PARA-BLASCHKE ISOPARAMETRIC HYPERSURFACES IN A UNIT SPHERE $S^{n+1}(1)$

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(Received 5 May 2011; accepted 24 September 2011; first published online 30 March 2012)

Abstract. Let $A = \rho^2 \sum_{i,j} A_{ij} \theta^i \otimes \theta^j$ and $B = \rho^2 \sum_{i,j} B_{ij} \theta^i \otimes \theta^j$ be the Blaschke tensor and the Möbius second fundamental form of the immersion $x$. Let $D = A + \lambda B$ be the para-Blaschke tensor of $x$, where $\lambda$ is a constant. If $x : M^n \mapsto S^{n+1}(1)$ is an $n$-dimensional para-Blaschke isoparametric hypersurface in a unit sphere $S^{n+1}(1)$ and $x$ has three distinct Blaschke eigenvalues one of which is simple or has three distinct Möbius principal curvatures one of which is simple, we obtain the full classification theorems of the hypersurface.

2001 Mathematics Subject Classification. 53A30, 53B25.

1. Introduction. In Möbius differential geometry, Wang [18] studied invariants of hypersurfaces in a unit sphere $S^{n+1}(1)$ under the Möbius transformation group. Let $x : M \mapsto S^{n+1}(1)$ be an $n$-dimensional immersed hypersurface without umbilical points in $S^{n+1}(1)$. We choose a local orthonormal basis $\{e_i\}$ for the induced metric $I = dx \cdot dx$ with dual basis $\{\theta^i\}$. Let $II = \sum_{i,j} h_{ij} \theta^i \otimes \theta^j$ be the second fundamental form and $H = \frac{1}{n} \sum_i h_{ii}$ the mean curvature of the immersion $x$. By putting $\rho^2 = \frac{n}{n-1} (\sum_{i,j} \delta_{ij}^2 - nH^2)$, Wang [18] defined the Möbius metric, the Möbius form, the Blaschke tensor and the Möbius second fundamental form of the immersion $x$ by $g = \rho^2 dx \cdot dx$, $\Phi = \rho \sum_i C_i \theta^i$, $A = \rho^2 \sum_{i,j} A_{ij} \theta^i \otimes \theta^j$ and $B = \rho^2 \sum_{i,j} B_{ij} \theta^i \otimes \theta^j$, respectively, where

$$C_i = -\rho^{-2} \left\{ H_i + \sum_j (h_{ij}-H\delta_{ij}) e_j(\log \rho) \right\},$$

$$A_{ij} = -\rho^{-2} \{ \text{Hess}_{ij}(\log \rho) - e_i(\log \rho)e_j(\log \rho) - Hh_{ij} \} \text{ (1.1)}$$

$$- \frac{1}{2} \rho^{-2} (|\nabla (\log \rho)|^2 - 1 + H^2) \delta_{ij},$$

$$B_{ij} = \rho^{-1} (h_{ij} - H\delta_{ij}). \text{ (1.3)}$$

*Project supported by NSF of Shaanxi Province (SJ08A31) and NSF of Shaanxi Educational Committee (11JK0479, 2010JK642).
and $\text{Hess}_g$, $\nabla$ are the Hessian matrix and the gradient with respect to the induced metric $dx \cdot dx$. It was proved that $g$, $\Phi$, $A$ and $B$ are the Möbius invariants (see [18]). We should notice that it is one of the important aims to characterize submanifolds in terms of Möbius invariants. Concerning this topic, there are many important results, one can see [1, 2 and 5–20]. Recently, by making use of the two important Möbius invariants, the Blaschke tensor $A$ and the Möbius second fundamental form $B$ of the immersion $x$, Cheng, Li and Qi [6] and Zhong and Sun [19] defined a symmetric $(0, 2)$ tensor $D = A + \lambda B$ which is so-called the para-Blaschke tensor of $x$, where $\lambda$ is a constant. An eigenvalue of the Blaschke tensor is called a Blaschke eigenvalue of $x$, an eigenvalue of the Möbius second fundamental form is called a Möbius principal curvature of $x$ and an eigenvalue of the para-Blaschke tensor is called a para-Blaschke eigenvalue of $x$. It is reasonable to introduce the definition: A hypersurface $x : M \hookrightarrow S^{n+1}(1)$ without umbilical points is called a Blaschke isoparametric hypersurface, or a Möbius isoparametric hypersurface, or a para-Blaschke isoparametric hypersurface, if the Möbius form $\Phi \equiv 0$ and the Blaschke eigenvalues, or the Möbius principal curvatures, or the para-Blaschke eigenvalues of the immersion $x$ are constants. In [11], Li and Wang investigated and completely classified hypersurfaces $x : M \hookrightarrow S^{n+1}(1)$ without umbilical points and with vanishing Möbius form $\Phi$ in $S^{n+1}(1)$, which satisfy $A + \lambda B + \mu g = 0$. Li and Zhang [12] generalized this topic to general submanifolds. It should be noted that the condition $A + \lambda B + \mu g = 0$ implies that the para-Blaschke eigenvalues of $x$ are all equal. If $x$ has two distinct constant para-Blaschke eigenvalues, the classification theorem was obtained by Zhong and Sun [19].

Let $H^{n+1}$ be an $(n + 1)$-dimensional hyperbolic space defined by

$$H^{n+1} = \{(y_0, y_1) \in \mathbb{R}^+ \times \mathbb{R}^{n+1} \mid -y_0^2 + y_1 \cdot y_1 = -1\}.$$ 

Let $\sigma : \mathbb{R}^{n+1} \hookrightarrow S^{n+1}(1) \setminus \{(-1, 0)\}$ and $\tau : H^{n+1} \hookrightarrow S^{n+1}_+(1)$ be defined by

$$\sigma(u) = \left(\frac{1 - |u|^2}{1 + |u|^2}, \frac{2u}{1 + |u|^2}\right), \quad u \in \mathbb{R}^{n+1},$$

$$\tau(y_0, y_1) = \left(\frac{1}{y_0}, \frac{y_1}{y_0}\right), \quad (y_0, y_1) \in H^{n+1},$$

respectively, where $S^{n+1}_+(1)$ is the open hemisphere in $S^{n+1}(1)$ whose first coordinate is positive.

If $\lambda = 0$, we notice that para-Blaschke isoparametric hypersurfaces reduce to Blaschke isoparametric hypersurfaces. Li and Peng [13] obtained the following:

**Theorem 1.1.** Let $x$ be an $n$-dimensional immersed Blaschke isoparametric hypersurface in a unit sphere $S^{n+1}(1)$ with three distinct Blaschke eigenvalues one of which is simple. Then, $x$ is locally Möbius equivalent to

1. $CSS(p, q, r)$ for some constants $p, q, r$, $p \neq q$ and $r \neq \frac{1}{\sqrt{2}}$, or
2. Cartan’s non-minimal isoparametric hypersurfaces in $S^4$ with three principal curvatures, that is, the non-minimal tube of constant radius over a standard Veronese minimal immersion of $S^2(\sqrt{3})$ into $S^4$, or
3. one of the hypersurfaces as indicated in Example 3.4 where $k = 3$ and $\tilde{y}_1 : M_1 \hookrightarrow S^4(r)$ is one of Cartan’s non-minimal isoparametric hypersurfaces with
three principal curvatures $\mu_1, \mu_2$ and $\mu_3$ satisfying $\lambda \mu_i = \frac{1}{\rho}$ for some $i \in \{1, 2, 3\}$.

If $\lambda \neq 0$, we consider the immersed para-Blaschke isoparametric hypersurfaces in a unit sphere $S^{n+1}(1)$ with three distinct Blaschke eigenvalues. We may obtain the following:

**Theorem 1.2.** Let $x : M \mapsto S^{n+1}(1)$ be an $n(n \geq 4)$-dimensional immersed para-Blaschke isoparametric hypersurface in a unit sphere $S^{n+1}(1)$ and $D = A + \lambda B$, ($\lambda \neq 0$), be the para-Blaschke tensor of $x$. If $x$ is of three distinct Blaschke eigenvalues one of which is simple, then $x$ is locally Möbius equivalent to:

1. a hypersurface with constant mean curvature and constant scalar curvature in $S^{n+1}(1)$, or
2. the image of $\sigma$ of a hypersurface with constant mean curvature and constant scalar curvature in $\mathbb{R}^{n+1}$, or
3. the image of $\tau$ of a hypersurface with constant mean curvature and constant scalar curvature in $H^{n+1}$, or
4. CSS($p, q, r$) for some constants $p, q, r, p \neq q$ and $r \neq \frac{1}{\sqrt{2}}$, or
5. one of the hypersurfaces as indicated in Example 3.4 where $k = 3$ and $\tilde{y}_1 : M_1 \mapsto S^4(r)$ is one of Cartan’s non-minimal isoparametric hypersurfaces with three principal curvatures satisfying $\lambda \mu_i = \frac{1}{\rho}$ for some $i \in \{1, 2, 3\}$.

