SOME EXTENSIONS

OF THE HAUSDORFF-YOUNG AND PALEY THEOREMS

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1. Introduction. Orthonormal sequences, o. n. s., $\{\phi_n\}$ defined on $[0, 1]$ and satisfying

$$(1) \quad \|\phi_n\|_\nu = \left(\int_0^1 |\phi_n|^\nu \, dx\right) \leq F_n, \qquad n = 1, 2, \ldots, \; 2 \leq \nu < \infty$$
$$= \max_{0 \leq x \leq 1} |\phi_n| \leq F_n, \qquad n = 1, 2, \ldots, \; \nu = \infty,$$

have been studied in [3] and [1]. One of the objects of this paper is to indicate that the methods used to study such o. n. s. can be used for a much wider class, and that, although there seems to be no super theorem to cover all cases, a knowledge of the results and methods of proof in some fairly broad special cases enables one to state and prove theorems for other classes of o. n. s.

An o. n. s. satisfying (1) can be considered as follows. Although $\{\|\phi_n\|_\nu\}$ is not a bounded sequence there is a function of n, namely F_n^{-1}, such that $\{\|\phi_n F_n^{-1}\|_\nu\}$ is bounded. In § 4 we consider o. n. s. for which there is a function of x, say $\psi(x)$, such that $\{\|\phi_n \psi\|_\nu\}$ is bounded.

In § 3, and § 5 other types of o. n. s. are considered. As has been mentioned many of the proofs are similar to those of known results and so will not be given in any detail.

2. Notation. Let us define the following notation

$$\|c_n\|_r = \left(\Sigma_1^\infty |c_n|^r\right)^{\frac{1}{r}}, \qquad 1 \leq r < \infty,$$

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$$= \max_n |c_n|, \qquad r = \infty,$$

$$\|c_n\|_r' = \left(\Sigma_1 |c_n|^r n^{\frac{r-2}{2-\nu'}}\right)^{\frac{1}{r}} = \left\| c_n n^{\frac{1-2/r}{2-\nu'}} \right\|_r$$

where $\nu > 2$, $1 < \nu' < r < \nu < \infty$ and ν, ν' are conjugate indices defined by $\frac{1}{\nu} + \frac{1}{\nu'} = 1$,

$$\|f\|_r' = \left(\int_0^1 |f|^r x^{\frac{r-2}{\nu'}} dx\right)^{\frac{1}{r}} = \left\| f . x^{\frac{1-2/r}{\nu'}} \right\|_r$$

where $\nu > 2$, $1 < \nu' < r < \nu < \infty$.

Also, given $\{c_n\}$ and $\{F_n\}$, define

(2) $$d_n(r) = c_n F_n^{\frac{\nu'(r-2)}{(2-\nu')r}}, \qquad 1 \leq r \leq \infty, \quad 2 \leq \nu \leq \infty.$$

Note that if r, s are related by

(3) $$\frac{\nu'}{r} + \frac{2-\nu'}{s} = 1$$

then $|d_n(r)|^s = |c_n|^s F_n^{2-s}$, and that $d_n(\nu') = c_n F_n^{-1}$. Further if $\nu = \infty$ (when $\nu' = 1$) then if r, s are related by (3) they are conjugate indices. Finally that if $F_n = 1$ for all n then $d_n = c_n$.

As has been pointed out in [1] the sequence $\{d_n\}$ plays, for o. n. s. satisfying (1), the role that $\{c_n\}$ plays for uniformly bounded o. n. s., i. e. o. n. s. satisfying (1) with $\nu = \infty$, $F_n = M$ for all n.

The results it is intended to extend in this paper are listed below. For further results and details of proofs the reader is referred to [1], [2], [3] and [7].

THEOREM A (Mercer). If $f \in L_{\nu'}$ then $d_n(\nu') = o(1)$.

THEOREM B (Hausdorff-Young). (i) If $f \in L_p$, $\nu' \leq p \leq 2$, then the Fourier coefficients

(4) $$c_n = \int_0^1 f \phi_n \, dx$$

satisfy the inequality

$$\| d_n(p) \|_q \leq \| f \|_p$$

where p, q are related by (3).

