline{(-)} \circ [2] \simeq \nu \circ \overline{(-)} :$ $\mathsf{add}\,T \to \mathsf{inj}\,\Lambda$ *of functors.*

Now we express $\mathrm{Ext}^1_{\mathcal{C}}(X, Y)$ in terms of the images $\overline{X}$ and $\overline{Y}$ in our fixed 2-CY tilted algebra Λ. We let

$$\langle X, Y \rangle_\Lambda = \langle X, Y \rangle := \dim_k \mathrm{Hom}_\Lambda(X, Y).$$

PROPOSITION 4.3. *Let X and Y be objects in $\mathcal{C}$. Assume that there are no non-zero indecomposable direct summands of $T[1]$ for X and Y.*

(a) *We have* $\overline{X[1]} \simeq \tau\overline{X}$ *and* $\overline{Y[1]} \simeq \tau\overline{Y}$ *as Λ-modules.*

(b) *We have an exact sequence*

$$0 \to D\,\mathrm{Hom}_\Lambda(\overline{Y}, \tau\overline{X}) \to \mathrm{Ext}^1_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_\Lambda(\overline{X}, \tau\overline{Y}) \to 0.$$

(c) $\dim_k \mathrm{Ext}^1_{\mathcal{C}}(X, Y) = \langle \overline{X}, \tau\overline{Y} \rangle_\Lambda + \langle \overline{Y}, \tau\overline{X} \rangle_\Lambda$.

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Proof. (a) This can be shown as in the proof of [BMR07, Proposition 3.2]. Here, we give a direct proof. Take a triangle

$$T_1 \xrightarrow{g} T_0 \xrightarrow{f} X \longrightarrow T_1[1] \tag{11}$$

with a minimal right $(\mathsf{add}\,T)$-approximation f and $T_0, T_1 \in \mathsf{add}\,T$. Applying $\overline{(\ )}$ to (11), we have an exact sequence

$$\overline{T_1} \xrightarrow{\overline{g}} \overline{T_0} \xrightarrow{\overline{f}} \overline{X} \longrightarrow 0. \tag{12}$$

This gives a minimal projective presentation of $\overline{X}$, since X has no non-zero indecomposable direct summands of $T[1]$. Applying the Nakayama functor to (12) and $\mathrm{Hom}_{\mathcal{C}}(T,-)$ to (11) and comparing them by Lemma 4.2, we have the following commutative diagram of exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & \tau\overline{X} \arrow[r] & \nu\overline{T_1} \arrow[r, "\nu\overline{g}"] \arrow[d, "\wr"] & \nu\overline{T_0} \arrow[d, "\wr"] \\
0 = \overline{T_0[1]} \arrow[r] & \overline{X[1]} \arrow[r] & \overline{T_1[2]} \arrow[r, "\overline{g[2]}"] & \overline{T_0[2]}
\end{tikzcd}$$

Thus we have $\tau\overline{X} \simeq \overline{X[1]}$.

(b) We have an exact sequence

$$0 \to [T[1]](X, Y[1]) \to \mathrm{Hom}_{\mathcal{C}}(X, Y[1]) \to \mathrm{Hom}_{\mathcal{C}/[T[1]]}(X, Y[1]) \to 0,$$

where $[T[1]]$ is the ideal of $\mathcal{C}$ consisting of morphisms that factor through $\mathsf{add}\,T[1]$. We have a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{C}/[T[1]]}(X, Y[1]) \simeq \mathrm{Hom}_\Lambda(\overline{X}, \overline{Y[1]}) \overset{\text{(a)}}{\simeq} \mathrm{Hom}_\Lambda(\overline{X}, \tau\overline{Y}). \tag{13}$$

On the other hand, the first of the following functorial isomorphisms was given in [Pal08, 3.3].

$$[T[1]](X, Y[1]) \simeq D\,\mathrm{Hom}_{\mathcal{C}/[T[1]]}(Y, X[1]) \overset{(13)}{\simeq} D\,\mathrm{Hom}_\Lambda(\overline{Y}, \tau\overline{X}).$$

Thus the assertion follows.

(c) This is immediate from (b). $\square$

We now consider the general case, where we allow indecomposable direct summands from $T[1]$ in X or Y.