For Möbius isoparametric hypersurface with three distinct Möbius principal curvatures in a unit sphere $S^{n+1}(1)$, Hu and co-authors [8] and [9] obtained the following:

**Theorem 1.3.** Let $x : M \mapsto S^{n+1}(1)$ be an $n(n \geq 4)$-dimensional immersed Möbius isoparametric hypersurface with three distinct Möbius principal curvatures one of which is simple. Then $x$ is locally Möbius equivalent to

1. CSS($p, q, r$) for some constants $p, q, r, r$,
2. an open part of the image of $\sigma$ of the cone $\tilde{x} : N^3 \times \mathbb{R}^+ \mapsto \mathbb{R}^5$ defined by $\tilde{x}(\varphi, t) = t\varphi$, where $t \in \mathbb{R}^+$ and $\varphi : N^3 \mapsto S^4 \hookrightarrow \mathbb{R}^5$ is minimal isoparametric immersion in $S^4$ with three principal curvatures, or
3. one of the hypersurfaces as indicated in Example 3.4 where $k = 3$, $r = \sqrt{\frac{5n}{n-1}}, \lambda = 0$ and $\tilde{y}_1 : M_1 \mapsto S^4(r)$ is Cartan’s minimal isoparametric hypersurfaces with vanishing scalar curvature and three principal curvatures of values $\pm \sqrt{\frac{n-1}{2n}}, 0$.

If $x$ is an immersed para-Blaschke isoparametric hypersurface in a unit sphere $S^{n+1}(1)$ with three distinct Möbius principal curvatures one of which is simple, we obtain the following:

**Theorem 1.4.** Let $x : M \mapsto S^{n+1}(1)$ be an $n(n \geq 4)$-dimensional immersed para-Blaschke isoparametric hypersurface in a unit sphere $S^{n+1}(1)$. If $x$ is of three distinct Möbius principal curvatures one of which is simple, then $x$ is locally Möbius equivalent to:

1. a hypersurface with constant mean curvature and constant scalar curvature in $S^{n+1}(1)$, or
2. M"obius invariants and fundamental formulas. In this section, we review the M"obius invariants and fundamental formulas on M"obius geometry of hypersurfaces in $S^{n+1}(1)$, for more details, see Wang [18].

Let $x : M \hookrightarrow S^{n+1}(1)$ be an $n$-dimensional hypersurface of $S^{n+1}(1)$ without umbilical points. We use the following range of indices throughout this paper:

$$1 \leq i, j, k \leq n.$$ 

For an immersed hypersurface $x : M \hookrightarrow S^{n+1}(1) \hookrightarrow \mathbb{R}^{n+2}$ of $S^{n+1}(1)$ without umbilical points, we define its M"obius position vector $Y : M \hookrightarrow L^{n+3}$ by $Y = \rho(1, x)$, where $\rho^2 = \frac{n}{n-1} \{ \sum_{i,j} h_{ij}^2 - nH^2 \}$. Let $\Delta$ be the Laplace–Beltrami operator of M"obius metric $g = \rho^2 dx \cdot dx$. We define $N = -\frac{1}{n} \Delta Y - \frac{1}{2n} \langle \Delta Y, \Delta Y \rangle Y$, then the structure equations on $M$ with respect to the M"obius metric $g$ can be written as follows:

$$dY_i = -\psi_i Y - \omega_{in+1}E_{n+1},$$ (2.3)

$$dE_{n+1} = -\phi Y - \sum_i \omega_{in+1}Y_i,$$ (2.4)

where $\{\psi_i, \omega_{ij}, \omega_{in+1}, \phi\}$ are 1-forms on $M$ with

$$\omega_{ij} + \omega_{ji} = 0.$$ (2.5)

By exterior differentiation of these equations, we get

$$\sum_i \omega_i \wedge \psi_i = 0, \quad \sum_i \omega_{in+1} \wedge \omega_i = 0,$$ (2.6)

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j.$$ (2.7)
\[ d\psi_i = \sum_j \omega_{ij} \wedge \psi_j + \omega_{in+1} \wedge \phi, \quad (2.8) \]
\[ d\phi = -\sum_i \omega_{in+1} \wedge \psi_i, \quad (2.9) \]
\[ d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \quad R_{ijkl} = -R_{jikl}, \quad (2.10) \]
\[ d\omega_{in+1} = \sum_j \omega_{ij} \wedge \omega_{jn+1} - \omega_i \wedge \phi, \quad (2.11) \]

where
\[ \psi_i = \sum_j A_{ij} \omega_j, \quad A_{ij} = A_{ji}, \quad \omega_{in+1} = \sum_j B_{ij} \omega_j, \quad B_{ij} = B_{ji}, \quad \phi = \sum_i C_i \omega_i, \quad (2.12) \]

and \( A_{ij}, B_{ij} \) and \( C_i \) are locally defined functions and satisfy (1.1), (1.2) and (1.3). We have

\[ d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \quad (2.13) \]
\[ \sum_i B_{ii} = 0, \quad \sum_{i,j} B_{ij}^2 = \frac{n-1}{n}, \quad \text{tr} A = \frac{1}{2n} (1 + n^2 R). \quad (2.14) \]

Let \( C_{ij}, A_{ij,k} \) and \( B_{ij,j} \) be the covariant derivative of \( C_i, A_{ij} \) and \( B_{ij} \). We define them by

\[ \sum_j C_{ij} \omega_j = dC_i + \sum_j C_j \omega_{ji}, \quad (2.15) \]
\[ \sum_k A_{ij,j} \omega_k = dA_{ij} + \sum_k A_{ik} \omega_{kj} + \sum_k A_{kj} \omega_{ki}, \quad (2.16) \]
\[ \sum_k B_{ij,k} \omega_k = dB_{ij} + \sum_k B_{ik} \omega_{kj} + \sum_k B_{kj} \omega_{ki}. \quad (2.17) \]

From the structure equations (2.1)–(2.4), we infer

\[ A_{ij,k} - A_{ik,j} = B_{ik} C_j - B_{ij} C_k, \quad (2.18) \]
\[ C_{ij} - C_{j,i} = \sum_k (B_{ik} A_{kj} - B_{ij} A_{ki}), \quad (2.19) \]
\[ B_{ij,k} - B_{ik,j} = \delta_{ij} C_k - \delta_{ik} C_j, \quad (2.20) \]
\[ R_{ijkl} = B_{ik} B_{jl} - B_{il} B_{jk} + \delta_{ik} A_{jl} + \delta_{ij} A_{lk} - \delta_{il} A_{jk} - \delta_{jk} A_{il}, \quad (2.21) \]

where \( R_{ijkl} \) denotes the curvature tensor with respect to the M"{o}bius metric \( g \) on \( M \) and \( n(n-1)R = \sum_{i,j} R_{ij} \) is the M"{o}bius scalar curvature of the immersion \( x : M \to S^{n+1}(1) \).
Since the M"obius form $\Phi = \sum_i C_i \omega_i E_{n+1} \equiv 0$, by (2.18)–(2.20), we have for all indices $i, j$ and $k$ that

$$A_{ij,k} = A_{ik,j}, \quad B_{ij,k} = B_{ik,j}, \quad \sum_k B_{ik} A_{kj} = \sum_k B_{kj} A_{ki}.$$  \hfill (2.22)

Denote by $D = \sum_{i, j} D_{ij} \omega_i \otimes \omega_j$ the $(0, 2)$ para-Blaschke tensor, then

$$D_{ij} = A_{ij}, \quad B_{ij}, \quad 1 \leq i, j \leq n, \quad (2.23)$$

where $\lambda$ is a constant. The covariant derivative of $D_{ij}$ is defined by

$$\sum_k D_{ij,k} \omega_k = dD_{ij} + \sum_k D_{ik} \omega_{kj} + \sum_k D_{kj} \omega_{ki}.$$ \hfill (2.24)

From (2.23), we have

$$D_{ij,k} = A_{ij,k} + \lambda B_{ij}, \quad D_{ij,k} - D_{ik,j} = A_{ij,k} - A_{ik,j} + \lambda (B_{ij,k} - B_{ik,j}).$$ \hfill (2.25)

From (2.22), we have for all indices $i, j$ and $k$ that

$$D_{ij,k} = D_{ik,j}.$$ \hfill (2.26)

3. Propositions and typical examples. Throughout this section, we shall make the following convention on the ranges of indices:

$1 \leq a, b \leq m_1, \quad m_1 + 1 \leq p, q \leq m_1 + m_2,$

$m_1 + m_2 + 1 \leq \alpha, \beta \leq m_1 + m_2 + m_3 = n, \quad 1 \leq i, j, k \leq n.$

We may prove the following:

**Proposition 3.1.** Let $\mathbf{x} : M \mapsto S^{n+1}(1)$ be an $n$-dimensional hypersurface with vanishing M"obius form in a unit sphere $S^{n+1}(1)$.

