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FINITE GROUPS WITH HEREDITARILY *G*-PERMUTABLE SCHMIDT SUBGROUPS

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Abstract

A subgroup A of a group G is said to be hereditarily G-permutable with a subgroup B of G, if $AB^x = B^xA$ for some element $x \in \langle A, B\rangle$. A subgroup A of a group G is said to be hereditarily G-permutable in G if A is hereditarily G-permutable with every subgroup of G. In this paper, we investigate the structure of a finite group G with all its Schmidt subgroups hereditarily G-permutable.

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1. Introduction

All groups considered in the paper are finite.

Recall that a group G is said to be a *minimal nonnilpotent group* or *Schmidt group* if G is not nilpotent and every proper subgroup of G is nilpotent. It is clear that every nonnilpotent group contains Schmidt subgroups, and their embedding has a strong structural impact (see, for example, [2, 3, 10]).

However, the following extensions of permutability turn out to be important in the structural study of groups and were introduced by Guo *et al.* in [6].

Definition 1.1. Let A and B be subgroups of a group G.

(1) A is said to be *G-permutable* with B if there exists some $g \in G$ such that $AB^g = B^gA$.
(2) A is said to be *hereditarily G-permutable* with B (or *G-h-permutable* with B, for short) if there exists some $g \in \langle A, B\rangle$ such that $AB^g = B^gA$.

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(3) A is said to be *G-permutable* in G if A is G-permutable with all subgroups of G.
(4) A is said to be *hereditarily G-permutable* (or *G-h-permutable*, for short) in G if A is hereditarily G-permutable with all subgroups of G.

It is clear that permutability implies G-permutability but the converse does not hold in general as the Sylow 2-subgroups of the symmetric group of degree 3 show.

Our main goal here is to complete the structural study of groups in which every Schmidt subgroup of a group G is G-h-permutable. This study was started in [2] where we prove the following important fact.

THEOREM 1.2 [2, Theorem B]. *If every Schmidt subgroup of a group G is G-h-permutable in G, then G is soluble.*

Observe that the alternating group of degree 4 is a nonsupersoluble Schmidt group.

Let $p_1 > p_2 > \cdots > p_r$ be the primes dividing $|G|$ and let P_i be a Sylow p_i-subgroup of G, for each $i = 1, 2, \ldots, r$. Then we say that G is *a Sylow tower group of supersoluble type* if all subgroups $P_1, P_1P_2, \ldots, P_1P_2 \cdots P_{r-1}$ are normal in G. The class of all Sylow tower groups of supersoluble type is denoted by $\mathfrak{D}$.

Recall that if $\mathfrak{F}$ is a nonempty class of groups and π is a set of primes, then $\mathfrak{F}_\pi$ is the class of all π-groups in $\mathfrak{F}$. In particular, if p is a prime, then $\mathfrak{N}_p$ is the class of all p-groups and $\mathfrak{D}_{\pi(p-1)}$ is the class of all Sylow tower groups G of supersoluble type such that every prime dividing $|G|$ also divides $p - 1$.

If G is a group, then $\mathrm{Soc}(G)$ is the product of all minimal normal subgroups of G and $\Phi(G)$ is the Frattini subgroup of G, that is, the intersection of all maximal subgroups of G.

Our main goal here is to describe completely the groups G with trivial Frattini subgroup which have their Schmidt subgroups G-h-permutable.

THEOREM 1.3. *Let G be a group with $\Phi(G) = 1$. Assume that $\mathfrak{F} = LF(F)$ is the saturated formation locally defined by the canonical local definition F such that $F(p) = \mathfrak{N}_p\mathfrak{D}_{\pi(p-1)}$ for every prime p. If every Schmidt subgroup of G is G-h-permutable in G, then the following statements hold:*

(1) *$G = [\mathrm{Soc}(G)]M$ is the semidirect product of $\mathrm{Soc}(G)$ with an $\mathfrak{F}$-group M;*
(2) *if $\Phi(M) = 1$, then M is supersoluble.*

We shall adhere to the notation and terminology of [1, 4].