PROPOSITION 4.4. *Let $X = X' \oplus X''$ and $Y = Y' \oplus Y''$ be objects in $\mathcal{C}$ such that X'' and Y'' are the maximal direct summands of X and Y respectively, which belong to $\mathsf{add}\,T[1]$. Then*

$$\dim_k \mathrm{Ext}^1_{\mathcal{C}}(X, Y) = \langle \overline{X'}, \tau\overline{Y'}\rangle_\Lambda + \langle \overline{Y'}, \tau\overline{X'}\rangle_\Lambda + \langle \overline{X''[-1]}, \overline{Y'}\rangle_\Lambda + \langle \overline{Y''[-1]}, \overline{X'}\rangle_\Lambda.$$

Proof. Since $\mathrm{Ext}^1_{\mathcal{C}}(X'', Y'') = 0$, we have

$$\dim_k \mathrm{Ext}^1_{\mathcal{C}}(X, Y) = \dim_k \mathrm{Ext}^1_{\mathcal{C}}(X', Y') + \dim_k \mathrm{Ext}^1_{\mathcal{C}}(X'', Y') + \dim_k \mathrm{Ext}^1_{\mathcal{C}}(X', Y'').$$

By Proposition 4.3, the first term equals $\langle \overline{X'}, \tau\overline{Y'}\rangle_\Lambda + \langle \overline{Y'}, \tau\overline{X'}\rangle_\Lambda$. Clearly, the second term equals $\langle \overline{X''[-1]}, \overline{Y'}\rangle_\Lambda$, and the third term equals $\langle \overline{Y''[-1]}, \overline{X'}\rangle_\Lambda$. $\square$

Now we are ready to prove Theorem 4.1.

By Proposition 4.4, we have that X is rigid if and only if $\widetilde{X}$ is a τ-rigid pair for Λ. Thus we have a bijection $\text{rigid}\,\mathcal{C} \leftrightarrow \tau\text{-rigid}\,\Lambda$, which induces a bijection $\text{m-rigid}\,\mathcal{C} \leftrightarrow \text{s}\tau\text{-tilt}\,\Lambda$ by Corollary 2.13(a)⇔(b).

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On the other hand, we show that a bijection c-tilt $\mathcal{C} \leftrightarrow$ sτ-tilt Λ is induced. Since c-tilt $\mathcal{C} \subseteq$ m-rigid $\mathcal{C}$, we only have to show that any $X \in \operatorname{rigid} \mathcal{C}$ satisfying that $\widetilde{X}$ is a support τ-tilting pair for Λ is a cluster-tilting object in $\mathcal{C}$. Assume that $Y \in \mathcal{C}$ satisfies $\operatorname{Ext}^1_{\mathcal{C}}(X, Y) = 0$. By Proposition 4.4, we have $\operatorname{Hom}_\Lambda(\overline{X'}, \tau\overline{Y'}) = 0$, $\operatorname{Hom}_\Lambda(\overline{Y'}, \tau\overline{X'}) = 0$, $\operatorname{Hom}_\Lambda(\overline{X''[-1]}, \overline{Y'}) = 0$ and $\operatorname{Hom}_\Lambda(\overline{Y''[-1]}, \overline{X'}) = 0$. By the first three equalities, we have $\overline{Y'} \in \mathsf{add}\,\overline{X'}$ by Corollary 2.13(a)$\Leftrightarrow$(d). By the last equality, we have $\overline{Y''[-1]} \in \mathsf{add}\,\overline{X''[-1]}$. Thus $Y \in \mathsf{add}\,X$ holds, which shows that X is a cluster-tilting object in $\mathcal{C}$.

The remaining statements follow immediately. □

Now, we recover the following results in [IY08, ZZ11].

COROLLARY 4.5. *Let $\mathcal{C}$ be a 2-CY triangulated category with a cluster-tilting object T.*

(a) [IY08] *Any basic almost complete cluster-tilting object is a direct summand of exactly two basic cluster-tilting objects. In particular, T is a mutation of V if and only if T and V have all but one indecomposable direct summand in common.*

(b) [ZZ11] *An object X in $\mathcal{C}$ is cluster-tilting if and only if it is maximal rigid if and only if it is rigid and $|X| = |T|$.*

Proof. (a) This is immediate from the bijections given in Theorem 4.1 and the corresponding result for support τ-tilting pairs given in Theorem 2.18.

