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TAIL BEHAVIOR OF NEGATIVELY ASSOCIATED HEAVY-TAILED SUMS

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Abstract

Consider a sequence $\{X_k,\ k \geq 1\}$ of random variables on $(-\infty, \infty)$. Results on the asymptotic tail probabilities of the quantities $S_n = \sum_{k=0}^{n} X_k$, $X_{(n)} = \max_{0 \leq k \leq n} X_k$, and $S_{(n)} = \max_{0 \leq k \leq n} S_k$, with $X_0 = 0$ and $n \geq 1$, are well known in the case where the random variables are independent with a heavy-tailed (subexponential) distribution. In this paper we investigate the validity of these results under more general assumptions. We consider extensions under the assumptions of having long-tailed distributions (the class L) and having the class D ∩ L, where D is the class of distribution functions with dominatedly varying tails. Some results are also given in the case where $X_k,\ k \geq 1$, are not necessarily identically distributed and/or independent.

Keywords: Asymptotics; subexponentiality; partial sum; tail probability; negative association

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1. Introduction

The asymptotic tail behavior of sums of heavy-tailed random variables has been studied by many authors. Recent applications cover areas from importance sampling [3], where tail probabilities of sums are estimated, and queueing systems [5], [23], to nonparametric regression [16].

Although there does not seem to be general agreement on the terminology, a common feature of heavy-tailed random variables X (or the corresponding distribution function F) which satisfy $\overline{F}(x) = \mathrm{P}\{X > x\} > 0$, for any $x \in (-\infty, \infty)$, is the property $\mathrm{E}\exp\{\gamma X\} = \infty$, for any $\gamma > 0$.

Early asymptotic results for the tail probability of the convolution of random variables with a regularly varying tail distribution (denoted by $F \in \mathrm{RV}$), i.e. for which there exists an $\alpha > 0$ such that

$$\overline{F}(tx) \sim t^{-\alpha}\overline{F}(x), \qquad t > 0,$$

are due to Feller [9]. Here and henceforth, all limiting relationships are for $x \to \infty$ unless stated otherwise; by using the symbol '$\sim$' we mean that the ratio of the two sides tends to 1.

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The study of the asymptotic behavior of tail sums was (see [13] and references therein) subsequently extended to the class of subexponential distributions, S. By definition, a distribution function F on $[0, \infty)$ is defined to be subexponential (denoted by $F \in \mathrm{S}$) if

$$\frac{\overline{F}^{*2}(x)}{\overline{F}(x)} \to 2, \tag{1.1}$$

where $\overline{F}^{*2}(x) = \mathrm{P}\{X_1 + X_2 > x\}$ is the tail probability of the convolution of two independent and identically distributed (i.i.d.) random variables with distribution function F. More generally, a distribution function F on $(-\infty, \infty)$ is said to be subexponential if $F^{+}(x) = F(x)\,\mathbf{1}_{\{0 \le x < \infty\}}$ is subexponential, where $\mathbf{1}_{\{\cdot\}}$ is the indicator function.

In this case it is possible to prove basic results, for example the fact that (1.1) implies that

$$\frac{\overline{F}^{*n}(x)}{\overline{F}(x)} \to n, \quad \text{for } n \ge 2, \tag{1.2}$$

for random variables on the real line; see, e.g. [10], [15], and [22]. For background information on subexponentiality and its applications, the reader is referred to [2], [4], [8], [18].

An important reason why subexponential distributions play a role in the fields of applied probability and risk theory is that, for a sequence of i.i.d. random variables with common distribution $F \in \mathrm{S}$, from (1.2) it holds, for each $n \ge 2$, that

$$\mathrm{P}\left\{\sum_{k=1}^{n} X_k > x\right\} \sim \mathrm{P}\big\{\max_{1 \le k \le n} X_k > x\big\},$$

which makes it clear that the class S is useful in modeling large losses.