(ii) Given a sequence $c_1, c_2, \ldots$ with $\| d_n(q) \|_p < \infty$, $1 \leq p \leq 2$, where q, p are related by (3), then there is an $f \in L_q$ satisfying (4) for all n and such that

$$\| f \|_q \leq \| d_n(q) \|_p .$$

For the next theorem only, we assume F_n to be monotonic increasing, i.e.

$$F_1 \leq F_2 \leq F_3 \ldots$$

THEOREM C (Paley). (i) If $f \in L_p$, $\nu' < p \leq 2$ with Fourier coefficients c_n given by (4) then

$$\| d_n(p) \|_p' \leq A_{p,\nu} \| f \|_p ,$$

where $A_{p,\nu}$ depends on p, ν only.

(ii) If $\{d_n\}$ satisfies $\| d_n \|_q' < \infty$, $2 \leq q < \nu$ then there is an $f \in L_q$ satisfying (4) for all n, where c_n is given by (2) with $r = q$, and such that

$$\| f \|_q \leq A_{q,\nu} \| d_n \|_q' .$$

THEOREM D (Integral analogue of C). (i) If $\| d_n \|_p < \infty$, $\nu' < p \leq 2$, then there is an f satisfying (4) for all n, where c_n is given by (2) with $r = q$, q, p being related as in (3), and

$$\| f \|_p' \leq A_{p,\nu} \| d_n \|_p .$$

(ii) If $\|f\|'_q < \infty$, $2 \leq q < \nu$, and if f has Fourier coefficients given by (4) then

$$\|d_n(p)\|_q \leq A_{q,\nu} \|f\|'_q$$

where p, q are related by (3).

Theorems C, D have so-called star extensions. If $d_n = o(1)$ then $\{d_n^*\}$ denotes $\{|d_n|\}$ arranged in a decreasing order. Similarly f^* will denote a non-increasing function equimeasurable with f, (for further details see [7]). Then C, D can be replaced by stronger theorems, theorems C^*, D^*. The truth of theorem C^* is only known when (5) is unimportant, i.e. if $F_n = M$ for all n, when it follows by applying theorem C to a rearrangement of the o.n.s. $\{\phi_n\}$. (For further discussion see [1].)

THEOREM C^*. (i) Under the hypothesis of theorem C (i) we have

$$\|d_n^*(p)\|'_p < A_{p,\nu} \|f\|_p.$$

(ii) If $\{d_n\}$ satisfies $d_n = o(1)$ and $\|d_n^*\|'_q < \infty$, $2 \leq q < \nu$ then we can deduce theorem C (ii) with

$$\|f\|_q < A_{q,\nu} \|d_n^*\|'_q.$$

THEOREM D^*. (i) Under the hypothesis of theorem D (i) we have

$$\|f^*\|'_p \leq A_{p,\nu} \|d_n\|_p.$$

(ii) If $\|f^*\|'_q < \infty$, $2 \leq q < \nu$ then we can deduce theorem D (ii) and

$$\|d_n(p)\|_q \leq A_{q,\nu} \|f^*\|'_q.$$

Theorem D^* is deduced from theorem D by applying theorem D to the o.n.s. $\{\psi_n\}$ obtained from o.n.s. $\{\phi_n\}$ by a transformation of $[0,1]$ that transforms f into f^*. (See [7], p. 125.) If $\{\phi_n\}$ satisfies (5) so does $\{\psi_n\}$.

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The proofs of theorems B-D depend on the following theorem. If E is a measure space with measure μ, define

$$\|f\|_{r,\mu} = \left(\int_E |f|^r d\mu\right)^{\frac{1}{r}}, \qquad 1 \leq r < \infty$$

$$= \inf\{M: |f| \leq M \text{ except on a set } A, \quad \mu(A) = 0\}, \; r = \infty.$$

THEOREM E (Riesz-Thorin). Let E_1 and E_2 be two measure spaces with measures μ_1, μ_2 respectively. Let T be a linear operation defined on E_1 to E_2. Suppose that

$$(6_1) \qquad \|Tf\|_{\mu_2, \frac{1}{\beta_1}} \leq M_1 \|f\|_{\mu_1, \frac{1}{\alpha_1}}$$

and

$$(6_2) \qquad \|Tf\|_{\mu_2, \frac{1}{\beta_2}} < M_2 \|f\|_{\mu_1, \frac{1}{\alpha_2}}$$

where (α_1, β_1) and (α_2, β_2) belong to the square $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$. If

$$\alpha = (1 - t)\alpha_1 + t\alpha_2, \quad \beta = (1 - t)\beta_1 + t\beta_2, \quad 0 < t < 1,$$

then

$$(7) \qquad \|Tf\|_{\mu_2, \frac{1}{\beta}} \leq M_1^{1-t} M_2^t \|f\|_{\mu_1, \frac{1}{\alpha}}.$$

The method of proving theorems B-D is as follows.