(b) This is the last equality in Theorem 4.1. □

Connections between cluster-tilting objects in $\mathcal{C}$ and tilting Λ-modules have been investigated in [FL09, Smi08]. It was shown that a tilting Λ-module always comes from a cluster-tilting object in $\mathcal{C}$, but the image of a cluster-tilting object is not always a tilting Λ-module. This is explained by Theorem 4.1 asserting that the Λ-modules corresponding to the cluster-tilting objects of $\mathcal{C}$ are the support τ-tilting Λ-modules, which are not necessarily tilting Λ-modules.

4.2 Two-term silting complexes and cluster-tilting objects

Throughout this section, let $\mathcal{C}$ be a 2-CY triangulated category with a cluster-tilting object T. Let $\Lambda := \operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ and let $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ be the homotopy category of bounded complexes of finitely generated projective Λ-modules. In this section, we shall show directly that there is a bijection between cluster-tilting objects in $\mathcal{C}$ and two-term silting complexes for Λ, and that the mutations are compatible with each other.

The following result will be useful, where we denote by $\mathsf{K}^2(\mathsf{proj}\,\Lambda)$ the full subcategory of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ consisting of two-term complexes for Λ.

PROPOSITION 4.6. *There exists a bijection*

$$\operatorname{iso} \mathcal{C} \longleftrightarrow \operatorname{iso} (\mathsf{K}^2(\mathsf{proj}\,\Lambda))$$

that preserves the number of non-isomorphic indecomposable direct summands.

Proof. For any object $U \in \mathcal{C}$, there exists a triangle

$$T_1 \xrightarrow{g} T_0 \xrightarrow{f} U \longrightarrow T_1[1],$$

where $T_1, T_0 \in \mathsf{add}\,T$ and f is a minimal right $(\mathsf{add}\,T)$-approximation. By Lemma 4.2, we have a two-term complex $\overline{T_1} \xrightarrow{\overline{g}} \overline{T_0}$ in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$.

Conversely, let $P_1 \xrightarrow{d} P_0$ be a two-term complex for Λ. By Lemma 4.2, there exists a morphism $g : T_1 \to T_0$ in $\mathsf{add}\,T$ such that $\overline{g} = d$. Taking the cone of g, we have an object U in $\mathcal{C}$. Then we

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can easily check that the correspondence gives a bijection and preserves the number of non-isomorphic indecomposable direct summands. □

Using this, we get the desired correspondence.

Theorem 4.7. *The bijection in Proposition 4.6 induces bijections*

$$\operatorname{rigid}\mathcal{C} \longleftrightarrow 2\text{-presilt}\,\Lambda \quad \textit{and} \quad \text{c-tilt}\,\mathcal{C} \longleftrightarrow 2\text{-silt}\,\Lambda.$$

Proof. (i) For any rigid object $U \in \mathcal{C}$, we have a triangle

$$T_1 \xrightarrow{g} T_0 \xrightarrow{f} U \xrightarrow{h} T_1[1],$$

where $T_1, T_0 \in \mathsf{add}\, T$ and f is a minimal right $(\mathsf{add}\, T)$-approximation. Let $a : T_1 \to T_0$ be an arbitrary morphism in $\mathcal{C}$. Since U is rigid, we have $fah[-1] = 0$. Thus we have a commutative diagram

$$\begin{tikzcd} U[-1] \arrow[r, "{h[-1]}"] \arrow[d] & T_1 \arrow[r, "g"] \arrow[d, "a"] & T_0 \arrow[r, "f"] \arrow[d, "b"] & U \arrow[d] \\ T_1 \arrow[r, "g"] & T_0 \arrow[r, "f"] & U \arrow[r, "h"] & T_1[1] \end{tikzcd}$$

of triangles in $\mathcal{C}$. Since $hb = 0$, there exists $k_0 : T_0 \to T_0$ such that $b = fk_0$. Since $f(a - k_0 g) = 0$, there exists $k_1 : T_1 \to T_1$ such that $gk_1 = a - k_0 g$. Therefore we have

$$\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}((\overline{T_1} \xrightarrow{\overline{g}} \overline{T_0}), (\overline{T_1} \xrightarrow{\overline{g}} \overline{T_0})[1]) = 0.$$

Thus $\overline{T_1} \xrightarrow{\overline{g}} \overline{T_0}$ is a presilting complex for Λ.