The class S is related to several other classes of functions. A well-known result is the inclusion $\mathrm{S} \subset \mathrm{L}$, where L is the class of long-tailed distribution functions F satisfying $\overline{F}(x+a) \sim \overline{F}(x)$, for $a \in \mathbb{R}$. In this case, convergence is uniform on compact subsets of $\mathbb{R}$.

There is a connection with functions of dominated variation as well: the inclusion $\mathrm{D} \cap \mathrm{L} \subset \mathrm{S}$. We write $F \in \mathrm{D}$ if the tail function $\overline{F}$ is of dominated variation, i.e. if $\limsup_{x\to\infty} \overline{F}(ax)/\overline{F}(x) < \infty$, for $0 < a < 1$. Convolution tails for dominatedly varying distributions were studied by Tang and Yan [20].

We use the following notation. For a sequence $\{X_k,\ k \ge 1\}$ of random variables with distribution functions on $(-\infty, \infty)$, using the convention $X_0 = 0$, we write

$$X_{(n)} = \max_{0 \le k \le n} X_k, \qquad S_n = \sum_{k=0}^{n} X_k, \qquad S_{(n)} = \max_{0 \le k \le n} S_k.$$

In this paper, under the assumption that the random variables $X_k,\ k \ge 1$, have heavy-tailed distribution functions, we aim to formulate the asymptotic relations

$$\mathrm{P}\{S_{(n)} > x\} \sim \mathrm{P}\{X_{(n)} > x\} \sim \mathrm{P}\{S_n > x\} \sim \sum_{k=1}^{n} \overline{F}_k(x). \tag{1.3}$$

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2. The case of independent random variables

Using the Pollaczek–Spitzer identity, Sgibnev [19] studied the asymptotic behavior of the tail probability of $S_{(n)}$ in the case of i.i.d. summands. In particular, when $F \in \mathrm{S}$, he proved that $\mathrm{P}\{S_{(n)} > x\} \sim n\overline{F}(x)$. Using a quantile inequality for S_n from [17], a simplified proof of this equivalence was given in [14]. See also [13, Theorem 3.1] for the i.i.d. case.

Since (1.2) holds for distribution functions $F \in \mathrm{S}$, we have

$$\mathrm{P}\{S_{(n)} > x\} \sim \mathrm{P}\{S_n > x\} \sim n\overline{F}(x).$$

Below we give a new proof which avoids the use of complicated technical results and is valid for nonidentically distributed random variables as well.

Theorem 2.1. *Suppose that X_k, $k = 1, \ldots, n$, are independent random variables with distribution functions $F_1, \ldots, F_n$ on $(-\infty, \infty)$.*

(i) *If $F_i \in \mathrm{L}$, for $i = 1, \ldots, n$, then $\mathrm{P}\{S_{(n)} > x\} \sim \mathrm{P}\{S_n > x\}$.*

(ii) *Suppose that the convolution $F_i * F_j$ is a member of S, for all $1 \le i, j \le n$, and one of the following conditions holds:*

(iia) *there exists a c such that $\mathrm{P}\{X_i > c\} = 1$, for $1 \le i \le n$,*

(iib) *$F_i \in \mathrm{L}$ for $i = 1, \ldots, n$.*

Then the asymptotic relations in (1.3) hold.

For every real number x we write $x^+ = x \vee 0 = \max\{x, 0\}$. To prove Theorem 2.1 we need the following lemma.