In all cases it can be shown that parts (i) and (ii) of the theorems are equivalent. So we prove only B (ii), C (ii) and D (ii), say.

Then it is noted that (4), or some similar equation, defines a linear operation between two measure spaces for which (7) is just the inequality to be proved.

Thus theorem E shows that we can prove our theorems if we can prove them in two special cases, corresponding to (6_1) and (6_2), provided only that the range of (7) is then wide enough.

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In the case of theorem B (ii) this "interpolation" is fairly easy since the theorem is true for a closed interval of p. Considering the cases $p = 1$, $p = 2$ gives the two special cases required and interpolation gives the result for $1 \leq p \leq 2$.

This simple procedure breaks down for theorems C (ii) and D (ii) since they are not true for a closed range of q. In this case we have to prove the theorems for all integral q and then interpolate between each pair of integers separately.

For further details the reader is referred to [3] and [7]. Extensions of theorem E can be used to avoid the lengthy proofs of theorems C and D and for details of this reference should be made to [5] and [7].

3. Relatively orthonormal sequences. The sequence of functions $\{\lambda_n\}$ is called a relatively o.n.s., r.o.n.s., with respect to the weight function ψ, [2, p. 276], if

$$\int_0^1 \lambda_n(x)\,\lambda_m(x)\,\psi(x)\,dx = 0, \qquad n \neq m$$

$$= 1, \qquad n = m.$$

We write $f \sim \Sigma c_n \lambda_n$ if

$$c_n = \int f \lambda_n \psi\, dx \tag{8}$$

The results of §2 can be extended to r.o.n.s. in various ways.

With any r.o.n.s. $\{\lambda_n\}$ we can associate an o.n.s. $\{\phi_n\}$

$$\phi_n(x) = (\psi(x))^{\frac{1}{2}}\,\lambda_n(x),$$

provided only that $\psi^{\frac{1}{2}}$ is well defined. Then if $g(x) = (\psi(x))^{\frac{1}{2}} f(x)$, $g \sim \Sigma c_n \phi_n$, c_n being given by (8).

Suppose now that ϕ_n satisfies (1), i.e.

$$\|\phi_n\|_\nu = \|\psi^{\frac{1}{2}}\lambda_n\|_\nu \leq F_n, \qquad n = 1, 2, \ldots, \quad 2 \leq \nu \leq \infty.$$

Then we can immediately, from theorems A-D, obtain theorems about the r. o. n. s. $\{\lambda_n\}$. The following gives a typical selection of results.

THEOREM 1. (i) If $\psi^{\frac{1}{2}} f \in L_{\nu'}$ then $d_n(\nu') = o(1)$.

(ii) If $\psi^{\frac{1}{2}} f \in L_p$, $\nu' \leq p \leq 2$ and if p, q are related by (3) then

$$\| d_n(p) \|_q \leq \| \psi^{\frac{1}{2}} f \|_p .$$

(iii) If $\psi^{\frac{1}{2}} f \in L_p$, $\nu' < p \leq 2$, and if (5) holds

$$\| d_n(p) \|'_p \leq A_{p,\nu} \| \psi^{\frac{1}{2}} f \|_p .$$

(iv) If $\| \psi^{\frac{1}{2}} f \|_q < \infty$, $2 \leq q < \nu$ and if p, q are related by (3) then

$$\| d_n(p) \| \leq A_{q,\nu} \| \psi^{\frac{1}{2}} f \|'_q .$$

We could, however, suppose that (1) applies directly to the r. o. n. s. $\{\lambda_n\}$ i. e (9) is replaced by

(10) $$\| \lambda_n \|_\nu \leq F_n, \qquad n = 1, 2, \ldots, \quad 2 \leq \nu \leq \infty .$$

The extension of theorems A-D in this case is not quite so immediate.