1. If the multiplicity of a Blaschke eigenvalue is constant and greater than 1, then this Blaschke eigenvalue is constant along its leaf.
2. If the multiplicity of a M"obius principal curvature is constant and greater than 1, then this M"obius principal curvature is constant along its leaf.

**Proof.** (1) Let $A_i$ be the Blaschke eigenvalues of $\mathbf{x}$ with constant multiplicities. We choose a local orthonormal frame $\{E_1, \ldots, E_n\}$ such that $E_i$ is a unit principal vector with respect to $A_i$. From (2.16), we have

$$A_{ij,k} = E_k (A_i) \delta_{ij} + \Gamma^i_{jk} (A_i - A_j),$$ \hfill (3.1)

where $\Gamma^i_{jk}$ is the Levi–Civita connection for the M"obius metric $g$ given by

$$\omega_{ij} = \sum_k \Gamma^i_{jk} \omega_k, \quad \Gamma^i_{jk} = -\Gamma^j_{ik}.$$ \hfill (3.2)

From (2.22), we know that $A_{ii,j} = A_{ji,i}$. Thus, from (3.1), we get

$$E_j (A_i) = \Gamma^i_{ij} (A_i - A_j), \quad \text{for } i \neq j.$$ \hfill (3.3)
Without loss of generality, we may assume that $A_1$ is the Blaschke eigenvalue of $x$ with constant multiplicity $m_1$ and $m_1 \geq 2$, that is, for $1 \leq a \leq m_1$, we have $A_a = A_1$. From (3.3), we have

$$E_a(A_1) = \Gamma_{1i}^{a_1}(A_1 - A_a) = 0, \quad \text{for } a \neq 1,$$

and

$$E_1(A_1) = E_1(A_a) = \Gamma_{1a}^1(A_a - A_1) = 0, \quad \text{for } a \neq 1.$$

Thus,

$$E_a(A_1) = 0, \quad \text{for any } a.$$

This implies that $A_1$ is constant along its leaf.

(2) Since the Möbius second fundamental form is also Codazzi tensor, by the same method, we see that (2) is true. We complete the proof of Proposition 3.1.

**Proposition 3.2.** Let $x : M \mapsto S^{n+1}(1)$ be an $n(n \geq 4)$-dimensional immersed para-Blaschke isoparametric hypersurface in a unit sphere $S^{n+1}(1)$ and $D = A + \lambda B$ be the para-Blaschke tensor of $x$.

1. If $x$ has three distinct Blaschke eigenvalues $A_1$, $A_2$ and $A_3$ one of which is simple and $\lambda \neq 0$, then either $A_1$, $A_2$ and $A_3$ are constants or $A_{ap,n} = 0$ for every $a, p$.

2. If $x$ has three distinct Möbius principal curvatures $B_1$, $B_2$ and $B_3$ one of which is simple, then either $B_1$, $B_2$ and $B_3$ are constants or $B_{ap,n} = 0$ for every $a, p$.

**Proof.** (1) Let $A$, $B$ and $D$ denote the $n \times n$-symmetric matrices $(A_y)$, $(B_y)$ and $(D_y)$, respectively, where $A_y$, $B_y$ and $D_y$ are defined by (1.2), (1.3) and (2.23). From (2.22) and (2.23), we know that $BA = AB$, $DA = AD$ and $BD = DB$. We may choose a local orthonormal basis $\{E_1, E_2, \ldots, E_n\}$ such that $A_y = A_i\delta_{ij}$, $B_y = B_i\delta_{ij}$ and $D_y = D_i\delta_{ij}$, where $A_i$, $B_i$ and $D_i$ are the Blaschke eigenvalues, the Möbius principal curvatures and the para-Blaschke eigenvalues of the immersion $x$.

From (2.13) and (3.2), the curvature tensor of $x$ may be given by (see [14])

$$R_{ijkl} = E_i(\Gamma_{jk}^l) - E_k(\Gamma_{ij}^l) + \sum_m \Gamma_{im}^j \Gamma_{lk}^m - \sum_m \Gamma_{im}^j \Gamma_{kl}^m + \sum_m \Gamma_{il}^m \Gamma_{jm}^l - \sum_m \Gamma_{il}^m \Gamma_{jm}^l. \quad (3.4)$$

Since $x$ has three distinct Blaschke eigenvalues $A_1$, $A_2$ and $A_3$ one of which is simple and $n \geq 4$, without loss of generality, we may assume that $m_3 = 1$, $m_1 m_2 \geq 2$ and $m_2 \geq 2$.

From (2.14) and (2.23), we have

$$m_1 A_1 + m_2 A_2 + m_3 A_3 = \operatorname{tr} D, \quad (3.5)$$

$$m_1 A_1^2 + m_2 A_2^2 + m_3 A_3^2 - 2 \left( \sum_a D_a \right) A_1 - 2 \left( \sum_p D_p \right) A_2$$

$$- 2 \left( \sum_a D_a \right) A_3 = \frac{n-1}{n} \chi^2 - \sum_i D_i^2. \quad (3.6)$$
Since we assume that $D_1, D_2, \ldots, D_n$ are constants, we get

\begin{align}
    m_1dA_1 + m_2dA_2 + m_3dA_3 &= 0, \\
    \xi_1dA_1 + \xi_2dA_2 + \xi_3dA_3 &= 0,
\end{align}

where $\xi_1 = m_1A_1 - \sum_a D_a$, $\xi_2 = m_2A_2 - \sum_p D_p$ and $\xi_3 = m_3A_3 - \sum_a D_a$. Since $\lambda \neq 0$, we know that all of $\xi_1, \xi_2$ and $\xi_3$ are not zero. We consider two cases:

If at least one of $m_2\xi_3 - m_3\xi_2, m_3\xi_1 - m_1\xi_3$ and $m_1\xi_2 - m_2\xi_1$ is zero, from (3.7) and (3.8), we easily know that $A_1, A_2$ and $A_3$ are constants.

If all of $m_2\xi_3 - m_3\xi_2, m_3\xi_1 - m_1\xi_3$ and $m_1\xi_2 - m_2\xi_1$ are not zero, from (3.7) and (3.8), we easily see that

\begin{equation}
    \frac{dA_1}{m_2\xi_3 - m_3\xi_2} = \frac{dA_2}{m_3\xi_1 - m_1\xi_3} = \frac{dA_3}{m_1\xi_2 - m_2\xi_1}.
\end{equation}

From Proposition 3.1 and (3.9), we have

\begin{equation}
    E_p(A_2) = E_p(A_1) = E_p(A_3) = 0,
\end{equation}

and from (3.1), we have

\begin{equation}
    \Gamma^p_{ab} = \Gamma^a_{ab} = 0, \quad a \neq b, \quad \Gamma^a_{pq} = 0, \quad p \neq q, \quad \Gamma^p_{aa} = \Gamma^p_{bb}, \quad \Gamma^a_{aa} = \Gamma^a_{bb},
\end{equation}

\begin{equation}
    \Gamma^p_{ac} = \frac{A_{aq, p}}{A_1 - A_2}, \quad \Gamma^a_{ap} = \frac{A_{aa, p}}{A_3 - A_1}, \quad \Gamma^a_{pa} = \frac{A_{pa, a}}{A_2 - A_3}.
\end{equation}

(i) If $m_1 \geq 2$, from Proposition 3.1 and (3.9), we have

\begin{equation}
    E_a(A_1) = E_a(A_2) = E_a(A_3) = 0.
\end{equation}

From (3.1), (3.3), (3.10) and (3.13), we have

\begin{equation}
    \Gamma^p_{ab} = \Gamma^a_{pq} = 0, \quad \Gamma^a_{mn} = \Gamma^p_{nn} = 0,
\end{equation}

\begin{equation}
    \Gamma^n_{aa} = \frac{E_n(A_1)}{A_1 - A_3}, \quad \Gamma^n_{pp} = \frac{E_n(A_2)}{A_2 - A_3}.
\end{equation}

