POSITIVE SOLUTIONS FOR A CLASS OF \(p(x)\)-LAPLACIAN PROBLEMS

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Abstract. We consider the system

\[
\begin{align*}
-\Delta_{p(x)} u &= \lambda_1 f(v) + \mu_1 h(u) \quad \text{in } \Omega \\
-\Delta_{q(x)} v &= \lambda_2 g(u) + \mu_2 \gamma(v) \quad \text{in } \Omega, \\
u &= v = 0 \quad \text{on } \partial \Omega
\end{align*}
\]

where \(p(x), q(x) \in C^1(R^N)\) are radial symmetric functions such that \(\sup |\nabla p(x)| < \infty, \sup |\nabla q(x)| < \infty\) and \(1 < \inf p(x) \leq \sup p(x) < \infty, 1 < \inf q(x) \leq \sup q(x) < \infty\), where \(-\Delta_{p(x)} u = -\text{div}(\nabla u |\nabla p(x)|^{-2} \nabla u), -\Delta_{q(x)} v = -\text{div}(\nabla v |\nabla q(x)|^{-2} \nabla v)\), respectively are called \(p(x)\)-Laplacian and \(q(x)\)-Laplacian, \(\lambda_1, \lambda_2, \mu_1\) and \(\mu_2\) are positive parameters and \(\Omega = B(0, R) \subset R^N\) is a bounded radial symmetric domain, where \(R\) is sufficiently large. We prove the existence of a positive solution when

\[
\lim_{u \to +\infty} \frac{f(M g(u))^{\frac{1}{p(x)-1}}}{u^{p(x)-1}} = 0,
\]

for every \(M > 0\), \(\lim_{u \to +\infty} \frac{h(u)}{u^{q(x)-1}} = 0\) and \(\lim_{u \to +\infty} \frac{\gamma(u)}{u^{q(x)-1}} = 0\). In particular, we do not assume any sign conditions on \(f(0), g(0), h(0)\) or \(\gamma(0)\).

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1. Introduction. The study of differential equations and variational problems with non-standard \(p(x)\)-growth conditions has been a new and interesting topic. Many results have been obtained on this kind of problem, for example, [3–8, 10, 11, 13]. In [6, 7] Fan and Zhao give the regularity of weak solutions for differential equations with non-standard \(p(x)\)-growth conditions. Zhang in [12] investigated the existence of positive solutions of the system

\[
\begin{align*}
-\Delta_{p(x)} u &= f(v) \quad \text{in } \Omega \\
-\Delta_{q(x)} v &= g(u) \quad \text{in } \Omega, \\
u &= v = 0 \quad \text{on } \partial \Omega
\end{align*}
\]
where \( p(x) \in C^1(R^N) \) is a function and \( \Omega \subset R^N \) is a bounded domain. The operator \(-\Delta_{p(x)} u = -\text{div}(|\nabla u|^{p(x)-2}\nabla u)\) is called \( p(x) \)-Laplacian. Especially, if \( p(x) \equiv p \) (a constant), (1) is the well-known \( p \)-Laplacian systems. There are many papers on the existence of solutions for \( p \)-Laplacian elliptic systems, for example, [1–9].

In [9] the authors consider the existence of positive weak solutions for the following \( p \)-Laplacian problems:

\[
\begin{aligned}
-\Delta_p u &= f(v) \quad \text{in } \Omega \\
-\Delta_p v &= g(u) \quad \text{in } \Omega, \\
u &= v = 0 \quad \text{on } \partial \Omega \\
\end{aligned}
\]

(2)

The first eigenfunction is used for constructing the subsolution of \( p \)-Laplacian problems successfully. On the condition of

\[
\lim_{u \to +\infty} \frac{f(M(g(u))^{\frac{1}{p-1}})}{u^{p-1}} = 0, \quad \forall M > 0,
\]

the authors show the existence of positive solutions for problem (2).

In this paper, we mainly consider the existence of positive solutions of the system

\[
\begin{aligned}
-\Delta_{p(x)} u &= \lambda_1 f(v) + \mu_1 h(u) \quad \text{in } \Omega \\
-\Delta_{q(x)} v &= \lambda_2 g(u) + \mu_2 \gamma(v) \quad \text{in } \Omega, \\
u &= v = 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

(3)

where \( p(x), q(x) \in C^1(R^N) \) are functions, \( \lambda_1, \lambda_2, \mu_1, \mu_2 \) are positive parameters and \( \Omega \subset R^N \) is a bounded domain.