However it is known, (see [3]), that the interval [0, 1] of theorems A-D can be replaced by a general measure space E, the measure of E being finite or infinite. (In the case that E is of infinite measure extra care is necessary at certain points, [3].) The results we want will follow from theorems A-D applied to the measure space $E = [0, 1]$ with measure

(11) $$\mu(x) = \int_0^x \psi(t) dt .$$

Then theorem 1 becomes

THEOREM 2. (i) If $\psi^{\frac{1}{\nu'}} f \in L_{\nu'}$ then $d_n(\nu') = o(1)$.

(ii) If $\psi^{\frac{1}{p}} f \in L_p$, $\nu' \leq p \leq 2$ and if p, q are related by (3), then

$$\| d_n(p) \|_q \leq \| \psi^{\frac{1}{p}} f \|_p .$$

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(iii) If $\psi^{\frac{1}{p}} f \in L_p$, $\nu' < p \leq 2$ and if (5) holds

$$\|d_n(p)\|'_p \leq A_{p,\nu} \|\psi^{\frac{1}{p}} f\|_p .$$

(iv) If $\|\psi^{\frac{1}{q}} f\|'_q < \infty$, $2 \leq q < \nu$ and if p, q are related as in (3) then

$$\|d_n(p)\|_q \leq A_{q,\nu} \|\psi^{\frac{1}{q}} f\|'_q .$$

That theorem 2 is the correct result is seen from the fact that hypothesis (i) is $f \in L_{\nu',\mu}$, hypotheses (ii), (iii) are $f \in L_{p,\mu}$, hypothesis (iv) is $\|f\|'_{q,\mu} < \infty$, where μ is defined by (11).

4. Suppose now that $\{\phi_n\}$ is an o. n. s. such that $\{\|\phi_n\|_\nu\}$ is not bounded but for which there exists a function $\psi(x)$ such that

(12) $$\|\psi\phi_n\|_\nu \leq 1, \qquad n = 1, 2, \ldots, \quad 2 \leq \nu \leq \infty.$$

Such o. n. s. were first considered by Rosskopf, [4]. Some restrictions on $\psi(x)$ seem inevitable and we will assume that $\psi \in L_2$. Further let us write $\chi(x) = \frac{1}{\psi(x)}$.

THEOREM 3. If $\chi f \in L_{\nu'}$ then $c_n = o(1)$.

The proof follows the lines of theorem A (see [1] and [2, p. 155]). From (4),

$$c_n = \int f \phi_n \, dx = \int_0^1 f \chi \psi \phi_n \, dx.$$

Given any f such that $\chi f \in L_{\nu'}$ and given any $\varepsilon > 0$ we can write $f = f_1 + f_2$ where

(i) χf_1 is bounded and hence $f_1 \in L_2$ since $\psi \in L_2$,

(ii) $\|\chi f_2\|_{\nu'} < \frac{\varepsilon}{2}$.

Therefore

$$|c_n| \leq \left|\int_0^1 f_1 \phi_n \, dx\right| + \left|\int_0^1 f_2 \chi \psi \phi_n \, dx\right|.$$

By the Riesz-Fischer theorem the first integral is less than $\frac{\varepsilon}{2}$ for all n large enough, and an application of Hölders inequality shows that the second integral cannot exceed $\frac{\varepsilon}{2}$. This completes the proof of theorem 3.

If we now restrict ψ further by

$$\psi \in L_2 \quad \text{and} \quad \chi \in L_2 \tag{13}$$

we can extend theorems B, C, D to this class of o.n.s.

THEOREM 4. (i) If $\chi^{\frac{2}{p}-1} f \in L_p$, $\nu' \leq p \leq 2$, then the Fourier coefficients, given by (4), satisfy the inequality

$$\|c_n\|_q \leq \|\chi^{\frac{2}{p}-1} f\|_p,$$

where p, q are related by (3).

(ii) If $\|c_n\|_p < \infty$, $1 \leq p \leq 2$, then there is an f satisfying (4) for all n and such that

$$\|\chi^{\frac{2}{q}-1} f\|_q < \|c_n\|_p,$$

where q, p are related by (3).