From (3.12), we have

\begin{equation}
    \Gamma^p_{an} = \frac{A_{ap, n}}{A_1 - A_2}, \quad \Gamma^p_{nb} = \frac{A_{bp, n}}{A_3 - A_2}, \quad \Gamma^p_{bq} = \frac{A_{bq, n}}{A_1 - A_3}, \quad \Gamma^n_{qb} = \frac{A_{bq, n}}{A_2 - A_3}.
\end{equation}

Thus, from (3.4), (3.11) and (3.14)–(3.16), we have

\begin{equation}
    R_{aqp} = E_q(\Gamma^p_{ab}) - E_b(\Gamma^p_{aq}) + \sum_m \Gamma^m_{an} \Gamma^m_{qb} - \sum_m \Gamma^m_{an} \Gamma^m_{bq} + \sum_m \Gamma^m_{ab} \Gamma^m_{mq} - \sum_m \Gamma^m_{aq} \Gamma^m_{mb} = 0.
\end{equation}
On the other hand, from (2.21), we have
\[ R_{qpbq} = (B_a B_p + A_a + A_p)\delta_{ab}\delta_{pq} = (B_a B_p + A_1 + A_2)\delta_{ab}\delta_{pq}. \] (3.18)

By (3.17) and (3.18), we have
\[ A_{ap,n} A_{bq,n} + A_{aq,n} A_{lp,n} = \{(A_1 - A_3)(A_2 - A_3)(B_a B_p + A_1 + A_2) + E_n(A_1)E_n(A_2)\}\delta_{ab}\delta_{pq}. \]

Putting
\[ q_{a,p} = (A_1 - A_3)(A_2 - A_3)(B_a B_p + A_1 + A_2) + E_n(A_1)E_n(A_2), \] (3.19)
we get
\[ A_{ap,n} A_{bq,n} + A_{aq,n} A_{lp,n} = q_{a,p}\delta_{ab}\delta_{pq}. \]

If \( a = b \), from \( B_p = \frac{1}{\lambda}(D_p - A_2) \), we have
\[ 2A_{ap,n} A_{aq,n} = q_{a,p}\delta_{pq}, \] (3.20)
and
\[ q_{a,p} = (A_1 - A_3)(A_2 - A_3)\left(\frac{B_a}{\lambda} D_p + A_1 + \left(1 - \frac{B_a}{\lambda}\right) A_2\right) + E_n(A_1)E_n(A_2). \] (3.21)

If \( p = q \), from \( B_a = \frac{1}{\lambda}(D_a - A_1) \), we have
\[ 2A_{ap,n} A_{lp,n} = q_{a,p}\delta_{ab}, \] (3.22)
and
\[ q_{a,p} = (A_1 - A_3)(A_2 - A_3)\left(\frac{B_p}{\lambda} D_a + A_2 + \left(1 - \frac{B_p}{\lambda}\right) A_1\right) + E_n(A_1)E_n(A_2). \] (3.23)

Since \( m_1 \geq 2 \) and \( m_2 \geq 2 \), we may consider two cases:

If at least one of \( B_a \) and \( B_p \) is zero, for example \( B_a = 0 \), from (3.21), we know that \( q_{a,p} \) is irrelevant to \( p \). Assume that exists one \( p_0 \) such that \( A_{ap,n} \neq 0 \) for any \( a, 1 \leq a \leq m_1 \). By (3.20), we have \( A_{ap,n} = 0 \) for \( p \neq p_0 \). By (3.20) again, if \( p = q \), then \( A_{ap,n}^2 = \frac{q_{a,p}}{2} \) for any \( p \). Since \( q_{a,p} \) is irrelevant to \( p \), we have \( A_{ap,n}^2 = \frac{q_{a,p}}{2} = A_{ap,n}^2 = 0 \) for \( p_0, p(p \neq p_0) \). Thus, \( A_{ap,n} = 0 \), this is a contradiction. Therefore, we have \( A_{ap,n} = 0 \) for any \( p \) and \( a \). If, for example \( B_p = 0 \), from (3.22), (3.23) and by the same assertion, we have \( A_{ap,n} = 0 \) for any \( a \) and \( p \).

If \( B_a \neq 0 \) and \( B_p \neq 0 \), from (3.21) and (3.23), we know that \( q_{a,p} \) depends on \( a, p \). If \( D_1 = D_2 = \cdots = D_n \), from \( B_a = \frac{1}{\lambda}(D_a - A_1) \) and \( B_p = \frac{1}{\lambda}(D_p - A_2) \), we know that for any \( a \), all \( B_a \) are equal and for any \( p \), all \( B_p \) are equal. From (3.21) and (3.23), we see that for any \( a \) and \( p \), all \( q_{a,p} \) are equal. By the same proof as above, we know that \( A_{ap,n} = 0 \) for any \( a \) and \( p \).

If at least two of \( D_1, D_2, \ldots, D_n \) are not equal, since \( m_2 \geq 2 \) and \( m_1 \geq 2 \), we may prove that there exists at most one \( p \) such that \( q_{a,p} \neq 0 \) for any \( a, 1 \leq a \leq m_1 \) and there exists at most one \( a \) such that \( q_{a,p} \neq 0 \) for any \( p, m_1 + 1 \leq p \leq m_1 + m_2 \).
In fact, assume that exists more than one $p$, for example $p_1, p_2$ ($p_1 \neq p_2$), such that $q_{a,p_1} \neq 0, q_{a,p_1} \neq 0$. By (3.20), we have $A_{q,p}^2 = \frac{q_{a,p}}{2}$ for any $p$. Thus, $A_{q,p_1}^2 = \frac{q_{a,p_1}}{2} \neq 0$, $A_{q,p_2}^2 = \frac{q_{a,p_2}}{2} \neq 0$. By (3.20) again, we see that $A_{q,p_1}A_{q,p_2} = 0$, this is a contradiction. Thus, we know that there exists at most one $p$ such that $q_{a,p} \neq 0$ for any $a, 1 \leq a \leq m_1$. By the same proof as above, we also know that there exists at most one $a$ such that $q_{a,p} \neq 0$ for any $p, m_1 + 1 \leq p \leq m_1 + m_2$.

If for all $p, q_{a,p} = 0, 1 \leq a \leq m_1$, by (3.20), we have $A_{q,p} = 0$ for any $p$ and $a$.

If there exists one $p_0$ such that $q_{a,p_0} \neq 0, q_{a,p_0} \neq 0$, in this case, we must have that there exists one $a_0$ such that $q_{a_0,p_0} \neq 0, q_{a_0,p_0} = 0, (a \neq a_0)$. In fact, if for all $a, q_{a,p_0} = 0, m_1 + 1 \leq p \leq m_1 + m_2$, this is in contradiction with $q_{a,p_0} \neq 0$. Thus, for $1 \leq a \leq m_1$, from (3.23), we have

$$
\frac{B_p}{\lambda} D_a + A_2 + \left(1 - \frac{B_p}{\lambda}\right) A_1 = - \frac{E_a(A_1)E_n(A_2)}{(A_1 - A_3)(A_2 - A_3)}, \quad p \neq p_0. \tag{3.24}
$$

Since $B_p \neq 0$, by (3.24), we know that $D_a = D_b$ for any $a, b, 1 \leq a, b \leq m_1$. Since $B_a = \frac{1}{\lambda}(D_a - A_1)$, we easily see that $B_a = B_b$.

By the same assertion as above, from (3.21), we have $D_l = D_q$ and $B_p = B_q$ for any $p, q, m_1 + 1 \leq p, q \leq m_1 + m_2$. Thus, from (3.21) and (3.23), we see that $q_{a,p_0} = q_{a,p}$. This is in contradiction with the assumption that $q_{a,p_0} \neq 0, q_{a,p} = 0, (p \neq p_0)$. Thus, the case that there exists one $p_0$ such that $q_{a,p_0} \neq 0, q_{a,p_0} = 0, (p \neq p_0)$ does not occur.