(ii) Let $P := (P_1 \xrightarrow{d} P_0)$ be a two-term presilting complex for Λ. There exists a unique $g : T_1 \to T_0$ in $\mathsf{add}\, T$ such that $\overline{g} = d$. We consider a triangle

$$T_1 \xrightarrow{g} T_0 \xrightarrow{f} U \xrightarrow{h} T_1[1]$$

in $\mathcal{C}$. We take a morphism $a : U \to U[1]$ in $\mathcal{C}$. Then we have the following commutative diagram.

$$\begin{tikzcd} T_1 \arrow[r, "g"] \arrow[d] & T_0 \arrow[r] \arrow[d, "{h[1]af}"] & 0 \arrow[d] \\ 0 \arrow[r] & T_1[2] \arrow[r, "{g[2]}"] & T_0[2] \end{tikzcd}$$

Applying $\overline{(-)}$, we have a commutative diagram.

$$\begin{tikzcd} P_1 \arrow[r, "d"] \arrow[d] & P_0 \arrow[r] \arrow[d, "\overline{h[1]af}"] & 0 \arrow[d] \\ 0 \arrow[r] & \nu P_1 \arrow[r, "\nu d"] & \nu P_0 \end{tikzcd}$$

Thus we have a morphism $P \to \nu P[-1]$ in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$. Since P is a presilting complex for Λ, we have

$$\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(P, \nu P[-1]) \simeq D \operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(P[-1], P) = 0.$$

Therefore $\overline{h[1]af} = 0$, and the morphism $h[1]af$ factors through $\mathsf{add}\, T[1]$. Hence we have $h[1]af = 0$. Thus we have a commutative diagram.

$$\begin{tikzcd}
T_1 \arrow[r, "g"] & T_0 \arrow[r, "f"] \arrow[d, "a_0"] & U \arrow[r, "h"] \arrow[d, "a"] & T_1[1] \\
T_1[1] \arrow[r, "g[1]"] & T_0[1] \arrow[r, "f[1]"] & U[1] \arrow[r, "h[1]"] & T_1[2]
\end{tikzcd}$$

Since $T_0 \in \mathsf{add}\, T$, we have $a_0 = 0$. Thus $af = 0$, so there exists $\varphi : T_1[1] \to U[1]$ such that $a = \varphi h$. Since $T_1 \in \mathsf{add}\, T$, we have $h[1]\varphi = 0$. Thus there exists $b : T_1[1] \to T_0[1]$ such that $\varphi = f[1]b$. Consequently, we have commutative diagrams.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & T_1 \arrow[r, "g"] \arrow[d, "{b[-1]}"] & T_0 \arrow[d] \\
T_1 \arrow[r, "g"] & T_0 \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] \arrow[d] & P_1 \arrow[r, "d"] \arrow[d, "\overline{b[-1]}"] & P_0 \arrow[d] \\
P_1 \arrow[r, "d"] & P_0 \arrow[r] & 0
\end{tikzcd}$$

Since P is a presilting complex for Λ, there exist $s : T_0[1] \to T_0[1]$ and $t : T_1[1] \to T_1[1]$ such that $b = sg[1] + g[1]t$. Therefore we have

$$a = \varphi h = f[1]bh = f[1]sg[1]h + f[1]g[1]th = 0.$$

Hence $\mathrm{Hom}_{\mathcal{C}}(U, U[1]) = 0$ (i.e. U is rigid) and the claim follows. □

COROLLARY 4.8. *The bijections in Theorems 3.2 and 4.7 induce isomorphisms of the following graphs.*

(a) *The underlying graph of the support τ-tilting quiver* $\mathrm{Q}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ *of* Λ.

(b) *The underlying graph of the two-term silting quiver* $\mathrm{Q}(2\text{-silt}\,\Lambda)$ *of* Λ.

(c) *The cluster-tilting graph* $\mathrm{G}(\text{c-tilt}\,\mathcal{C})$ *of* $\mathcal{C}$.

Proof. The graphs (a) and (b) are the same by Corollary 3.9.

We show that (b) and (c) are the same. Let U and V be cluster-tilting objects in $\mathcal{C}$. Let P and Q be the two-term silting complexes for Λ corresponding, respectively, to U and V by Theorem 4.7. By Corollary 4.5(a), the following conditions are equivalent.

(a) There exists an edge between U and V in the exchange graph.

(b) U and V have all but one indecomposable direct summand in common.