In order to deal with \( p(x) \)-Laplacian problems, we need some theories on spaces \( L^{p(x)}(\Omega) \), and \( W^{1,p(x)}(\Omega) \) and properties of \( p(x) \)-Laplacian which we will use later (see [8]). If \( \Omega \subset R^N \) is an open domain, then

\[
C_+(\Omega) = \{ h \mid h \in C(\Omega), h(x) > 1 \text{ for } x \in \Omega \},
\]

\[
h^+ = \sup_{x \in \Omega} h(x), \quad h^- = \inf_{x \in \Omega} h(x), \quad \text{for any } h \in C(\Omega),
\]

\[
L^{p(x)}(\Omega) = \{ u \mid u \text{ is a measurable real-valued function, } \int_\Omega |u|^{p(x)} dx < \infty \}.
\]

Throughout the paper, we will assume that \( p, q \in C_+(\Omega) \) and

\[
1 < \inf_{x \in R^N} p(x) \leq \sup_{x \in R^N} p(x) < N,
\]

\[
1 < \inf_{x \in R^N} q(x) \leq \sup_{x \in R^N} q(x) < N.
\]

We can introduce the norm on \( L^{p(x)}(\Omega) \) by

\[
|u|_{p(x)} = \inf \{ \lambda > 0 \mid \int_\Omega \frac{|u|}{\lambda}^{p(x)} dx \leq 1 \},
\]

and \((L^{p(x)}(\Omega), \cdot \cdot_{p(x)})\) becomes a Banach space, which we call generalised Lebesgue space.

The space \((L^{p(x)}(\Omega), \cdot \cdot_{p(x)})\) is a separable, reflexive and uniformly convex Banach space (see [8, Theorem 1.10, 1.14]).

The space \(W^{1,p(x)}(\Omega)\) is defined by

\[
W^{1,p(x)}(\Omega) = \{ u \in L^{p(x)}(\Omega) ||\nabla u||_{p(x)} \in L^{p(x)}(\Omega) \}.
\]
and it can be equipped with the norm
\[ ||u|| = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega). \]

We denote by \( W^{1,p(x)}_0(\Omega) \) the closure of \( C_0^\infty(\Omega) \) in \( W^{1,p(x)}(\Omega) \). \( W^{1,p(x)}(\Omega) \) and \( W^{1,p(x)}_0(\Omega) \) are separable, reflexive and uniformly convex Banach spaces (see \([8, \text{Theorem 2.1}]\)). We define that if
\[ (L(u), v) = \int_{\Omega^N} |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx, \quad \forall u, v \in W^{1,p(x)}(\Omega), \]
then \( L : W^{1,p(x)}(\Omega) \to (W^{1,p(x)}(\Omega))^* \) is a continuous, bounded and strictly monotone operator and is also a homeomorphism (see \([4, \text{Theorem 3.11}]\)). If \( u, v \in W^{1,p(x)}_0(\Omega) \), \( (u, v) \) is called a weak solution of (3) which satisfies
\[
\begin{align*}
\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \xi \, dx &= \int_\Omega (\lambda_1 f(u) + \mu_1 h(u)) \xi \, dx, \quad \forall \xi \in W^{1,p(x)}_0(\Omega), \\
\int_\Omega |\nabla v|^{q(x)-2} \nabla v \nabla \xi \, dx &= \int_\Omega (\lambda_2 g(u) + \mu_2 \gamma(v)) \xi \, dx, \quad \forall \xi \in W^{1,p(x)}_0(\Omega).
\end{align*}
\]

We make the following assumptions

(H.1) \( p(x), q(x) \in C^1(R^N) \) are radial symmetric and \( \sup |\nabla p(x)| < \infty, \sup |\nabla q(x)| < \infty \).

(H.2) \( \Omega = B(0, R) = \{x ||x| < R\} \) is a ball, where \( R > 0 \) is a sufficiently large constant.