Lemma 2.1. *The following statements hold.*

(i) *If $F_1, F_2 \in \mathrm{L}$ then $F_1 * F_2 \in \mathrm{L}$.*

(ii) *If $X_i \ge 0$ almost surely for $i = 1, \ldots, n$ and $F_i * F_j \in \mathrm{S}$ for all $1 \le i, j \le n$, then $F_k \in \mathrm{S}$ and $F_1 * F_2 \cdots * F_k \in \mathrm{S}$ for $1 \le k \le n$, and $\mathrm{P}\{S_n > x\} \sim \sum_{k=1}^{n} \overline{F}_k(x)$.*

(iii) *If $F_1, F_2 \in \mathrm{L}$ then $\mathrm{P}\{X_1^+ + X_2 > x\} \sim \mathrm{P}\{X_1 + X_2 > x\}$.*

Proof. Part (i) is from [6]; see also [14]. The convolution closure of S is from [7]. The other results of parts (ii) and (iii) are from [11] and [10], respectively.

Proof of Theorem 2.1. We use the notation $f(x) \lesssim g(x)$ to mean $\limsup f(x)/g(x) \le 1$, and define the reverse relation in the natural way. We use induction to prove that

$$\mathrm{P}\{S_{(n)} > x\} \sim \mathrm{P}\{S_n > x\}. \tag{2.1}$$

Suppose that $F_i \in \mathrm{L}$, for $i = 1, \ldots, n$, and that (2.1) holds for all values of subscript less than or equal to $n - 1$. The inequality $\mathrm{P}\{S_{(n)} > x\} \gtrsim \mathrm{P}\{S_n > x\}$ is trivial, so we only need to find

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an upper estimate. Note that

$$\begin{aligned}
\mathrm{P}\{S_{(n)} \geq x\} &= \sum_{k=1}^{n} \mathrm{P}\{S_0 \leq x, \dots, S_{k-1} \leq x, S_k > x\} \\
&\leq \sum_{k=1}^{n} \mathrm{P}\{S_{k-1} \leq x, X_k + S_{k-1} > x\} \\
&= \sum_{k=1}^{n} [\mathrm{P}\{S_k > x\} - \mathrm{P}\{S_{k-1} > x\} + \mathrm{P}\{S_{k-1} > x, S_k \leq x\}] \\
&= \mathrm{P}\{S_n > x\} + \sum_{k=1}^{n} \mathrm{P}\{S_{k-1} > x, S_k \leq x\}.
\end{aligned}$$

Since the class L is closed under convolution by Lemma 2.1(i), we may apply Lemma 2.1(iii) to find that

$$\mathrm{P}\{S_{k-1} > x, X_k + S_{k-1} \leq x\} = \mathrm{P}\{S_{k-1} + X_k^+ > x\} - \mathrm{P}\{S_{k-1} + X_k > x\} = o(\mathrm{P}\{S_k > x\}).$$

Hence,

$$\mathrm{P}\{S_{(n)} \geq x\} \leq \mathrm{P}\{S_n > x\} + o\bigg(\sum_{k=1}^{n} \mathrm{P}\{S_k > x\}\bigg). \tag{2.2}$$

Since $\mathrm{P}\{S_{(k)} > x\}$ is nondecreasing in k for all x, using the induction hypothesis $\mathrm{P}\{S_k > x\} \sim \mathrm{P}\{S_{(k)} > x\}$ for $k \leq n-1$ it follows that

$$o\bigg(\sum_{k=1}^{n} \mathrm{P}\{S_k > x\}\bigg) = o(\mathrm{P}\{S_n > x\}) + o(\mathrm{P}\{S_{(n)} > x\}).$$

Combining this with (2.2) gives $\mathrm{P}\{S_{(n)} \geq x\} \lesssim \mathrm{P}\{S_n \geq x\}$, completing the proof of part (i).

In order to prove part (ii) under the assumption given by part (iia), note that (2.1) holds, since the assumption $F_i * F_j \in$ S, for all $1 \leq i, j \leq n$, implies that $F_i \in \mathrm{S} \subset \mathrm{L}$ (consider Lemma 2.1(ii) applied to $X_i - c$).