The proof follows the lines indicated in §2. It is sufficient to note how we can relate this theorem to theorem E.

We remark that (4) can be written

$$c_n = \int_0^1 (f\psi)(\phi_n\psi)\chi^2\, dx,$$

that

$$\|\chi^{\frac{2}{p}-1} f\|_p = \int_0^1 |f|^p \chi^{2-p}\, dx = \int_0^1 |f\psi|^p \chi^2\, dx$$

and that, following condition (13), we can define

$$X(x) = \int_0^x \chi^2(t)\, dt.$$

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Then we see that $\{c_n\}$ is obtained by a linear transformation from $L_{p,X}$ acting on $\psi f \in L_{p,X}$. Thus theorem E can be applied to prove this theorem.

Since (12) does not depend on the order of the o. n. s. ϕ_n we can extend theorem C^* to cover such o. n. s.

THEOREM 5. (i) If $\chi^{\frac{2}{p}-1} f \in L_p$, $\nu' < p \leq 2$, with Fourier coefficients given by (4) then

$$\|c_n^*\|_p' \leq A_{p,\nu} \|\chi^{\frac{2}{p}-1} f\|_p,$$

where $A_{p,\nu}$ depends on p, ν only.

(ii) If $c_n = o(1)$ and $\|c_n^*\|_q' < \infty$, $2 \leq q < \nu$ then there is an f satisfying (4) for all n and such that

$$\|\chi^{\frac{2}{q}-1} f\|_q \leq A_{q,\nu} \|c_n^*\|_q',$$

where $A_{q,\nu}$ depends on q, ν only.

Because of the remarks following theorem 4 the proof of this follows exactly the lines indicated in §2.

When a generalization of theorem D is attempted a further restriction of ψ is needed, analogous to (5),

(14) if $0 \leq x \leq y \leq 1$ then $|\chi(x)| \leq |\chi(y)|$.

THEOREM 6. (i) If $\|c_n\|_p < \infty$, $\nu' < p \leq 2$ then there exists a function f satisfying (4) for all n and such that

$$\|\chi^{(1-\frac{2}{p})(\frac{2}{\nu'}-1)} f\|_p' \leq A_{p,\nu} \|c_n\|_p$$

where $A_{p,\nu}$ depends on p, ν only.

(ii) If $\|\chi^{(1-\frac{2}{q})(\frac{2}{\nu'}-1)} f\|_q < \infty$, $2 \leq q < \nu$ and if Fourier coefficients of f are given by (4) then

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$$\|c_n\|_q \leq A_{q,\nu} \| \quad ^{(1-\frac{2}{q})(\frac{2}{\nu'}-1)} \quad f\|_q,$$

where $A_{q,\nu}$ depends on q, ν only.

Following previous remarks it will suffice to indicate how (14) is needed to put through the proof of this theorem.

Let

$$\begin{aligned} f_j(x) &= f(x), \qquad 2^{-j} < x < 2^{-j+1} \\ &= 0, \qquad \text{elsewhere,} \end{aligned}$$

and $f_j(x) \sim \Sigma\, c_n^{(j)} \phi_n(x)$, then $c_n = \Sigma_{j=0}^{\infty} c_n^{(j)}$.

Let

$$\eta_j = \int_{2^{-j}}^{2^{-j+1}} |f|^q \left(\chi^{\frac{2-\nu'}{\nu'}} x^{\frac{1}{\nu'}} \right)^{q-2} dx$$

then the crucial lemma [3] is the inequality

$$\Sigma_{r=1}^{N} |c_r^{(j)} c_n^{(k)}| \leq B_{q,\nu}\, \eta_j^{\frac{1}{2}} \eta_k^{\frac{1}{2}}\, 2^{-(j-k)\frac{2-\nu'}{2\nu'}}$$

where $j > k$ and $B_{q,\nu}$ depends on q, ν only.

Following the proof in [3] one can easily prove the inequality but with the r.h.s. multiplied by the factor

$$\max_{2^{-j} < x < 2^{-j+1}} |\chi(x)| \Big/ \max_{2^{-k} < x < 2^{-k+1}} |\chi(x)|$$

and it is to replace this that (14) is needed.

For alternative proofs of this theorem reference should be made to [7, p. 125].