2. Definitions and preliminary results

Our first lemma collects some basic properties of G-h-permutable subgroups which are very useful in induction arguments. Its proof is straightforward.

LEMMA 2.1. *Let A, B and K be subgroups of G with K normal in G. Then, the following statements hold.*

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(1) *If A is G-h-permutable with B, then AK/K is G/K-h-permutable with BK/K in G/K.*
(2) *If $K \subseteq A$, then A/K is G/K-h-permutable with BK/K in G/K if and only if A is G-h-permutable with B in G.*
(3) *If A is G-h-permutable in G, then AK/K is G/K-h-permutable in G/K.*
(4) *If $A \subseteq B$ and A is G-h-permutable in G, then A is B-h-permutable in B.*

The following result describes the structure of Schmidt groups.

LEMMA 2.2 [5, 8]. *Let S be a Schmidt group. Then S satisfies the following properties:*

(1) *the order of S is divisible by exactly two prime numbers p and q;*
(2) *S is a semidirect product $S = [P]\langle a\rangle$, where P is a normal Sylow p-subgroup of S and $\langle a\rangle$ is a nonnormal Sylow q-subgroup of S and $\langle a^q\rangle \in \mathrm{Z}(S)$;*
(3) *P is the nilpotent residual of S, that is, the smallest normal subgroup of S with nilpotent quotient;*
(4) *$P/\Phi(P)$ is a noncentral chief factor of S and $\Phi(P) = P' \subseteq \mathrm{Z}(S)$;*
(5) *$\Phi(S) = \mathrm{Z}(S) = P' \times \langle a^q\rangle$;*
(6) *$\Phi(P)$ is the centraliser $C_P(a)$ of a in P;*
(7) *if $\mathrm{Z}(S) = 1$, then $|S| = p^m q$, where m is the order of p modulo q.*

In what follows, $\mathrm{Sch}(G)$ denotes the set of all Schmidt subgroups of a group G. Following [3], a Schmidt group with a normal Sylow p-subgroup will be called an $S_{\langle p,q\rangle}$*-group*.

The proof of Theorem 1.3 follows after a series of lemmas. They give us an interesting picture of the groups with supersoluble Schmidt subgroups.

LEMMA 2.3. *Let $\mathfrak{F} = \{H \mid \mathrm{Sch}(H) \subseteq \mathfrak{U}\}$, where $\mathfrak{U}$ is the class of all supersoluble groups. Then, $\mathfrak{F}$ satisfies the following properties:*

(1) *if $G \in \mathfrak{F}$, then G is a Sylow tower group of supersoluble type; in particular, G is a soluble group;*
(2) *$\mathfrak{F}$ is a subgroup-closed saturated Fitting formation;*
(3) *$\mathfrak{U} \subseteq \mathfrak{F}$;*
(4) *$\mathfrak{F} = LF(F)$, where F is the canonical local definition such that $F(p) = \mathfrak{N}_p\mathfrak{D}_{\pi(p-1)}$ for every prime p.*

PROOF. Statements (1), (2) and (3) follow from [7, Lemma 4 and Theorem 2].

Let $\mathfrak{H} = LF(F)$ be a local formation defined by the formation function F with $F(p) = \mathfrak{N}_p\mathfrak{D}_{\pi(p-1)}$ for every prime p. Assume that $\mathfrak{F} \not\subseteq \mathfrak{H}$. Let G be a group in $\mathfrak{F} \setminus \mathfrak{H}$ of minimal order. Since $\mathfrak{F}$ is a saturated formation, it follows that G is a primitive soluble group. Let $N = \mathrm{Soc}(G)$ be the unique minimal normal subgroup of G. Then $G/N \in \mathfrak{H}$. Since G is a Sylow tower group of supersoluble type and $\mathrm{C}_G(N) = N$, we see that N is a Sylow p-subgroup of G, where p is the largest prime dividing $|G|$.