(ii) If $m_1 = 1$, from (3.3) and (3.11), we have

$$
\Gamma_{pq}^1 = \Gamma_{pq}^a = 0, \quad \Gamma_{mn}^0 = \Gamma_{11}^0 = 0, \tag{3.25}
$$

$$
\Gamma_{mn}^1 = \frac{E_n(A_3)}{A_3 - A_1}, \quad \Gamma_{pp}^1 = \frac{E_n(A_2)}{A_2 - A_1}, \tag{3.26}
$$

$$
\Gamma_{nn}^n = \frac{E_n(A_1)}{A_1 - A_3}, \quad \Gamma_{pp}^n = \frac{E_n(A_2)}{A_2 - A_3}. \tag{3.27}
$$

From (4.4), (3.11), (3.12), (3.25) and (3.26), by the similar calculation as in (i), we have

$$
2A_{1p,n}A_{q,n} = \nu_p \delta_{pq}, \tag{3.28}
$$

for any $p$ and $q$, where

$$
\nu_p = (A_1 - A_2)(A_1 - A_3) \left\{ \frac{B_n}{\lambda} D_p + A_3 + \left(1 - \frac{B_n}{\lambda}\right) A_2 \right\}
$$

$$
+ \frac{E_1(A_2)E_1(A_3)}{(A_1 - A_2)(A_1 - A_3)} + \frac{[E_n(A_2) - E_n(A_3)]E_n(A_2)}{(A_2 - A_3)^2} - \frac{E_n(E_n(A_2))}{A_2 - A_3} + \frac{E_n(A_2)}{(A_2 - A_3)^2}. \tag{3.29}
$$

Since $m_1 = 1$ and $m_2 \geq 2$, we may consider two cases:

If $B_p = 0$, from (3.28), we know that $\nu_p$ is irrelevant to $p$. By the same proof as in (i), we see that $A_{1p,n} = 0$ for any $p$.

If $B_p \neq 0$, from (3.28), we know that $\nu_p$ depends on $p$. If $D_1 = D_2 = \cdots = D_n$, from (3.28), we see that for any $p$, all $\nu_p$ are equal. By the same proof as in (i), we see that $A_{1p,n} = 0$ for any $p$.

If at least two of $D_1, D_2, \ldots, D_n$ are not equal, since $m_2 \geq 2$, by the same proof as in (i), we easily know that there exists at most one $p$ such that $\nu_p \neq 0$.

If for any $p, \nu_p = 0$, by (3.27), we have $A_{1p,n} = 0$. 


If there is \( p_0 \), such that \( \nu_{p_0} \neq 0 \) and \( \nu = 0 \), for other \( p(p \neq p_0) \), we have

\[
\nu_{p_0} = \nu_{p_0} - \nu = (A_1 - A_2)(A_1 - A_3) \frac{B_n}{\lambda} (D_{p_0} - D_p).
\]  

(3.29)

On the other hand, since \( m_1 = 1, m_3 = 1 \) and \( A_{i,j} \) is symmetric for all indices \( i, j \) and \( k \), interchanging 1 and \( n \) in the above equations, we also have

\[
2A_{\nu,1} A_{\nu,1} = \omega_p \delta_{pq},
\]  

(3.30)

where

\[
\omega_p = (A_3 - A_2)(A_3 - A_1) \left\{ \frac{B_1}{\lambda} D_p + A_1 + \left(1 - \frac{B_1}{\lambda}\right) A_2 \right. \\
+ \frac{E_n(A_2)E_n(A_1)}{(A_3 - A_2)(A_3 - A_1)} + \frac{[E_1(A_2) - E_1(A_1)]E_1(A_2)}{(A_2 - A_1)^2} - \frac{E_1(E_1(A_2))}{A_2 - A_1} + \frac{E_1(A_2)}{(A_2 - A_1)^2} \right\}.
\]  

(3.31)

If \( B_1 = 0 \), from (3.31), we know that \( \omega_p \) is irrelevant to \( p \). By the same assertion as above, we know that \( A_{1p,n} = 0 \) for any \( p \).

If \( B_1 \neq 0 \), from (3.31), we know that \( \omega_p \) depends on \( p \). If \( D_1 = D_2 = \cdots = D_n \), from (3.31), we see that for any \( p \), all \( \omega_p \) are equal. By the same assertion as above, we see that \( A_{1p,n} = 0 \) for any \( p \).

If at least two of \( D_1, D_2, \ldots, D_n \) are not equal, since \( m_2 \geq 2 \), by the same assertion as above, we know that there exists at most one \( p \) such that \( \omega_p \neq 0 \).

If for any \( p, \omega_p = 0 \), by (3.30), we have \( A_{1p,n} = 0 \). Otherwise, we may prove that \( \omega_{p_0} \neq 0 \) for the above \( p_0 \) in (3.29). In fact, by (3.27), we have \( \lambda_{1p,n} = \frac{\nu_{p_0}}{2} \neq 0 \). On the other hand, by (3.30), we have \( A_{2p_0,1} = \frac{\omega_{p_0}}{2} \). Since \( A_{1p_0,n} = A_{p_0,1} \), we have \( \omega_{p_0} = \nu_{p_0} = 0 \). By (3.31), we also have

\[
\nu_{p_0} = \nu_{p_0} = \nu_{p_0} - \nu = (A_3 - A_2)(A_3 - A_1) \frac{B_1}{\lambda} (D_{p_0} - D_p).
\]  

(3.32)

Thus, from (3.29) and (3.32), we have

\[
(A_1 - A_3) \left\{ \frac{(A_1 - A_2)B_n}{\lambda} + (A_3 - A_2) \frac{B_1}{\lambda} \right\} (D_{p_0} - D_p) = 0,
\]

that is

\[
\{(A_1 - A_2)(D_{p_0} - D_3) + (A_3 - A_2)(D_1 - A_1)\} (D_{p_0} - D_p) = 0.
\]

If \( D_{p_0} = D_p \), by (3.29), we have \( \nu_{p_0} = \nu_p \), this contradicts with \( \nu_{p_0} \neq 0, \nu_p = 0, (p \neq p_0) \). Therefore,

\[
(A_1 - A_2)(D_{p_0} - A_3) + (A_3 - A_2)(D_1 - A_1) = 0.
\]  

(3.33)

Thus,

\[
(2A_3 - A_2 - D_n)dA_1 - (A_3 + A_1 - D_n - D_1)dA_2 + (2A_1 - A_2 - D_1)dA_3 = 0.
\]  

(3.34)
If there is a point such that at this point,
\begin{align*}
2A_3 - A_2 - D_n &= 0, \\ A_3 + A_1 - D_n - D_1 &= 0, \\ 2A_1 - A_2 - D_1 &= 0.
\end{align*}
(3.35-1)
(3.35-2)
(3.35-3)

From (3.35-1)–(3.35-3), we have $A_3 - A_2 = A_2 - A_1$ at this point. By (3.33), we have
\[ A_1 - A_3 - D_1 + D_n = 0, \]
(3.36)

at this point. From (3.35-1)–(3.35-3) and (3.36), we see that $A_1 = A_2 = A_3$ at this point. This contradicts with the assumption of (1) in Proposition 3.2. Thus, the coefficients of (3.34) are not simultaneously zero at any point. From (3.9) and (3.34), we easily know that $dA_1 = dA_2 = dA_3 = 0$, that is $A_1$, $A_2$ and $A_3$ are constants.

(2) Since $x$ has three distinct Möbius principal curvatures $B_1$, $B_2$ and $B_3$ one of which is simple and $n \geq 4$, without loss of generality, we may assume that $m_3 = 1$, $m_1m_2 \geq 2$ and $m_2 \geq 2$.

From (2.14), we have
\[ \frac{m_1dB_1}{B_3 - B_2} = \frac{m_2dB_2}{B_1 - B_3} = \frac{m_3dB_3}{B_2 - B_1}. \]
(3.37)

(i) If $m_1 \geq 2$, from Proposition 3.1 and (3.37), by the same method in the proof of (1), we will obtain that
\[ 2B_{ap,n}B_{aq,n} = \varrho'_{a,p}\delta_{pq}, \]
(3.38)

where
\[ \varrho'_{a,p} = (B_1 - B_3)(B_2 - B_3) \left( D_a + D_p + B_1B_2 - \lambda(B_1 + B_2) \right) + E_n(B_1)E_n(B_2). \]
(3.39)

Since $m_1 \geq 2$ and $m_2 \geq 2$, if $D_1 = D_2 = \cdots = D_n$, from (3.39), we see that for any $a$ and $p$, all $\varrho'_{a,p}$ are equal. By the same proof as in (1), we know that $B_{ap,n} = 0$ for any $a$ and $p$.

If at least two of $D_1, D_2, \ldots, D_n$ are not equal, by the same method in the proof of (1), we may obtain that there exists at most one $p$ such that $\varrho'_{a,p} \neq 0$ for any $a$, $1 \leq a \leq m_1$.

If for all $p$, $\varrho'_{a,p} = 0$, $1 \leq a \leq m_1$, by (3.39), we easily see that $B_{ap,n} = 0$ for any $a$ and $p$. Otherwise, by the same method in the proof of (1), we also conclude.