(H.3) \( f, g, h, \gamma : [0, \infty) \to R \) are \( C^1 \), monotone functions such that
\[
\lim_{u \to +\infty} f(u) = \lim_{u \to +\infty} g(u) = \lim_{u \to +\infty} h(u) = \lim_{u \to +\infty} \gamma(u) = +\infty.
\]

(H.4) \( \lim_{u \to +\infty} \frac{f(Mg(u)^{1-1/p})}{u^{1-1/p}} = 0 \), for every \( M > 0 \).

(H.5) \( \lim_{u \to +\infty} \frac{h(u)}{u^{1-1/p}} = \lim_{u \to +\infty} \gamma(u) = 0 \).

We shall establish the following theorem.

2. Main results.

THEOREM 1. If (H.1)–(H.5) hold, then (3) has a positive solution.

Proof. We shall establish Theorem 1 by constructing a positive subsolution \((\phi_1, \phi_2)\) and supersolution \((z_1, z_2)\) of (3), such that \( \phi_1 \leq z_1 \) and \( \phi_2 \leq z_2 \). That is, \((\phi_1, \phi_2)\) and \((z_1, z_2)\) satisfy
\[
\begin{align*}
&\int_\Omega |\nabla \phi_1|^{p(x)-2} \nabla \phi_1 \cdot \nabla \xi \, dx \leq \lambda_1 \int_\Omega f(\phi_2) \xi \, dx + \mu_1 \int_\Omega h(\phi_1) \xi \, dx, \\
&\int_\Omega |\nabla \phi_2|^{q(x)-2} \nabla \phi_1 \cdot \nabla \xi \, dx \leq \lambda_2 \int_\Omega g(\phi_1) \xi \, dx + \mu_2 \int_\Omega \gamma(\phi_2) \xi \, dx,
\end{align*}
\]
and
\[
\begin{align*}
&\int_\Omega |\nabla \phi_1|^{p(x)-2} \nabla \phi_1 \cdot \nabla \xi \, dx \geq \lambda_1 \int_\Omega f(\phi_2) \xi \, dx + \mu_1 \int_\Omega h(\phi_1) \xi \, dx, \\
&\int_\Omega |\nabla \phi_2|^{q(x)-2} \nabla \phi_1 \cdot \nabla \xi \, dx \geq \lambda_2 \int_\Omega g(\phi_1) \xi \, dx + \mu_2 \int_\Omega \gamma(\phi_2) \xi \, dx,
\end{align*}
\]
for all \( \xi \in W^{1,p(x)}_0(\Omega) \) with \( \xi \geq 0 \). Then (3) has a positive solution.
**Step 1.** We construct a subsolution of (3).

Denote

\[
as_1 = \inf \frac{p(x) - 1}{4(\sup |\nabla p(x)| + 1)}, \quad R_1 = \frac{R - a_1}{2}, \quad a_2 = \inf \frac{q(x) - 1}{4(\sup |\nabla q(x)| + 1)}, \quad R_2 = \frac{R - a_2}{2},\n\]

and let \(k_0 > 0\) such that \(f(u), g(u), h(u), \gamma(u) \geq -k_0\) for all \(u \geq 0\), and let

\[
\phi_1(r) = \begin{cases} 
    e^{-k(r-R)} - 1, & 2R_1 < r \leq R, \\
    e^{a_1k} - 1 + \int_{r}^{2R_1} (ke^{a_1k})^{\frac{p(R_1)^{\gamma-1}}{p(R_1)}} \left[ \frac{(2R_1)^{\gamma-1}}{p(R_1)} \sin \left( \frac{\varepsilon_1(r - 2R_1) + \frac{\pi}{2}}{\frac{\pi}{2\varepsilon_1}} \right) k_0(\lambda_1 + \mu_1) \right] \frac{1}{\varepsilon_1} dr, \\
    e^{a_2k} - 1 + \int_{2R_1}^{2R_2} (ke^{a_2k})^{\frac{p(R_2)^{\gamma-1}}{p(R_2)}} \left[ \frac{(2R_2)^{\gamma-1}}{p(R_2)} \sin \left( \frac{\varepsilon_2(r - 2R_2) + \frac{\pi}{2}}{\frac{\pi}{2\varepsilon_2}} \right) k_0(\lambda_2 + \mu_2) \right] \frac{1}{\varepsilon_2} dr, \\
    0, & r \leq 2R_1 - \frac{\pi}{2\varepsilon_1},
\end{cases}
\]