The attempt to prove a star extension of theorem 6 is prevented by (14). The method of proof, suggested after theorem D^*, would transform χ into θ that would not necessarily satisfy (14). This is just the difficulty that (5) causes when attempting to prove theorem C^*. We conjecture the following star extension of theorem 6. As in [1], it can be shown to hold if the function $g(x)$, defined below, takes only the values $0, 1, -1$. Also the weaker theorem, with p, q replaced by $p+\varepsilon$, $q-\varepsilon$ respectively, can easily be proved.

THEOREM 6*. (i) If $\|c_n\|_p < \infty$, $\nu' < p \leq 2$ then there is an f satisfying (4) for all n and such that if

$$g = \chi^{(1-\frac{2}{p})(\frac{2}{\nu'}-1)} f \tag{15}$$

Then

$$\|g^*\|_p' \leq A_{p,\nu} \|c_n\|_p$$

(ii) If $\|g^*\|_q < \infty$, $2 \leq q < \nu$, and if c_n are defined by (4), with f defined by (15) with p replaced by q, then

$$\|c_n\|_q \leq A_{q,\nu} \|g^*\|_q'.$$

5. The above results could be used to state and prove theorems for o.n.s. satisfying various combinations of the above conditions; e.g. both (1) and (12), $\|\psi\phi_n\| \leq F_n$. Although as has been mentioned earlier such theorems are not implied by those already proved.

Further o.n.s. can be considered by simple changes of variable. Thus putting $y^{\frac{-\nu'}{2-\nu'}}$ for x in (1) enables us to state theorems for o.n.s. $\{\lambda_n\}$ satisfying $\left(\int_1^\infty |\lambda_n|^\nu y^{\nu-1} dy\right)^{\frac{1}{\nu}} \leq F_n$.

In particular the second inequality of theorem D would then read,

$$\|d_n(p)\|_q \leq A_{q,\nu}\left(\int_1^\infty |f|^q dy\right)^{\frac{1}{q}}.$$

Also we have so far been assuming the $\{\phi_n\}$ to be real valued but by changing (4) to

$$c_n = \int_0^1 f\bar{\phi}_n dx$$

we can easily extend the results to complex valued o. n. s.

6. Some examples of o. n. s. to illustrate these theorems can be given.

Let P_n be the Legendre polynomials [6]. Then

$$\phi_n(x) = (2n+1)^{\frac{1}{2}} P_n(x)$$

is an o. n. s. on $[-1, 1]$. We can either consider $\{\phi_n\}$ to be an o. n. s. satisfying (1) because

$$|\phi_n| \leq (2n+1)^{\frac{1}{2}},$$

or satisfying (12), because

$$|(1-x^2)^{\frac{1}{4}} \phi_n(x)| \leq 1.$$

However this o. n. s. is a particular case of Jacobi polynomials $\pi_n^{(\alpha,\beta)}(x)$, a r. o. n. s. on $[-1, 1]$ with respect to $(1-x)^{\alpha}(1+x)^{\beta}$ [6]. Two particular cases are of interest,

(i) $\lambda_n(x) = \pi_n^{(\alpha, o)}(x)$

(ii) $\mu_n(x) = \pi_n^{(\alpha,\alpha)}(x)$.

It is known [6] that

$$|\lambda_n| \leq Kn^{\alpha+\frac{1}{2}}$$

and hence $\{\lambda_n\}$ is a r. o. n. s. satisfying (10) with $F_n = Kn^{\alpha+\frac{1}{2}}$, $\nu = \infty$ and weight function $\psi(x) = (1-x)^{\alpha}$. Then theorem 2 becomes for $\{\lambda_n\}$:

(i) If $(1-x)^{\alpha} f(x) \in L$ then $c_n = o(n^{\alpha+\frac{1}{2}})$,

(ii) If $(1-x)^{\frac{\alpha}{p}} f(x) \in L_p$, $1 \leq p \leq 2$ then

$$\left(\Sigma |c_n|^{p'} n^{(\alpha+\frac{1}{2})(2-p')}\right)^{\frac{1}{p'}} \leq K^{\frac{2-p}{p}} \left(\int_{-1}^{1} |f|^p (1-x)^{\alpha} dx\right)^{\frac{1}{p}},$$