(ii) If $m_1 = 1$, by the same method in the proof of (1), we have
\[ 2B_{1p,n}B_{1q,n} = \nu'_p\delta_{pq}, \]
(3.40)

for any $p$ and $q$, where
\[ \nu'_p = (B_1 - B_2)(B_1 - B_3) \left\{ D_p + D_n + B_2B_3 - \lambda(B_2 + B_3) \right\} + \frac{E_1(B_2)E_1(B_3)}{(B_1 - B_2)(B_1 - B_3)} + \frac{E_n(B_2)}{(B_2 - B_3)^2} - \frac{E_n(E_n(B_2))}{B_2 - B_3} + \frac{E_n(B_2)}{(B_2 - B_3)^2}. \]
(3.41)
Since \( m_1 = 1 \) and \( m_2 \geq 2 \), if \( D_1 = D_2 = \ldots = D_n \), from (3.41), we see that for any \( p \), all \( v_\rho \) are equal. By the same proof as in (1), we know that \( B_{1,p,n} = 0 \) for any \( p \).

If at least two of \( D_1, D_2, \ldots, D_n \) are not equal, we also see that there exists at most one \( p \) such that \( v_\rho \neq 0 \).

If for any \( p, v_\rho = 0 \), by (3.41), we have \( B_{1,p,n} = 0 \). Otherwise, by the same method in the proof of (1), we see that \( B_1, B_2 \) and \( B_3 \) are constants. This completes the proof of Proposition 3.2.

**Example 3.3.** [7, 8]. For any natural number \( p, q, p + q < n \) and real number \( r \in (0, 1) \), consider the immersed hypersurface \( u : S^q(r) \times S^q(\sqrt{1 - r^2}) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \mapsto \mathbb{R}^{n+1} \)

\[
u' \in S^q(r) \subset \mathbb{R}^{n+1}, \quad u' \in S^q(\sqrt{1 - r^2}) \subset \mathbb{R}^{n+1}, \quad u'' \in \mathbb{R}^{n-p-q-1},
\]

then \( x = u(S^q(r) \times S^q(\sqrt{1 - r^2}) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1}) \mapsto S^{n+1}(1) \) is a hypersurface in \( S^{n+1}(1) \) without umbilical points and with vanishing Möbius form, it is denoted by \( CSS(p, q, r) \). From [7] and [8], by a direct calculation, we know that \( CSS(p, q, r) \) has three distinct Möbius principal curvatures. In particular, if \( p \neq q \) and \( r \neq \frac{1}{\sqrt{2}} \), then \( CSS(p, q, r) \) has exactly three distinct Blaschke eigenvalues.

**Example 3.4.** [7, 19]. Let \( \lambda \in \mathbb{R} \). For any integers \( n \) and \( k \) satisfying \( n \geq 3 \) and \( 2 \leq k \leq n-1 \), let \( \tilde{y} : M_1 \mapsto S^{k+1}(r) \subset \mathbb{R}^{k+2} \) be an immersed hypersurface without umbilical points such that the scalar curvature \( S_1 \) and the mean curvature \( H_1 \) of it satisfy

\[ S_1 = \{nk(k-1) - (n-1)r^2\}/mr^2 + n(n-1)\lambda^2, \quad H_1 = -\frac{n}{k} \lambda. \]

Let \( \tilde{y} = (\tilde{y}_0, \tilde{y}_2) : H^{n-k}(-1/r^2) \mapsto R_1^{n-k+1} \) be the canonical embedding and \( \tilde{M}^n = \tilde{M}_1 \times H^{n-k}(-1/r^2) \), \( \tilde{Y} = (\tilde{y}_0, \tilde{y}_1, \tilde{y}_2) \). We have that \( \tilde{Y} : \tilde{M}^n \mapsto R_1^{n+3} \) is an immersion, satisfying \( (\tilde{Y}, \tilde{Y})_1 = 0 \) and inducing a Riemannian metric \( g = (d\tilde{Y}, d\tilde{Y})_1 = -d\tilde{y}_0^2 + d\tilde{y}_1^2 + d\tilde{y}_2^2 \)

Obviously, \( (\tilde{M}^n, g) = (\tilde{M}_1, d\tilde{y}_1^2) \times (H^{n-k}(-1/r^2), (d\tilde{y}, d\tilde{y})_1) \) is a Riemannian manifold. Define \( \tilde{x}_1 = \tilde{y}_1/\tilde{y}_0, \tilde{x}_2 = \tilde{y}_2/\tilde{y}_0 \) and \( \tilde{x} = (\tilde{x}_1, \tilde{x}_2) \), then \( |\tilde{x}|^2 = 1 \). Thus, \( \tilde{x} : \tilde{M}^n \mapsto S^{n+1} \) defines an immersed hypersurface without umbilical points. From [7] and [19], we know that the components of the Blaschke tensor, the Möbius second fundamental form and the para-Blaschke tensor of \( \tilde{x} \) are

\[
A_{\tilde{y}_i} = \left( \frac{1}{2\tilde{r}^2} - \frac{\tilde{\lambda}^2}{2} \right) \delta_{\tilde{y}_i} - \lambda \delta_{\tilde{y}_i}, \quad B_{\tilde{y}_i} = h_{\tilde{y}_i} + \lambda \delta_{\tilde{y}_i},
\]

\[
D_{\tilde{y}_i} = \left( \frac{1}{2\tilde{r}^2} + \frac{\tilde{\lambda}^2}{2} \right) \delta_{\tilde{y}_i}, \quad 1 \leq i, j \leq k,
\]

\[
A_{\tilde{y}_i} = \left( -\frac{1}{2\tilde{r}^2} - \frac{\tilde{\lambda}^2}{2} \right) \delta_{\tilde{y}_i}, \quad B_{\tilde{y}_i} = \lambda \delta_{\tilde{y}_i},
\]

\[
D_{\tilde{y}_i} = \left( -\frac{1}{2\tilde{r}^2} + \frac{\tilde{\lambda}^2}{2} \right) \delta_{\tilde{y}_i}, \quad k + 1 \leq i, j \leq n,
\]

\[
A_{\tilde{y}_i} = 0, \quad B_{\tilde{y}_i} = 0, \quad D_{\tilde{y}_i} = 0,
\]

\[
1 \leq i \leq k, \quad k + 1 \leq j \leq n, \quad \text{or} \quad 1 \leq j \leq k, \quad k + 1 \leq i \leq n.
\]

Thus, if \( \lambda = 0 \), \( \tilde{x} \) has exactly two distinct constant Blaschke eigenvalues.
In addition, from [7], we know that $\mathbf{x}$ is Blaschke isoparametric with three Blaschke eigenvalues and four Möbius principal curvatures if and only if the corresponding hypersurface $\tilde{y}_1$ is a non-minimal Euclidean isoparametric with three distinct principal curvatures $\mu_1, \mu_2$ and $\mu_3$ satisfying $\lambda \mu_i = \frac{1}{k}$ for some $i \in \{1, 2, 3\}$. From [3], we know that such hypersurfaces $\tilde{y}_1$ do exist. If $\mathbf{x}$ has a simple Blaschke eigenvalue, then $k = 3$.

**Example 3.5.** [7, 19]. Let $\lambda \in \mathbb{R}$. For any integers $n$ and $k$ satisfying $n \geq 3$ and $2 \leq k \leq n - 1$, let $\tilde{y} = (\tilde{y}_0, \tilde{y}_1) : M_1 \mapsto H^{k+1}(-1/r^2) \subset \mathbb{R}^{k+2}_1$ be an immersed hypersurface without umbilical points such that the scalar curvature $S_1$ and the mean curvature $H_1$ of it satisfy

$$S_1 = -nk(k - 1) + (n - 1)r^2/nr^2 + n(n - 1)\lambda^2, \quad H_1 = -\frac{n}{k}\lambda.$$ 

Let $\tilde{y}_2 : S^{n-k}(r) \mapsto \mathbb{R}^{n-k+1}_1$ be the canonical embedding and $\tilde{M}^n = M_1 \times S^{n-k}(r)$, $\tilde{Y} = (\tilde{y}_0, \tilde{y}_1, \tilde{y}_2)$. $\tilde{Y} : \tilde{M}^n \mapsto \mathbb{R}^{n+3}$ is an immersion satisfying $\langle \tilde{Y}, \tilde{Y} \rangle_1 = 0$ and inducing a Riemannian metric $g = \langle d\tilde{Y}, d\tilde{Y} \rangle_1 = -d\tilde{y}_0^2 + d\tilde{y}_1^2 + d\tilde{y}_2^2$. Define $\tilde{x}_1 = \tilde{y}_1/\tilde{y}_0, \tilde{x}_2 = \tilde{y}_2/\tilde{y}_0, \tilde{x} = (\tilde{x}_1, \tilde{x}_2), |\tilde{x}|^2 = 1, \tilde{\mathbf{x}} : \tilde{M}^n \mapsto S^{n+1}$ is an immersed hypersurface without umbilical points. From [7] and [19], we know that

$$A_y = -\left(\frac{1}{2r^2} + \frac{\lambda^2}{2}\right)\delta_y - \lambda h_y, \quad B_y = h_y + \lambda \delta_y,$$
$$D_y = \left(-\frac{1}{2r^2} + \frac{\lambda^2}{2}\right)\delta_y, \quad 1 \leq i, j \leq k,$$
$$A_y = \left(\frac{1}{2r^2} - \frac{\lambda^2}{2}\right)\delta_y, \quad B_y = \lambda \delta_y,$$
$$D_y = \left(\frac{1}{2r^2} + \frac{\lambda^2}{2}\right)\delta_y, \quad k + 1 \leq i, j \leq n,$$
$$A_y = 0, \quad B_y = 0, \quad D_y = 0,$$
$$1 \leq i \leq k, \quad k + 1 \leq j \leq n, \quad \text{or} \quad 1 \leq j \leq k, \quad k + 1 \leq i \leq n.$$ 

Thus, if $\lambda = 0$, $\tilde{x}$ has exactly two distinct constant Blaschke eigenvalues.