where \(R_1\) is sufficiently large and \(\varepsilon_1\) is a small positive constant which satisfies

\[
R_1 \leq 2R_1 - \frac{\pi}{2\varepsilon_1},
\]

and let

\[
\phi_2(r) = \begin{cases} 
    e^{-k(r-R)} - 1, & 2R_2 < r \leq R, \\
    e^{a_2k} - 1 + \int_{r}^{2R_2} (ke^{a_2k})^{\frac{p(R_2)^{\gamma-1}}{p(R_2)}} \left[ \frac{(2R_2)^{\gamma-1}}{p(R_2)} \sin \left( \frac{\varepsilon_2(r - 2R_2) + \frac{\pi}{2}}{\frac{\pi}{2\varepsilon_2}} \right) k_0(\lambda_2 + \mu_2) \right] \frac{1}{\varepsilon_2} dr, \\
    0, & 2R_2 - \frac{\pi}{2\varepsilon_2} < r \leq 2R_2,
\end{cases}
\]

where \(R_2\) is sufficiently large and \(\varepsilon_2\) is a small positive constant which satisfies

\[
R_2 \leq 2R_2 - \frac{\pi}{2\varepsilon_2}.
\]

In the following, we will prove that \((\phi_1, \phi_2)\) is a subsolution of (3).

Since

\[
\phi'_1(r) = \begin{cases} 
    e^{-k(r-R)} - 1, & 2R_1 < r \leq R, \\
    -(ke^{a_1k})^{\frac{p(R_1)^{\gamma-1}}{p(R_1)}} \left[ \frac{(2R_1)^{\gamma-1}}{p(R_1)} \sin \left( \frac{\varepsilon_1(r - 2R_1) + \frac{\pi}{2}}{\frac{\pi}{2\varepsilon_1}} \right) k_0(\lambda_1 + \mu_1) \right] \frac{1}{\varepsilon_1} dr, \\
    0, & 0 \leq r \leq 2R_1 - \frac{\pi}{2\varepsilon_1},
\end{cases}
\]

and

\[
\phi'_2(r) = \begin{cases} 
    e^{-k(r-R)} - 1, & 2R_2 < r \leq R, \\
    -(ke^{a_2k})^{\frac{p(R_2)^{\gamma-1}}{p(R_2)}} \left[ \frac{(2R_2)^{\gamma-1}}{p(R_2)} \sin \left( \frac{\varepsilon_2(r - 2R_2) + \frac{\pi}{2}}{\frac{\pi}{2\varepsilon_2}} \right) k_0(\lambda_2 + \mu_2) \right] \frac{1}{\varepsilon_2} dr, \\
    0, & 0 \leq r \leq 2R_2 - \frac{\pi}{2\varepsilon_2},
\end{cases}
\]
it is easy to see that $\phi_1, \phi_2 \geq 0$ are decreasing and $\phi_1, \phi_2 \in C^1([0, R]), \phi_1(x) = \phi_1(|x|) \in C^1(\Omega)$ and $\phi_2(\cdot) = \phi_2(|\cdot|) \in C^1(\Omega)$.

Let $r = |x|$. By computation,

$$-\Delta_{p(x)} \phi_1 = -\text{div}(\nabla \phi_1)|p(x)-2 \nabla \phi_1| = -(r^{N-1}|\phi_1'(r)|p(r)-2 \phi_1'(r))'/r^{N-1},$$

so then

$$-\Delta_{p(x)} \phi_1 = \begin{cases} (ke^{-k(p-R)}p^{(p-1)} \left[ -k(p(r) - 1) + p'(r) \ln k - kp'(r)(r - R) + \frac{N-1}{r} \right], \\ 2R_1 < r \leq R, \\ e_1(2R_1)N^{-1}(ke^{a_1 k})p(2R_1)^{-1} \cos \left( e_1(r - 2R_1) + \frac{\pi}{2} \right) (\lambda_1 + \mu_1), \\ 2R_1 - \frac{\pi}{2\varepsilon_1} < r \leq 2R_1, 0 \leq r \leq 2R_1 - \frac{\pi}{2\varepsilon_1}. \end{cases}$$