Clearly, (b) is equivalent to the following condition.

(c) P and Q have all but one indecomposable direct summand in common.

Now (c) is equivalent to the following condition by Corollary 3.8(b).

(d) There exists an edge between P and Q in the underlying graph of the silting quiver.

Therefore, the exchange graph of $\mathcal{C}$ and the underlying graph of the silting full subquiver consisting of two-term complexes for Λ coincide. □

We end this section with the following application.

COROLLARY 4.9. *If* $\mathrm{G}(\text{c-tilt}\,\mathcal{C})$ *has a finite connected component* C, *then* $\mathrm{G}(\text{c-tilt}\,\mathcal{C}) = C$.

Proof. This is immediate from Corollaries 2.38 and 4.8. □

5. Numerical invariants

In this section, we introduce g-vectors following [AR85, DK08]. We show that g-vectors of indecomposable direct summands of support τ-tilting modules form a basis of the Grothendieck group (Theorem 5.1). Moreover, we observe that non-isomorphic τ-rigid pairs have different g-vectors (Theorem 5.5). In [DWZ10], the authors defined what they called E-invariants of finite-dimensional decorated representations of Jacobian algebras, and used this to solve several conjectures from [FZ07]. In the case of finite-dimensional Jacobian algebras, they showed that the E-invariants were given by formulas that we were led to in § 4.1, by considering $\dim_k \operatorname{Ext}^1_{\mathcal{C}}(T, T)$ for a cluster-tilting object T in $\mathcal{C}$. We here consider E-invariants for any finite-dimensional algebra, using the same formula, and show that they can be expressed in terms of homomorphism spaces, dimension vectors and g-vectors. We give some further results on the case of 2-CY tilted algebras, including a comparison for neighbouring 2-CY tilted algebras (Theorem 5.7).

In the rest of this paper, we assume that our base field k is algebraically closed. Let Λ be a finite-dimensional k-algebra.

5.1 g-vectors and E-invariants for finite-dimensional algebras

Recall from [DK08] that the g-vectors are defined as follows. Let $K_0(\mathsf{proj}\,\Lambda)$ be the Grothendieck group of the additive category $\mathsf{proj}\,\Lambda$. Then the isomorphism classes $P(1), \ldots, P(n)$ of indecomposable projective Λ-modules form a basis of $K_0(\mathsf{proj}\,\Lambda)$. Consider M in $\mathsf{mod}\,\Lambda$ and let

$$P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be its minimal projective presentation in $\mathsf{mod}\,\Lambda$. Then we write

$$P_0 - P_1 = \sum_{i=1}^{n} g_i^M P(i),$$

where by definition $g^M = (g_1^M, \ldots, g_n^M)$ is the *g-vector* of M. The element $P_0 - P_1$ is also called an *index* of M, which was investigated in [AR85], in connection with studying modules determined by their composition factors, and in [DK08].

Another useful vector associated with M is the dimension vector $c^M = (c_1^M, \ldots, c_n^M)$. Denote by $S(i)$ the simple top of $P(i)$. Then c_i^M is, by definition, the multiplicity of the simple module $S(i)$ as composition factor of M. This vector has played an important role in cluster theory for the acyclic case, since the denominators of cluster variables are determined by dimension vectors of indecomposable rigid modules over path algebras [BMRT07, CK06]. Now, this result is not true in general [BMR06].

We have the following result on g-vectors of support τ-tilting modules.

Theorem 5.1. *Let (M, P) be a support τ-tilting pair for Λ with $M = \bigoplus_{i=1}^{\ell} M_i$ and $P = \bigoplus_{i=\ell+1}^{n} P_i$ with M_i and P_i indecomposable. Then $g^{M_1}, \ldots, g^{M_\ell}, g^{P_{\ell+1}}, \ldots, g^{P_n}$ form a basis of the Grothendieck group $K_0(\mathsf{proj}\,\Lambda)$.*

Proof. By Theorem 3.2, we have a corresponding silting complex $Q = \bigoplus_{i=1}^{n} Q_i$ for Λ with indecomposable Q_i, where the vectors $g^{M_1}, \ldots, g^{M_\ell}, g^{P_{\ell+1}}, \ldots, g^{P_n}$ are exactly the classes of $Q_1, \ldots, Q_n$ in the Grothendieck group $K_0(\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)) = K_0(\mathsf{proj}\,\Lambda)$. By Proposition 1.6(b), we have the assertion. □

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This gives a result below due to Dehy–Keller. Recall that for a cluster-tilting object $T \in \mathcal{C}$ and an object $X \in \mathcal{C}$, there exists a triangle

$$T'' \to T' \to X \to T''[1]$$

in $\mathcal{C}$ with $T', T'' \in \mathsf{add}\, T$. We call $\mathrm{ind}_T(X) := T' - T'' \in K_0(\mathsf{add}\, T)$ the *index* of X.