In addition, from [7], we know that $\mathbf{x}$ is Blaschke isoparametric with three Blaschke eigenvalues and four Möbius principal curvatures if and only if the corresponding hypersurface $\tilde{y}_1$ is a non-minimal Euclidean isoparametric with three distinct principal curvatures $\mu_1, \mu_2$ and $\mu_3$ satisfying $\lambda \mu_i = \frac{1}{k}$ for some $i \in \{1, 2, 3\}$. But, from [4], we know that such a hypersurface $\tilde{y}_1$ does not exist, since if $\tilde{y}_1$ is isoparametric, then it has at most two principal curvatures.

**4. Proof of theorems.** We firstly state an important result due to Li and Wang [11]:

**Theorem 4.1.** For an immersed hypersurface $\mathbf{x} : M \mapsto S^{n+1}(1)$ without umbilical points and with vanishing Möbius form, if the para-Blaschke tensor $D$ satisfies $D = fg$ for some function $f$ on $M$, then $f$ is constant and $\mathbf{x}$ is locally Möbius equivalent to one of the following:
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(1) an immersed hypersurface $\mathbf{x} : M \mapsto S^{n+1}(1)$ with constant scalar curvature and constant mean curvature, or

(2) the image under $\sigma$ of an immersed hypersurface in $\mathbb{R}^{n+1}$ with constant scalar curvature and constant mean curvature, or

(3) the image under $\tau$ of an immersed hypersurface in $\mathbb{H}^{n+1}$ with constant scalar curvature and constant mean curvature.

**Proof of Theorem 1.2.** Let $A_1, A_2$ and $A_3$ be the three distinct Blaschke eigenvalues with multiplicities $m_1, m_2$ and $m_3$ and one of which is simple. We consider two cases:

(1) If all of the para-Blaschke eigenvalues of $\mathbf{x}$ are equal, that is $D_1 = D_2 = \ldots = D_n$, by Theorem 4.1, we know that $\mathbf{x}$ is locally Möbius equivalent to one of an immersed hypersurface $\mathbf{x} : M \mapsto S^{n+1}(1)$ with constant scalar curvature and constant mean curvature, or the image under $\sigma$ of an immersed hypersurface in $\mathbb{R}^{n+1}$ with constant scalar curvature and constant mean curvature, or the image under $\tau$ of an immersed hypersurface in $\mathbb{H}^{n+1}$ with constant scalar curvature and constant mean curvature.

(2) If not all of the para-Blaschke eigenvalues of $\mathbf{x}$ are equal, by Proposition 3.2, we know that $A_1, A_2$ and $A_3$ are constants, or $A_{a,p,n} = 0$ for any $a$ and $p$. We may prove that if $A_{q,p,n} = 0$ for any $a$ and $p$, then $A_1, A_2$ and $A_3$ are also constants. If fact, without loss of generality, we assume that $m_2 \geq 2$.

(i) If $m_1 = 1$, since $A_{1p,n} = 0$ for any $p$, putting $p = q$ in (3.27), we have $\nu_p = 0$. By (3.28),

$$
\frac{B_n}{\lambda} D_p + A_3 + \left(1 - \frac{B_n}{\lambda}\right) A_2 = -\frac{E_1(A_2)E_1(A_3)}{(A_1 - A_2)(A_1 - A_3)} - \frac{[E_n(A_2) - E_n(A_3)]E_n(A_2)}{(A_2 - A_3)^2} + \frac{E_n(A_n(A_2))}{A_2 - A_3} - \frac{E_n(A_3)}{(A_2 - A_3)^2}.
$$

If $B_n = 0$, we see that $A_3 = D_n$ is constant. By (3.9), we have that $A_1$ and $A_2$ are constants.

If $B_n \neq 0$, by (4.1), we know that for any $p$, all $D_p$ are equal. Thus, $\mathbf{x}$ has at most three distinct para-Blaschke eigenvalues $D_1, D_p$ and $D_n$ with multiplicities $1, m_2$ and $1$.

(ii) If $m_1 \geq 2$, putting $p = q$ in (3.20) and $a = b$ in (3.22), we have $\varrho_{a,q} = 0$. Thus, for any $a$ and $p$, by (3.21) and (3.23), we have

$$
\frac{B_n}{\lambda} D_p + A_1 + \left(1 - \frac{B_n}{\lambda}\right) A_2 = -\frac{E_n(A_1)E_n(A_2)}{(A_1 - A_3)(A_2 - A_3)},
$$

$$
\frac{B_n}{\lambda} D_n + A_2 + \left(1 - \frac{B_n}{\lambda}\right) A_1 = -\frac{E_n(A_1)E_n(A_2)}{(A_1 - A_3)(A_2 - A_3)}.
$$

If at least one of $B_n$ and $B_n$ is zero, we easily see that $A_1, A_2$ and $A_3$ are constants. If all of $B_n$ and $B_n$ are not zero, by (4.2) and (4.3), we easily see that for any $a$ and $b$, $D_a = D_b$, and for any $p$ and $q$, $D_p = D_q$. Thus, $\mathbf{x}$ has at most three distinct para-Blaschke eigenvalues $D_a, D_p$ and $D_n$ with multiplicities $m_1, m_2$ and 1.

Let $D_a, D_p$ and $D_n$ be the three constant para-Blaschke eigenvalues with multiplicities $m_1, m_2$ and 1. We choose a local orthonormal basis $\{E_1, \ldots, E_n\}$ such that $E_i$ is the unit para-Blaschke tensor of $D_i$. By (2.24),

$$
D_{i,k} = E_k(D_i)\delta_{i,j} + \Gamma_{i,k}^j(D_i - D_j),
$$
where $\Gamma^i_{\ell k}$ is the Levi–Civita connection of $g$ given by $\omega^\ell = \sum_k \Gamma^i_{\ell k} \omega_k$, $\Gamma^i_{\ell k} = -\Gamma^i_{k \ell}$. By (2.26), we have $D_{i,j} = D_{j,i}$. Thus,

$$E_i(D_i) = \Gamma^i_{j i}(D_i - D_j), \quad i \neq j.$$  \hfill (4.4)

If the number of the distinct para-Blaschke eigenvalues of $D_1, D_p$ and $D_n$ is two, when $m_1 = 1$, without loss of generality, we assume that $D_1 = D_n \neq D_p$. By (4.4),

$$0 = E_1(D_p) = \Gamma^{1}_{pp}(D_p - D_a), \quad 0 = E_n(D_p) = \Gamma^{n}_{pp}(D_p - D_n).$$  \hfill (4.5)

Thus, $\Gamma^{1}_{pp} = \Gamma^{n}_{pp} = 0$. On the other hand, by (2.16) and $A_{ii,j} = A_{j,i, i}$, we have

$$E_i(A_i) = \Gamma^i_{j i}(A_i - A_j), \quad i \neq j,$$  \hfill (4.6)

where $\{E_1, \ldots, E_n\}$ a local orthonormal basis such that $E_i$ is the unit Blaschke tensor of $A_i$. Thus,

$$E_1(A_2) = \Gamma^{1}_{pp}(A_2 - A_1) = 0, \quad E_n(A_2) = \Gamma^{n}_{pp}(A_2 - A_3) = 0.$$

From Proposition 3.1, we have $E_p(A_2) = 0$, $2 \leq p \leq 1 + m_2$. Thus, $A_2$ is constant. By (3.9), we have that $A_1$ and $A_3$ are constants.