Let $q \in \pi(G)$ with $q \neq p$ and let Q be a Sylow q-subgroup of G. Since $N = \mathrm{C}_G(N)$, it follows that PQ is not nilpotent. Hence, G has an $S_{\langle p,q\rangle}$-subgroup S, which is

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supersoluble p-closed because $G \in \mathfrak{F}$. Then, by statements (4) and (5) of Lemma 2.2, $|S/\operatorname{Z}(S)| = pq$ and therefore, by statement (7) of Lemma 2.2, q divides $p-1$. Since G is a Sylow tower group of supersoluble type, it follows that

$$G/N = G/\operatorname{C}_G(N) \in \mathfrak{D}_{\pi(p-1)},$$

and thus $G \in \mathfrak{H}$, which is a contradiction. Hence, $\mathfrak{F} \subseteq \mathfrak{H}$.

Assume that $\mathfrak{F} \neq \mathfrak{H}$, and let G be a group in $\mathfrak{H} \setminus \mathfrak{F}$ of minimal order. Since $\mathfrak{H}$ is a saturated formation and $F(p)$ is a formation of soluble groups for all primes p, it follows that G is a primitive soluble group. Let N be a unique minimal normal subgroup of G. The choice of G yields $G \in \mathfrak{H}$ and $G/N \in \mathfrak{F}$. Since G is soluble, N is a p-group for some prime p, and from $G \in \mathfrak{H}$, it follows that

$$G/N = G/\operatorname{C}_G(N) \in \mathfrak{N}_p\mathfrak{D}_{\pi(p-1)}.$$

We conclude that $G/N \in \mathfrak{D}_{\pi(p-1)}$ because $\operatorname{O}_p(G/N) = 1$ by [4, Lemma A.13.6].

Let S be an $S_{\langle r,q\rangle}$-subgroup of G. If $r \neq p$, then S is contained in some Hall p'-subgroup H of G. Since $H \cong G/N \in \mathfrak{F}$, we see that $S \in \mathfrak{U}$. If $r = p$, then from $G/N \in \mathfrak{D}_{\pi(p-1)}$, it follows that q divides $p-1$. Thus, by Lemma 2.2, $S \in \mathfrak{U}$. Consequently, every Schmidt subgroup of G is supersoluble, which is a contradiction. Hence, $\mathfrak{F} = \mathfrak{H}$. □

The following examples show that groups in Lemma 2.3 may not be supersoluble.

Example 2.4. Let

$$Q = \langle a, b \mid a^4 = b^4 = 1,\ a^2 = b^2,\ b^{-1}ab = a^{-1}\rangle$$

be the quaternion group of order 8. Then G has a faithful and irreducible module A over the field of 5 elements of dimension 2. Let $G = [A]Q$ be the corresponding semidirect product. Then G is not supersoluble and $C = [A]\langle a\rangle$ and $D = [A]\langle b\rangle$ are supersoluble and normal subgroups of $G = CD$. By Lemma 2.3, $G \in \mathfrak{F} = \{H \mid \operatorname{Sch}(H) \subseteq \mathfrak{U}\}$.

Example 2.5. Assume that M is a nonabelian group of order 21. Then M has a faithful and irreducible module N over GF(43), the field of 43 elements (see, for example, [4, Corollary B.11.8]). Consider the semidirect product $G = [N]M$. It is obvious that G is not supersoluble. By Lemma 2.3, $G \in \mathfrak{F} = \{H \mid \operatorname{Sch}(H) \subseteq \mathfrak{U}\}$.

The following result is of interest although it is not needed for the proof of Theorem 1.3.