Next we prove part (ii) under the assumption given by part (iib). Note that application of Lemma 2.1(iii) gives $\mathrm{P}\{X_1 + X_2 > x\} \sim \mathrm{P}\{X_1^+ + X_2^+ > x\}$, which implies that the distribution function of $X_1^+ + X_2^+$ is subexponential (since the tail is asymptotic to $\mathrm{P}\{S_2 > x\}$). Using part (iia) above, the right-hand side of (2.1) is asymptotic to

$$\sum_{k=1}^{2} \mathrm{P}\{X_k^+ > x\} = \sum_{k=1}^{2} \mathrm{P}\{X_k > x\}.$$

The proof for $n > 2$ follows by induction.

In view of Lemma 2.1(ii) and $\mathrm{P}\{X_{(n)} > x\} = 1 - \prod_{k=1}^{n}(1 - \overline{F}_k(x)) \sim \sum_{k=1}^{n} \overline{F}_k(x)$, the proof is complete.

3. The case of negatively associated random variables

A finite family of random variables $\{X_k,\ 1 \leq k \leq n\}$ is said to be negatively associated (NA) if, for every pair of disjoint subsets A_1 and A_2 of $\{1, 2, \dots, n\}$,

$$\operatorname{cov}\{f_1(X_{k_1}, k_1 \in A_1), f_2(X_{k_2}, k_2 \in A_2)\} \leq 0,$$

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whenever f_1 and f_2 are coordinatewise increasing such that the covariance exists. An infinite family is NA if each of its finite subfamilies is NA. This dependence structure was first introduced in [1] and [12].

Theorem 3.1. *Suppose that the random variables X_k, $k = 1, \ldots, n$, are NA with distribution functions*

$$F_k \in \mathrm{D} \cap \mathrm{L}, \qquad k \geq 1,$$

and there exists a constant $c > -\infty$ such that $\mathrm{P}\{X_k > c\} = 1$, $k = 1, \ldots, n$. Then (1.3) holds.

Wang and Tang [21] established the same result under the additional conditions that $X_1, \ldots, X_n$ are identically distributed and $\mathrm{E}\, X_1^r < \infty$, for some $r > 1$.

A closer look at the proof of Theorem 3.1 shows that it is sufficient to assume the asymptotic version of the NA property given below, because the inequalities need to be satisfied for large values of x only. A finite family of random variables $\{X_k,\ 1 \leq k \leq n\}$ is said to be *asymptotically NA* if there exists a constant c_0 such that, for every pair of disjoint subsets A_1 and A_2 of $\{1, 2, \ldots, n\}$,

$$\operatorname{cov}\{f_1(X_{k_1},\ k_1 \in A_1),\ f_2(X_{k_2},\ k_2 \in A_2)\} \leq 0,$$

for functions f_1 and f_2 which have support (c_0, ∞) and are coordinatewise increasing such that the covariance exists.

Proof of Theorem 3.1. The proof consists of three parts.

(i) We start by proving that $\mathrm{P}\{S_n > x\} \sim \sum_{k=1}^n \overline{F}_k(x)$, which amounts to a conjunction of the two asymptotic relations

$$\mathrm{P}\{S_n > x\} \gtrsim \sum_{k=1}^n \overline{F}_k(x) \quad \text{and} \quad \mathrm{P}\{S_n > x\} \lesssim \sum_{k=1}^n \overline{F}_k(x). \tag{3.1}$$

Choose functions $a_k(x) \uparrow \infty$ such that $\overline{F}_k(x \pm a_k(x)) \sim \overline{F}_k(x)$, for $k = 1, \ldots, n$. It follows that, with $a(x) = \min_{1 \leq k \leq n} a_k(x)$,

$$\overline{F}_k(x \pm a(x)) \sim \overline{F}_k(x), \qquad k = 1, \ldots, n. \tag{3.2}$$