COROLLARY 5.2 [DK08, Theorem 2.4]. *Let $\mathcal{C}$ be a 2-CY triangulated category, and T and $U = \bigoplus_{i=1}^{n} U_i$ be basic cluster-tilting objects with U_i indecomposable. Then the indices $\mathrm{ind}_T(U_1), \ldots, \mathrm{ind}_T(U_n)$ form a basis of the Grothendieck group $K_0(\mathsf{add}\, T)$ of the additive category $\mathsf{add}\, T$.*

Proof. We can assume that $U_i \notin \mathsf{add}\, T[1]$ for $1 \leqslant i \leqslant \ell$, and $U_i \in \mathsf{add}\, T[1]$ for $\ell + 1 \leqslant i \leqslant n$. Then $(\bigoplus_{i=1}^{\ell} \overline{U_i}, \bigoplus_{i=\ell+1}^{n} \overline{U_i[-1]})$ is a support τ-tilting pair for Λ by Theorem 4.1. The equivalence $\mathrm{Hom}_{\mathcal{C}}(T, -) : \mathsf{add}\, T \to \mathsf{proj}\, \Lambda$ gives an isomorphism $K_0(\mathsf{add}\, T) \simeq K_0(\mathsf{proj}\, \Lambda)$. This sends $\mathrm{ind}_T(U_i)$ to $g^{\overline{U_i}}$ for $1 \leqslant i \leqslant \ell$, and to $-g^{\overline{U_i[-1]}}$ for $\ell + 1 \leqslant i \leqslant n$. Thus the assertion follows from Theorem 5.1. $\square$

Now, we consider a pair $M = (X, P)$ of a Λ-module X and a projective Λ-module P. We regard a Λ-module X as a pair $(X, 0)$. For such pairs $M = (X, P)$ and $N = (Y, Q)$, let

$$\begin{aligned} g^M &:= g^X - g^P, \\ E'_\Lambda(M, N) &:= \langle X, \tau Y \rangle + \langle P, Y \rangle, \\ E_\Lambda(M, N) &:= E'_\Lambda(M, N) + E'_\Lambda(N, M), \\ E_\Lambda(M) &:= E_\Lambda(M, M). \end{aligned}$$

We call g^M the *g-vector* of M, and $E_\Lambda(M, N)$ the *E-invariant* of M and N. Clearly, a pair (X, P) is τ-rigid if and only if $E_\Lambda(M) = 0$.

There is the following relationship between E-invariants and g-vectors, where we denote by $a \cdot b$ the standard inner product $\sum_{i=1}^{n} a_i b_i$ for vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$.

PROPOSITION 5.3. *Let Λ be a finite-dimensional algebra, and let X and Y be in $\mathsf{mod}\, \Lambda$. Then we have the following.*

$$\begin{aligned} E'_\Lambda(X, Y) &= \langle Y, X \rangle - g^Y \cdot c^X, \\ E_\Lambda(X, Y) &= \langle Y, X \rangle + \langle X, Y \rangle - g^Y \cdot c^X - g^X \cdot c^Y, \\ E_\Lambda(X) &= 2(\langle X, X \rangle - g^X \cdot c^X). \end{aligned}$$

Proof. We only have to show the first equality. Since $P_0 - P_1 = \sum_{i=1}^{n} g_i^Y P(i)$, then $\langle P_0, X \rangle - \langle P_1, X \rangle = g^Y \cdot c^X$. By Proposition 2.4(a), we have

$$E'_\Lambda(X, Y) = \langle X, \tau Y \rangle = \langle Y, X \rangle + \langle P_1, X \rangle - \langle P_0, X \rangle = \langle Y, X \rangle - g^Y \cdot c^X. \qquad \square$$

The following more general description of E-invariants is also clear.