Proposition 2.6. *Let $\mathfrak{F} = \{H \mid \operatorname{Sch}(H) \subseteq \mathfrak{U}\}$. Then, for every $n \in \mathbb{N}$, there exists a group $G \in \mathfrak{F}$ of nilpotent length n.*

Proof. Let $n \geq 2$ and let $p_1, p_2, \ldots, p_n$ be primes such that $p_1 < p_2 < \cdots < p_n$ and p_i divides $p_j - 1$ for all $i < j$, where $i = 1, 2, \ldots, n-1$, $j = 2, \ldots, n$. By Dirichlet's theorem, there exists an infinite set of primes of the form

$$p_1 p_2 \cdots p_{n_0} + 1,$$

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where $n_0 \in \mathbb{N}$. Assume that p_{n+1} is one of them. It is obvious that p_i divides $p_{n+1} - 1$ for any $i = 1, 2, \ldots, n$.

Assume that G_1 is a cyclic group of order p_1. Assume that $i \geq 2$ and G_{i-1} is in $\mathfrak{F}$ and of nilpotent length $i - 1$. By [4, Corollary B.11.8], G_{i-1} has a faithful and irreducible module V_{p_i} over the field of p_i elements. Let $G_i = [V_{p_i}]G_{i-1}$ be the corresponding semidirect product. Then $\mathrm{F}(G_i) = V_{p_i}$ and hence the nilpotent length of G_i is equal to i. Furthermore, by Lemma 2.3, $G_i \in \mathfrak{F}$. In particular, G_n is an $\mathfrak{F}$-group of nilpotent length n. □

The following subgroup embedding property was introduced by Vasil'ev, Vasil'eva and Tyutyanov in [9].

Definition 2.7. A subgroup H of a group G is said to be $\mathbb{P}$-*subnormal* in G if there exists a chain of subgroups

$$H = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_{n-1} \subseteq H_n = G$$

such that for every $i = 1, 2, \ldots, n$, either $|H_i : H_{i-1}| \in \mathbb{P}$ or H_{i-1} is normal in H_i.

Note that $\mathbb{P}$-subnormality coincides with K-$\mathfrak{U}$-subnormality (see [1, Ch. 6]) in the soluble universe.

Lemma 2.8. *Let A be a G-h-permutable subgroup of a soluble group G. Then, A is $\mathbb{P}$-subnormal in G. In particular, the supersoluble residual $A^{\mathfrak{U}}$ of A is subnormal in G.*

Proof. Let G be a group of smallest order for which the lemma is not true, and let L be a minimal normal subgroup of G. Since G is soluble, $|L| = p^n$ for some prime $p \in \pi(G)$ and $n \geq 1$. Suppose that $G = AL$. Then A is a maximal subgroup of G and $A \cap L = 1$. Let L_1 be a subgroup of prime order of L. Then, $AL_1^x = L_1^x A$ for some $x \in G$. Consequently, AL_1^x is a subgroup of G. Since A is maximal in G and $A \neq AL_1^x$, we see that $AL_1^x = G$. Because

$$|G : A| = |AL_1^x|/|A| = |L_1^x|/|A \cap L_1^x| = |L_1^x|,$$

we conclude that $|G : A| = p$ and then A is $\mathbb{P}$-subnormal in G, which is a contradiction. Hence, $G \neq AL$. Since $|AL| < |G|$, by Lemma 2.1, it follows that A is $\mathbb{P}$-subnormal in AL. By Lemma 2.1, AL/L is (G/L)-h-permutable in G/L, and from $|G/L| < |G|$, it follows that AL/L is $\mathbb{P}$-subnormal in G/L. In particular, AL is $\mathbb{P}$-subnormal in G by [1, Lemma 6.1.6]. However, then A is a $\mathbb{P}$-subnormal subgroup of G by [1, Lemma 6.1.7], which is a contradiction. Consequently, A is $\mathbb{P}$-subnormal in G. Applying [1, Lemma 6.1.9], we conclude that $A^{\mathfrak{U}}$ is subnormal in G. □

Example 2.9. Let G be a group isomorphic to the alternating group of degree 6. Since G does not have maximal subgroups of prime index, the identity subgroup 1 of G is G-h-permutable but not $\mathbb{P}$-subnormal in G. Thus, the solubility of the group G in Lemma 2.8 is essential.