If $k$ is sufficiently large, when $2R_1 < r \leq R$, then we have

$$-\Delta_{p(x)} \phi_1 \leq -k \left[ \inf p(x) - 1 - \sup |\nabla p(x)| \left( \frac{\ln k}{k} + R - r \right) + \frac{N-1}{kr} \right] \leq -ka_1.$$ 

As $a_1$ is a constant dependent only on $p(x)$, if $k$ is big enough, such that

$$-ka_1 \leq -\left( \lambda_1 + \mu_1 \right) k_0,$$

then we have

$$-\Delta_{p(x)} \phi_1 \leq -\left( \lambda_1 + \mu_1 \right) k_0 \leq \lambda_1 f(\phi_2) + \mu_1 h(\phi_1), \quad 2R_1 < |x| \leq R. \quad (4)$$

If $k$ is sufficiently large, then

$$f(e^{a_1 k} - 1) \geq 1, \quad h(e^{a_1 k} - 1) \geq 1, \quad g(e^{a_1 k} - 1) \geq 1, \quad \gamma(e^{a_1 k} - 1) \geq 1,$$

where $k$ is dependent on $f$, $h$, $g$, $\gamma$ and $p$, $q$ and independent on $R$. Since

$$-\Delta_{p(x)} \phi_1 = e_1 \left( \frac{2R_1}{r} \right)^{N-1} (ke^{a_1 k})p(2R_1)^{-1} \cos \left( e_1(r - 2R_1) + \frac{\pi}{2} \right) (\lambda_1 + \mu_1)$$

$$\leq e_1(\lambda_1 + \mu_1)2^{N}k^{p_0}e^{a_1 k p_0}, \quad 2R_1 - \frac{\pi}{2\varepsilon_1} < |x| \leq 2R_1.$$ 

let

$$e_1 = 2^{-N}k^{-p_0}e^{-a_1 k p_0}.$$ 

Then we have

$$-\Delta_{p(x)} \phi_1 \leq \lambda_1 + \mu_1 \leq \lambda_1 f(\phi_2) + \mu_1 h(\phi_1), \quad 2R_1 - \frac{\pi}{2\varepsilon_1} < |x| \leq 2R_1. \quad (5)$$

Obviously,

$$-\Delta_{p(x)} \phi_1 = 0 \leq \lambda_1 + \mu_1 \leq \lambda_1 f(\phi_2) + \mu_1 h(\phi_1), \quad |x| \leq 2R_1 - \frac{\pi}{2\varepsilon_1}. \quad (6)$$

Since $\phi_1(x) \in C^1(\Omega)$, combining (4), (5) and (6), we have

$$-\Delta_{p(x)} \phi_1 \leq \lambda_1 f(\phi_2) + \mu_1 h(\phi_1), \quad \text{for a.e. } x \in \Omega.$$
Since $\phi_1(x), \phi_2(x) \in C^1(\bar{\Omega})$, it is easy to see that $(\phi_1, \phi_2)$ is a subsolution of (3).

Step 2. We construct a supersolution of (3).

Let $z_1$ be a radial solution of

$$-\Delta_{p(x)} z_1(x) = (\lambda_1 + \mu_1)\mu,$$

in $\Omega$, $z_1 = 0$ on $\partial \Omega$.

We denote that if $z_1 = z_1(r) = z_1(|x|)$, then $z_1$ satisfies

$$-(r^{\beta-1} z_1') z_1' = r^{\lambda_1 + \mu_1}, \quad z_1(R) = 0, \quad z_1'(0) = 0,$$

and so

$$z_1' = -\left| \frac{r^{\lambda_1 + \mu_1}}{N} \right|^{\frac{1}{p(x) - 1}} \tag{7}$$

and

$$z_1 = \int_r^R \left| \frac{r^{\lambda_1 + \mu_1}}{N} \right|^{\frac{1}{p(x) - 1}} dr.$$