PROPOSITION 5.4. *For any pair $M = (X, P)$ and $N = (Y, Q)$, we have*

$$E_\Lambda(M, N) = \langle Y, X \rangle + \langle X, Y \rangle - g^M \cdot c^Y - g^N \cdot c^X.$$

We end this subsection with the following analogue of [DK08, Theorem 2.3], which was also observed by Plamondon (private communication).

THEOREM 5.5. *The map $M \mapsto g^M$ gives an injection from the set of isomorphism classes of τ-rigid pairs for Λ to $K_0(\mathsf{proj}\, \Lambda)$.*

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Proof. The proof is based on Propositions 2.4(c) and 2.5, and is the same as that of [DK08, Theorem 2.3]. □

5.2 *E*-invariants for 2-CY tilted algebras

In the rest of this section, let $\mathcal{C}$ be a 2-CY triangulated k-category and let T be a cluster-tilting object in $\mathcal{C}$. Let $\Lambda := \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$. For any object $X \in \mathcal{C}$, we take a decomposition $X = X' \oplus X''$, where X'' is a maximal direct summand of X that belongs to $\mathsf{add}\, T[1]$, and define a pair by

$$\widetilde{X}_\Lambda := (\overline{X'}, \overline{X''[-1]}),$$

where $\overline{(-)}$ is an equivalence $\mathrm{Hom}_{\mathcal{C}}(T, -) : \mathcal{C}/[T[1]] \to \mathsf{mod}\, \Lambda$ given in (10).

We have the following interpretation of E-invariants.

Proposition 5.6. *We have* $E_\Lambda(\widetilde{X}_\Lambda, \widetilde{Y}_\Lambda) = \dim_k \mathrm{Ext}^1_{\mathcal{C}}(X, Y)$ *for any* $X, Y \in \mathcal{C}$.

Proof. This is immediate from Proposition 4.4 and our definition of E-invariants. □

Now let T' be a cluster-tilting mutation of T. Then we refer to the 2-CY tilted algebras $\Lambda = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ and $\Lambda' = \mathrm{End}_{\mathcal{C}}(T')^{\mathrm{op}}$ as *neighbouring* 2-CY tilted algebras. We define a pair $\widetilde{X}_{\Lambda'}$ for Λ' in a similar way to $\widetilde{X}_\Lambda$ by using the equivalence $\mathrm{Hom}_{\mathcal{C}}(T', -) : \mathcal{C}/[T'[1]] \to \mathsf{mod}\, \Lambda'$.

By our approach to the E-invariant, the following is now a direct consequence.

Theorem 5.7. *With the above notation, let M and N be objects in $\mathcal{C}$. Then* $E_\Lambda(\widetilde{M}_\Lambda, \widetilde{N}_\Lambda) = E_{\Lambda'}(\widetilde{M}_{\Lambda'}, \widetilde{N}_{\Lambda'})$.

Proof. This is clear from Proposition 5.6, since both sides are equal to $\dim_k \mathrm{Ext}^1_{\mathcal{C}}(M, N)$. □

In particular, $\widetilde{M}_\Lambda$ is τ-rigid if and only if $\widetilde{M}_{\Lambda'}$ is τ-rigid.

This result is analogous to the corresponding result for (neighbouring) Jacobian algebras proved in [DWZ10], in a larger generality. It is, however, not clear whether the two concepts of neighbouring algebras coincide for 2-CY tilted algebras which are Jacobian algebras. See [BIRS11] for more information.

6. Examples

In this section, we illustrate some of our work with easy examples.

Example 6.1. Let Λ be a local finite-dimensional k-algebra. Then we have $\mathrm{s}\tau\text{-tilt}\, \Lambda = \{\Lambda, 0\}$, since the condition $\mathrm{Hom}_\Lambda(M, \tau M) = 0$ implies either $M = 0$ or $\tau M = 0$ (i.e. M is projective). We have $\mathrm{Q}(\mathrm{s}\tau\text{-tilt}\, \Lambda) = (\ \Lambda \longrightarrow 0\ )$, $\mathrm{Q}(\text{f-tors}\, \Lambda) = (\ \mathsf{mod}\, \Lambda \longrightarrow 0\ )$ and $\mathrm{Q}(2\text{-silt}\, \Lambda) = (\ \Lambda \longrightarrow \Lambda[1]\ )$.