When $m_1 \geq 2$, without loss of generality, we assume that $D_1 = D_p \neq D_n$. From (4.4),

$$0 = E_n(D_a) = \Gamma^{n}_{aa}(D_a - D_n).$$

Thus, $\Gamma^{n}_{aa} = 0$. On the other hand, by (4.6),

$$E_n(A_1) = \Gamma^{n}_{aa}(A_1 - A_3) = 0.$$  

From Proposition 3.1, we have $E_a(A_1) = 0$, $1 \leq a \leq m_1$ and $E_p(A_2) = 0$, $m_1 + 1 \leq p \leq m_1 + m_2$. By (3.5) and (3.6), we have

$$\frac{E_i(A_1)}{m_2 \xi_3 - m_3 \xi_2} = \frac{E_i(A_2)}{m_3 \xi_1 - m_1 \xi_3} = \frac{E_i(A_3)}{m_1 \xi_2 - m_2 \xi_1}.$$  \hfill (4.7)

Thus, by (4.7), we have $E_a(A_3) = E_p(A_3) = E_n(A_3) = 0$ and $A_3$ is constant. By (3.9) again, we know that $A_1$ and $A_2$ are constants.

If the number of the distinct para-Blaschke eigenvalues of $D_1, D_p$ and $D_n$ is three, when $m_1 = 1$, by (4.4),

$$0 = E_1(D_p) = \Gamma^{1}_{pp}(D_p - D_1), \quad 0 = E_n(D_p) = \Gamma^{n}_{pp}(D_p - D_n).$$  \hfill (4.8)

Thus, $\Gamma^{1}_{pp} = \Gamma^{n}_{pp} = 0$. On the other hand, by (4.6),

$$E_1(A_2) = \Gamma^{1}_{pp}(A_2 - A_1) = 0, \quad E_n(A_2) = \Gamma^{n}_{pp}(A_2 - A_3) = 0.$$

From Proposition 3.1, we have $E_p(A_2) = 0$, $2 \leq p \leq 1 + m_2$. Thus, $A_2$ is constant. By (3.9) again, we know that $A_1$ and $A_3$ are constants.

When $m_1 \geq 2$, by (4.4),

$$0 = E_n(D_a) = \Gamma^{n}_{aa}(D_a - D_n).$$
Therefore, $\Gamma^a_{\alpha\alpha} = 0$. On the other hand, by (4.6),

$$E_n(A_1) = \Gamma^a_{\alpha\alpha}(A_1 - A_3) = 0.$$

From Proposition 3.1, we have $E_\alpha(A_1) = 0$, $1 \leq \alpha \leq m_1$ and $E_\alpha(A_2) = 0$, $m_1 + 1 \leq \alpha \leq m_1 + m_2$. From (4.7), we have $E_\alpha(A_3) = E_\alpha(A_3) = E_n(A_3) = 0$ and $A_3$ is constant. By (3.9) again, we know that $A_1$ and $A_2$ are constants.

Since $A_1$, $A_2$ and $A_3$ are constants and $n \geq 4$, from Theorem 1.1, we see that $x$ is locally M"obius equivalent to $CSS(p, q, r)$ for some constants $p, q, r$, $p \neq q$ and $r \neq \frac{1}{\sqrt{2}}$, or one of the hypersurfaces as indicated in Example 3.4 where $k = 3$ and $\tilde{\gamma}_1 : M_1 \hookrightarrow S^4(r)$ is one of Cartan's non-minimal isoparametric hypersurfaces with three principal curvatures satisfying $\lambda = \frac{1}{\sqrt{2}}$ for some $i \in \{1, 2, 3\}$. This completes the proof of Theorem 1.2.

**Proof of Theorem 1.4.** Let $B_1$, $B_2$ and $B_3$ be the three distinct Blaschke eigenvalues with multiplicities $m_1$, $m_2$ and $m_3$ and one of which is simple. We consider two cases:

(1) If all of the para-Blaschke eigenvalues of $x$ are equal, by Theorem 4.1, we see that Theorem 1.4 is true.

(2) If not all of the para-Blaschke eigenvalues of $x$ are equal, by Proposition 3.2, we know that $B_1$, $B_2$ and $B_3$ are constants, or $B_{p,n} = 0$ for any $a$ and $p$. In the latter case, without loss of generality, we assume that $m_2 \geq 2$.

(i) If $m_1 = 1$, since $B_{1p,n} = 0$ for any $p$, putting $p = q$ in (3.40), we have $\nu_p = 0$. By (3.41),

$$D_p + D_n + B_2 B_3 - \lambda(B_2 + B_3) = -\frac{E_1(B_2)E_1(B_3)}{(B_1 - B_2)(B_1 - B_3)} - \frac{E_n(B_2)E_n(B_3)}{(B_2 - B_3)^2} + \frac{E_n(E_n(B_2))}{B_2 - B_3} - \frac{E_n(B_2)}{(B_2 - B_3)^2}. \quad (4.9)$$

Thus, we know that for any $p$, all $D_p$ are equal and $x$ has at most three distinct para-Blaschke eigenvalues $D_1$, $D_p$ and $D_n$ with multiplicities $1$, $m_2$ and $1$.

(ii) If $m_1 \geq 2$, putting $p = q$ in (3.38), we have $\nu_{a,q} = 0$. By (3.39), we have

$$D_a + D_p + B_1 B_2 - \lambda(B_1 + B_2) = -\frac{E_n(B_1)E_n(B_2)}{(B_1 - B_2)(B_2 - B_3)}. \quad (4.10)$$

By (4.10), we easily see that $x$ has at most three distinct para-Blaschke eigenvalues $D_a$, $D_p$ and $D_n$ with multiplicities $m_1$, $m_2$ and $1$.

If the number of the distinct para-Blaschke eigenvalues of $D_a$, $D_p$ and $D_n$ is two, when $m_1 = 1$, without loss of generality, we assume that $D_a = D_n \neq D_p$. By (4.5), we have $\Gamma_1^{pp} = \Gamma_n^{pp} = 0$. On the other hand, by (2.17) and $B_{ij} = B_{ji}$, we have

$$E_j(B_i) = \Gamma_0^{ij}(B_i - B_j), \quad i \neq j. \quad (4.11)$$

Thus,

$$E_1(B_2) = \Gamma_1^{pp}(B_2 - B_1) = 0, \quad E_n(B_2) = \Gamma_n^{pp}(B_2 - B_3) = 0.$$

From Proposition 3.1 and (3.37), by the similar proof of Theorem 1.2, we know that $B_1$, $B_2$ and $B_3$ are constants.
When \( m_1 \geq 2 \), without loss of generality, we assume that \( D_a = D_p = D_n \). From (4.4), we have \( \Gamma_{aa}^n = 0 \). On the other hand, by (4.11),

\[
E_n(B_1) = \Gamma_{aa}^n(B_1 - B_3) = 0.
\]

From Proposition 3.1, we have \( E_a(B_1) = 0, 1 \leq a \leq m_1 \) and \( E_p(B_2) = 0, m_1 + 1 \leq p \leq m_1 + m_2 \). Combining with (3.37) and by the similar proof of Theorem 1.2, we know that \( B_1, B_2 \) and \( B_3 \) are constants.

If the number of the distinct para-Blaschke eigenvalues of \( D_1, D_p \) and \( D_n \) is three, when \( m_1 = 1 \), by (4.8), we have \( \Gamma_{pp}^1 = \Gamma_{pp}^n = 0 \). On the other hand, by (4.11),

\[
E_1(B_2) = \Gamma_{pp}^1(B_2 - B_1) = 0, \quad E_n(B_2) = \Gamma_{pp}^n(B_2 - B_3) = 0.
\]

From Proposition 3.1 and (3.37), by the similar proof of Theorem 1.2, we know that \( B_1, B_2 \) and \( B_3 \) are constants.

When \( m_1 \geq 2 \), by (4.4), \( \Gamma_{aa}^n = 0 \). On the other hand, by (4.11),

\[
E_n(B_1) = \Gamma_{aa}^n(B_1 - B_3) = 0.
\]

From Proposition 3.1, we have \( E_a(B_1) = 0, 1 \leq a \leq m_1 \) and \( E_p(B_2) = 0, m_1 + 1 \leq p \leq m_1 + m_2 \). Combining with (3.37), we know that \( B_1, B_2 \) and \( B_3 \) are constants.

Since \( B_1, B_2 \) and \( B_3 \) are constants, from Theorem 1.3, we see that Theorem 1.4 is true. This completes the proof of Theorem 1.4.

Acknowledgement. The authors would like to thank the referee for his many valuable remarks and suggestions that really improve the paper.

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