Note that

$$\begin{aligned}
\mathrm{P}\{S_n > x\} &\geq \mathrm{P}\{S_n > x,\ X_{(n)} > x + a(x)\} \\
&\geq \sum_{k=1}^n \mathrm{P}\{S_n > x,\ X_k > x + a(x)\} \\
&\quad - \sum_{1 \leq k < l \leq n} \mathrm{P}\{S_n > x,\ X_k > x + a(x),\ X_l > x + a(x)\} \\
&=: J_1 - J_2.
\end{aligned}$$

Since $\mathrm{P}\{X_k > c\} = 1$, for $k = 1, \ldots, n$, it holds for x sufficiently large that

$$J_1 \geq \sum_{k=1}^n \mathrm{P}\{S_n - X_k > -a(x),\ X_k > x + a(x)\} = \sum_{k=1}^n \overline{F}_k(x + a(x)).$$

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In view of (3.2), we have $J_1 \gtrsim \sum_{k=1}^n \overline{F}_k(x)$. Using the NA property, it follows that

$$J_2 \le \sum_{1\le k<l\le n} \mathrm{P}\{X_k > x + a(x),\, X_l > x + a(x)\} \le \sum_{1\le k<l\le n} \overline{F}_k(x + a(x))\overline{F}_l(x + a(x));$$

hence, $J_2 = o(\sum_{k=1}^n \overline{F}_k(x))$. This proves the first relation in (3.1).

To prove the second relation in (3.1) in terms of the function $a(x)$ in (3.2), we derive

$$\begin{aligned}\mathrm{P}\{S_n > x\} &\le \mathrm{P}\biggl\{\bigcup_{k=1}^n X_k > x - a(x)\biggr\} + \mathrm{P}\biggl\{S_n > x,\, \bigcap_{k=1}^n [X_k \le x - a(x)]\biggr\}\\ &\le \sum_{k=1}^n \mathrm{P}\{X_k > x - a(x)\}\\ &\quad + \mathrm{P}\biggl\{S_n > x,\, \bigcup_{k=1}^n \biggl[X_k > \frac{x}{n}\biggr],\, \bigcap_{k=1}^n [X_k \le x - a(x)]\biggr\}.\end{aligned}$$

By the choice of the function a, the first term on the right-hand side is asymptotic to $\sum_{k=1}^n \overline{F}_k(x)$. Clearly, the second term is dominated by $\sum_{k=1}^n \mathrm{P}\{S_n - X_k > a(x),\, X_k > x/n\}$. Using the NA property, it follows that this expression is further dominated by

$$\sum_{k=1}^n \mathrm{P}\{S_n - X_k > a(x)\}\, \mathrm{P}\biggl\{X_k > \frac{x}{n}\biggr\} = o\biggl(\sum_{k=1}^n \overline{F}_k(x)\biggr),$$

the last equality being true since $F_k \in \mathrm{D}$, for $1 \le k \le n$. This proves the second relation in (3.1).

(ii) Next we prove that

$$\mathrm{P}\{X_{(n)} > x\} \sim \sum_{k=1}^n \overline{F}_k(x).$$

The inequality $\mathrm{P}\{X_{(n)} > x\} \le \sum_{k=1}^n \overline{F}_k(x)$, for all x, is trivial. To prove the reverse relation, by the NA property we have

$$\mathrm{P}\{X_{(n)} > x\} \ge \sum_{k=1}^n \mathrm{P}\{X_k > x\} - \sum_{1\le k\ne l\le n} \mathrm{P}\{X_k > x,\, X_l > x\} \sim \sum_{k=1}^n \overline{F}_k(x).$$

(iii) Finally we prove (2.1). Note that part (i) of this proof implies that

$$\mathrm{P}\{S_{(n)} > x\} \le \mathrm{P}\biggl\{\sum_{i=1}^n X_i^+ > x\biggr\} \lesssim \sum_{i=1}^n \mathrm{P}\{X_i^+ > x\} = \sum_{i=1}^n \overline{F}_i(x).$$

Since $\mathrm{P}\{S_{(n)} > x\} \ge \mathrm{P}\{S_n > x\}$, the proof is complete.

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