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(iii) If $(1-x)^{\frac{\alpha}{p}} f(x) \in L_p$, $1 < p \leq 2$ then

$$\Sigma \; c_n^{*\,p} \, n^{(\alpha+\frac{3}{2})(p-2)} \leq A_p^p \int_{-1}^{1} |f|^p (1-x)^{\alpha} \, dx,$$

(iv) If $(1-x)^{\frac{\alpha}{q}} (1+x)^{1-\frac{2}{q}} f \in L_q$, $2 \leq q < \infty$, then

$$\Sigma \, |c_n|^q \, n^{(\alpha+\frac{1}{2})(2-q)} \leq A_q^q \, K^{q-2} \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{q-2} |f|^q \, dx$$

where $c_n = \int_{-1}^{1} f(x) \, \lambda_n(x) (1-x)^{\alpha} dx.$

If for example $f(x) = (1-x)^{-\mu}$ then the conditions of the above results are satisfied if $\mu < \frac{1+\alpha}{p}$, which implies the convergence of the series in (ii) and (iii) if $\mu < \frac{1+\alpha}{p}$. In fact it is known [6] that $c_n \sim n^{2\mu-\alpha-3/2}$ which shows the series converge under the same condition.

Considering now the r.o.n.s. $\{\mu_n\}$ it is known that [6]

$$|(1-x^2)^{\frac{2\alpha+1}{4}} \mu_n(x)| \leq K$$

i.e.

$$|(1-x^2)^{\alpha/2} (1-x^2)^{\frac{1}{4}} \mu_n(x)| \leq K.$$

Then we can consider this as a r.o.n.s. satisfying (12) with $\psi(x) = \frac{1}{K}(1-x^2)^{\frac{2\alpha+1}{4}}$, $\nu = \infty$, and weight function $(1-x^2)^{\alpha}$.

In this way we combine results in §4 with theorem 2 to get required theorems. Or we can take it to be a r.o.n.s. satisfying a combination of (9) and (12) with the function

$(1 - x^2)^{\alpha/2}$ being the $\psi^{\frac{1}{2}}$ of (9) and the rest $\frac{1}{K}(1 - x^2)^{\frac{1}{4}}$, the function required in §4. Then combining the results in §4 with theorem 2 gives the desired theorems. As both methods give same results let us consider the first. Then theorem 2 becomes

(i) If $(1 - x^2)^{\frac{\alpha}{2} - \frac{1}{4}} f(x) \in L_p$ then $c_n = o(1)$.

(ii) If $(1 - x^2)^{\frac{\alpha}{2} + \frac{1}{4} - \frac{1}{2p}} f(x) \in L_p$, $1 \leq p \leq 2$ then

$$\left(\Sigma\, |c_n|^{p'}\right)^{\frac{1}{p'}} \leq K^{\frac{2-p}{p}} \left(\int_{-1}^{1} |f|^p (1 - x^2)^{p(\frac{\;}{2} + \frac{1}{4}) - \frac{1}{2}} dx\right)^{\frac{1}{p}}.$$

(iii) If $(1 - x^2)^{\frac{\alpha}{2} + \frac{1}{4} - \frac{1}{2p}} f(x) \in L_p$, $1 < p \leq 2$, then

$$\Sigma\, c_n^{*p}\, n^{p-2} \leq A_p^p \int_{-1}^{1} |f|^p (1 - x^2)^{p(\frac{\alpha}{2} + \frac{1}{4}) - \frac{1}{2}} dx.$$

(iv) If $(1 - x^2)^{\frac{\alpha}{2} + \frac{1}{4} - \frac{1}{2q}} (1 + x)^{1 - \frac{2}{q}} f(x) \in L_q$, then

$$\Sigma\, |c_n|^q \leq A_q^q K^{q-2} \int_{-1}^{1} |f|^q (1 - x^2)^{q(\frac{\alpha}{2} + \frac{1}{4}) - \frac{1}{2}} (1 + x)^{q-2} dx$$

where

$$c_n = \int_{-1}^{1} f(x) \mu_n(x) (1 - x^2)^{\alpha} dx.$$

Again the function $f(x) = (1 - x)^{-\mu}$ can be considered as illustration for these results.

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