On a problem of Hahn

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It is shown that for any almost periodic linear differential
system asymptotic stability and uniform stability together imply
uniform asymptotic stability.

The question was asked by Hahn ([3], p. 202) whether, for almost
periodic linear differential equations, asymptotic stability implies
uniform stability. Conley and Miller [1] showed, by a fairly complicated
example, that the answer was "no". The object of the present note is to
point out that for an almost periodic linear differential equation
asymptotic stability and uniform stability on the half-line R^+ together
imply uniform asymptotic stability on the whole line R. This follows
without difficulty from other recent work, but the present strong and
convenient formulation seems to have been overlooked. An immediate
consequence is that if $a(t)$ is *any* almost periodic function with mean
value 0 such that

$$\int_0^t a(s)ds \to \infty \quad \text{as} \quad t \to \infty,$$

then the scalar differential equation

$$x' = -a(t)x$$

is asymptotically stable but not uniformly stable. For example (see [2],
p. 72), one can take

$$a(t) = \sum_{n=1}^{\infty} n^{-3/2} \sin \pi t/n.$$

Let

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(1) $x' = A(t)x$

be any linear differential equation which is uniformly stable on the whole line R. Then there exists a constant $K > 0$ such that, for every solution $x(t)$ of (1),

$$|x(t)| \leq K|x(s)| \quad \text{for} \quad -\infty < s \leq t < \infty .$$

It follows that if there exists a sequence $t_n \to -\infty$ such that $|x(t_n)| \to 0$ then $x(t) \equiv 0$. On the other hand, if there exists a sequence $t_n \to \infty$ such that $|x(t_n)| \to 0$ then $|x(t)| \to 0$ as $t \to \infty$. Hence either

$$\inf |x(t)| > 0$$

or $|x(t)| \to 0$ as $t \to \infty$. Again, if

$$\sup |x(t)| = \infty$$

then, since $x(t)$ is bounded for $t \geq 0$, there exists a sequence $t_n \to -\infty$ such that $|x(t_n)| \to \infty$ and hence $|x(t)| \to \infty$ as $t \to -\infty$.

If, in addition, the coefficient matrix $A(t)$ is almost periodic then Nakajima [4] has shown that $x(t) \not\equiv 0$ and

$$\sup |x(t)| < \infty$$

imply

$$\inf |x(t)| > 0 .$$

Suppose now that $A(t)$ is almost periodic and (1) is uniformly stable on the half-line R^+. Then there exists a constant $K > 0$ such that, if $X(t)$ is the fundamental matrix of (1) for which $X(0) = I$, then

$$|X(t)X^{-1}(s)| \leq K \quad \text{for} \quad 0 \leq s \leq t < \infty .$$

Let $B(t)$ be any almost periodic function in the hull of $A(t)$. Then there exists a sequence $t_n \to \infty$ such that $A(t+t_n) \to B(t)$ uniformly on R. If $Y(t)$ is the fundamental matrix of the equation

(2) $y' = B(t)y$

for which $Y(0) = I$, then

$$Y(t) = \lim_{n \to \infty} X(t+t_n)X^{-1}(t_n)$$

for every real t and hence

$$|Y(t)Y^{-1}(s)| \le K \quad \text{for} \quad -\infty < s \le t < \infty .$$

Thus (2), and in particular (1), is uniformly stable on the whole line R .

   Assume (2) has a non-trivial bounded solution $y(t)$. Then, by Nakajima's result, $\inf|y(t)| > 0$. We have $B(t-t_n) \to A(t)$ uniformly on R , and by restricting attention to a subsequence we may suppose that $y(-t_n) \to \xi \ne 0$. Hence $y(t-t_n) \to x(t)$ for every real t , where $x(t)$ is the solution of (1) such that $x(0) = \xi$. Evidently $x(t)$ is bounded and $\inf|x(t)| > 0$.

   It follows that if the equation (1) is also asymptotically stable then no equation (2) has a nontrivial bounded solution. Therefore, by a fundamental result of Sacker and Sell [5], the equation (1) possesses an exponential dichotomy on R . Since $X(t)$ is bounded for $t \ge 0$ this is possible only if there exist positive constants L, α such that

$$|X(t)X^{-1}(s)| \le Le^{-\alpha(t-s)} \quad \text{for} \quad -\infty < s \le t < \infty .$$

In other words, the equation (1) is uniformly asymptotically stable on R .

References

[1] C.C. Conley and R.K. Miller, "Asymptotic stability without uniform stability: almost periodic coefficients", *J. Differential Equations* 1 (1965), 333-336.

[2] W.A. Coppel, *Stability and asymptotic behavior of differential equations* (Heath, Boston, 1965).

[3] Wolfgang Hahn, "The present state of Lyapunov's direct method", *Nonlinear problems*, 195-205 (Proceedings of a Symposium conducted by the Mathematics Research Center, United States Army, University of Wisconsin, Madison, 1962. University of Wisconsin Press, Madison, 1963).

[4] Fumio Nakajima, "Separation conditions and stability properties in almost periodic systems", *Tôhoku Math. J.* 26 (1974), 305-314.

[5]   Robert J. Sacker and George R. Sell, "Existence of dichotomies and
          invariant splittings for linear differential systems I", *J.
          Differential Equations* 15 (1974), 429-458.

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