Lemma 2.10. *Let $G \in \mathfrak{F} = \{H \mid \mathrm{Sch}(H) \subseteq \mathfrak{U}\}$. If $\Phi(G) = 1$ and every Schmidt subgroup of G is G-h-permutable in G, then G is supersoluble.*

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PROOF. We argue by induction on $|G|$. Let N be a minimal normal subgroup of G. Since G is soluble by Lemma 2.3, it follows that N is p-elementary abelian for some prime p. Since $\Phi(G) = 1$, it follows that $G = NM$ for some maximal subgroup M of G and $N \cap M = 1$.

Suppose that $NM_{p'}$ is p-nilpotent. Then $NM_{p'} \subseteq C_G(N)$. Then $G/C_G(N)$ is a p-group. Since $O_p(G/C_G(N)) = 1$ by [4, Lemma A.13.6], we have $N \subseteq Z(G)$. Then $G = N \times M$. Now, M belongs to $\mathfrak{F}$ and $\Phi(M) \subseteq \Phi(G) = 1$ by [4, Theorem A.9.2]. By induction, M is supersoluble. Hence, G is supersoluble.

Assume that $NM_{p'}$ is not p-nilpotent. Consequently, $NM_{p'}$ contains a minimal non-p-nilpotent group X. By [1, Corollary 6.4.5], X is an $S_{\langle p,q\rangle}$-subgroup $X = [P]Q$ and $P \subseteq N$. We can assume without loss of generality that $Q \subseteq M_{p'}$. Since the subgroup $[P]Q$ is G-h-permutable, we may assume that $([P]Q)M = PM$ is a subgroup of G. Consequently, $P = N$ and NQ is an $S_{\langle p,q\rangle}$-subgroup G. By hypothesis, NQ is supersoluble. Hence, in view of Lemma 2.2, $|N/\Phi(N)| = p$ by Lemma 2.2. Since $\Phi(N) = 1$, it follows that $|N| = p$.

Consequently, we may assume that every minimal normal subgroup of G is cyclic. Then, by [4, Theorem A.10.6], $F(G)$ is a direct product of normal subgroups of G of prime order and so $G/C_G(F(G))$ is abelian. Since $C_G(F(G)) \subseteq F(G)$ by [4, Theorem A.10.6], it follows that $G/F(G)$ is abelian. In particular, G is supersoluble. □

3. Proof of Theorem 1.3

Since G is soluble and $\Phi(G) = 1$, we conclude that $F(G) = \mathrm{Soc}(G)$ and $G = [\mathrm{Soc}(G)]M$ for some subgroup M of G, that is, $\mathrm{Soc}(G) \cap M = 1$ by [4, Theorem A.10.6].

Let S be a Schmidt subgroup of M. Suppose that S is an $S_{\langle p,q\rangle}$-subgroup. Then, by hypothesis, S is G-h-permutable in G. Consequently, by Lemma 2.8, S is $\mathbb{P}$-subnormal in G and $S^{\mathfrak{U}}$ is subnormal in G. In view of Lemma 2.2, we see that either $S^{\mathfrak{U}} \neq 1$ is a p-subgroup of S or $S^{\mathfrak{U}} = 1$. Assume that $S^{\mathfrak{U}} \neq 1$. Then,

$$S^{\mathfrak{U}} \subseteq F(G) \cap M = 1,$$

which is a contradiction. Therefore, every Schmidt subgroup of M is supersoluble.

By Lemma 2.3, it follows that $M \in \mathfrak{F} = LF(F)$, where F is the formation function given by $F(r) = \mathfrak{N}_r\mathfrak{D}_{\pi(r-1)}$ for any prime r.

By Lemma 2.10, M is supersoluble provided that $\Phi(M) = 1$.

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