We denote that if $\beta = \beta((\lambda_1 + \mu_1)\mu) = \max_{0 \leq r \leq R} z_1(r)$, then

$$\beta((\lambda_1 + \mu_1)\mu) = \int_0^R \left| \frac{r^{\lambda_1 + \mu_1}}{N} \right|^{\frac{1}{p(x) - 1}} dr = ((\lambda_1 + \mu_1)\mu)^{\frac{1}{p(x) - 1}} \int_0^R \left| \frac{r}{N} \right|^{\frac{1}{p(x) - 1}} dr,$$

where $t \in [0, 1]$. Since $\int_0^R \frac{r}{N} \left| \frac{r}{N} \right|^{\frac{1}{p(x) - 1}} dr$ is a constant, then there exists a positive constant $C \geq 1$ such that

$$\frac{1}{C}((\lambda_1 + \mu_1)\mu)^{\frac{1}{p(x) - 1}} \leq \beta((\lambda_1 + \mu_1)\mu) = \max_{0 \leq r \leq R} z_1(r) \leq C((\lambda_1 + \mu_1)\mu)^{\frac{1}{p(x) - 1}}. \tag{8}$$

We consider

$$\begin{cases}
-\Delta_{p(x)} z_1 = (\lambda_1 + \mu_1)\mu & \text{in } \Omega \\
-\Delta_{p(x)} z_2 = (\lambda_2 + \mu_2) g(\beta((\lambda_1 + \mu_1)\mu)) & \text{in } \Omega, \\
z_1 = z_2 = 0 & \text{on } \partial \Omega
\end{cases}$$

and then we shall prove that $(z_1, z_2)$ is a supersolution for (3).

For $\xi \in W^{1,p(x)}(\Omega)$ with $\xi \geq 0$ it is easy to see that

$$\int_{\Omega} |\nabla z_1|^{p(x)-2} \nabla z_2 \cdot \nabla \xi \, dx = \int_{\Omega} (\lambda_2 + \mu_2) g(\beta((\lambda_1 + \mu_1)\mu)) \xi \, dx$$

$$\geq \int_{\Omega} \lambda_2 g(z_1) \xi \, dx + \int_{\Omega} \mu_2 g(\beta((\lambda_1 + \mu_1)\mu)) \xi \, dx.$$

By (H.5) for $\mu$ large enough, we have

$$g(\beta((\lambda_1 + \mu_1)\mu)) \geq \gamma \left( [(\lambda_2 + \mu_2) g(\beta((\lambda_1 + \mu_1)\mu))]^{\frac{1}{p(x) - 1}} \right) \geq \gamma(z_2).$$
Hence
\[
\int\limits_{\Omega} |\nabla z_2|^{p(x)-2} \nabla z_2 \cdot \nabla \xi \, dx \geq \int\limits_{\Omega} \lambda_2 g(z_1) \xi \, dx + \int\limits_{\Omega} \mu_2 \gamma(z_2) \xi \, dx.
\] (9)

Also
\[
\int\limits_{\Omega} |\nabla z_1|^{p(x)-2} \nabla z_1 \cdot \nabla \xi \, dx = \int\limits_{\Omega} (\lambda_1 + \mu_1) \mu \xi \, dx.
\]

Similar to (8), we have
\[
\max_{0 \leq r \leq R} z_2(r) \leq C[(\lambda_2 + \mu_2)g(\beta((\lambda_1 + \mu_1)\mu))]^{1/(q-1)}.
\]

By (H.4) and (H.5), when \(\mu\) is sufficiently large, according to (8), we have
\[
(\lambda_1 + \mu_1)\mu \geq \left[\frac{1}{C} \beta((\lambda_1 + \mu_1)\mu)\right]^{p-1} \geq \lambda_1 f \left[ C((\lambda_2 + \mu_2)g(\beta((\lambda_1 + \mu_1)\mu))]^{1/(q-1)} \right] + \mu_1 h(\beta((\lambda_1 + \mu_1)\mu) \geq \lambda_1 f(z_2) + \mu_1 h(z_1),
\]

and so
\[
\int\limits_{\Omega} |\nabla z_1|^{p(x)-2} \nabla z_1 \cdot \nabla \xi \, dx \geq \int\limits_{\Omega} \lambda_1 f(z_2) \xi \, dx + \int\limits_{\Omega} \mu_1 h(z_1) \xi \, dx \quad (10)
\]

According to (9) and (10), we can conclude that \((z_1, z_2)\) is a supersolution of (3).

Let \(\mu\) be sufficiently large; then from (7) and the definition of \((\phi_1, \phi_2)\), it is easy to see that \(\phi_1 \leq z_1\) and \(\phi_2 \leq z_2\). This completes the proof. \(\square\)

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