Example 6.2. Let Λ be a finite-dimensional k-algebra given by the quiver

$$1 \underset{a}{\overset{a}{\rightleftarrows}} 2,$$

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with relations $a^2 = 0$. Then Q(sτ-tilt Λ), Q(f-tors Λ) and Q(2-silt Λ) are as follows.

$$\begin{tikzcd}
\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[r] \arrow[d] & \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1 \arrow[r] & 1 \arrow[d] \\
2 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[r] & 2 \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{mod}\,\Lambda \arrow[r] \arrow[d] & \mathsf{add}\,(\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1) \arrow[r] & \mathsf{add}\,1 \arrow[d] \\
\mathsf{add}\,(2 \oplus \begin{smallmatrix}2\\1\end{smallmatrix}) \arrow[r] & \mathsf{add}\,2 \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
& \Lambda \arrow[r] \arrow[d] & \left[\begin{smallmatrix}2\\1\end{smallmatrix} \xrightarrow{[a\ 0]} \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix}\right] \arrow[r] & \left[\begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \xrightarrow{[a\ 0]} \begin{smallmatrix}1\\2\end{smallmatrix}\right] \arrow[d] \\
& \left[\begin{smallmatrix}1\\2\end{smallmatrix} \xrightarrow{[a\ 0]} \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix}\right] \arrow[r] & \left[\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \xrightarrow{[a\ 0]} \begin{smallmatrix}2\\1\end{smallmatrix}\right] \arrow[r] & \Lambda[1]
\end{tikzcd}$$

Example 6.3. Let Λ be a finite-dimensional k-algebra given by the quiver

$$\begin{tikzcd}
& 2 \arrow[dr, "a"] & \\
1 \arrow[ur, "a"] & & 3, \arrow[ll, "a"]
\end{tikzcd}$$

with relations $a^2 = 0$. Then Λ is a cluster-tilted algebra of type A_3, and there are 14 elements in c-tilt $\mathcal{C}$ for the cluster category $\mathcal{C}$ of type A_3. By our bijections, we know that there are 14 elements in each set sτ-tilt Λ, f-tors Λ and 2-silt Λ.

$$\begin{tikzcd}
& \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 2 \arrow[rr] \arrow[dr] & & \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 2 \arrow[dr] & & \\
& & \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 2 \arrow[rr] \arrow[dr] & & 2 \arrow[ddr] & \\
\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\end{smallmatrix} \arrow[uur] \arrow[r] \arrow[ddr] & \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1 \oplus \begin{smallmatrix}3\\1\end{smallmatrix} \arrow[rr] \arrow[dr] & & \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1 \arrow[dr] & & \\
& & \begin{smallmatrix}3\\1\end{smallmatrix} \oplus 1 \arrow[rr] \arrow[dr] & & 1 \arrow[r] & 0 \\
& 3 \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\end{smallmatrix} \arrow[rr] \arrow[dr] & & \begin{smallmatrix}3\\1\end{smallmatrix} \oplus 3 \arrow[dr] & & \\
& & \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3 \arrow[rr] \arrow[uuuuur] & & 3 \arrow[uur] &
\end{tikzcd}$$

Example 6.4. Let $\Lambda = kQ/\langle\beta\alpha\rangle$, where Q is the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. Then $T = S_1 \oplus P_1 \oplus P_3$ is a τ-tilting module that is not a tilting module. Here, S_i denotes the simple Λ-module associated with the vertex i, and P_i denotes the corresponding indecomposable projective Λ-module.

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In this case, there are 12 basic support τ-tilting Λ-modules, and $\mathrm{Q}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ is as follows.

$$\begin{tikzcd}
 & {\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 2} \arrow[r] \arrow[drr] & {\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 2} \arrow[r] \arrow[drr] & {\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1} \arrow[r] & 1 \arrow[dr] & \\
{\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3} \arrow[ur] \arrow[r] \arrow[dr] & {\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1 \oplus 3} \arrow[r] \arrow[urr] & 1 \oplus 3 \arrow[urr] \arrow[drr] & {\begin{smallmatrix}2\\3\end{smallmatrix} \oplus 2} \arrow[r] & 2 \arrow[r] & 0 \\
 & {\begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3} \arrow[urr] \arrow[rrr] & & & 3 \arrow[ur] &
\end{tikzcd}$$

We refer to [Ada13, Jas13, Miz13, Zha12] for more examples of support τ-tilting modules.

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