J. Austral. Math. Soc. (*Series A*) **57** (1994), 76–80

CHARACTER DEGREES AND NILPOTENCE CLASS IN p-GROUPS

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(Received 13 May 1993)

Communicated by L. G. Kovács

Abstract

Work of Isaacs and Passman shows that for some sets X of integers, p-groups whose set of irreducible character degrees is precisely X have bounded nilpotence class, while for other choices of X, the nilpotence class is unbounded. This paper presents a theorem which shows some additional sets of character degrees which bound nilpotence class within the family of metabelian p-groups. In particular, it is shown that if the non-linear irreducible character degrees of G lie between p^a and p^b, where $a \leq b \leq 2a-2$, then the nilpotence class of G is bounded by a function of p and $b-a$.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 20C15.

1. Introduction

Given a finite group G, there are many interesting theorems relating the internal structure of G to its set of irreducible character degrees $cd(G)$. In particular, if G is a finite p-group, Isaacs and Passman [1] showed that if $cd(G) = \{1, p^e\}$ with $e > 1$, then G has nilpotence class at most p. On the other hand, if $cd(G) = \{1, p\}$, the class of G can be arbitrarily large. Thus we see that some sets of character degrees bound the nilpotence class of a p-group and some do not. This raises the question of characterizing those sets which bound. As mentioned above, the set $\{1, p\}$ does not bound and by a direct product construction one sees that if the powers in X consist of all the numbers in some set of degrees Y unioned with p times these numbers (that is, $X = \{1, p\} \cdot Y$), then X does not bound. Currently the sets described above are the only ones for which the question of bounding nilpotence class has been settled.

In this paper we will prove

THEOREM 1.1. *Let G be a finite metabelian p-group and let $a \leq b \leq 2a-2$ be*

https://doi.org/10.1017/S1446788700036065 Published online by Cambridge University Press

such that

$$p^a \leq \chi(1) \leq p^b$$

for all non-linear χ *in* $\mathrm{Irr}(G)$. *Then the nilpotence class of* G *is bounded by a function of* p *and* $b - a$.

Because of the hypothesis that G is metabelian, this theorem does not really give us new bounding sets, but gives some evidence that these sets may bound. Not only has the author been unable to remove the metabelian hypothesis or show it is needed, but he has found no example of a non-metabelian p-group with the character degrees described in the theorem. Especially in light of [2], it may be the case that these character degree sets force G to be metabelian.

Finally, note that Taketa's Theorem says that if $cd(G) = \{1, p^e\}$, then G is metabelian, so our theorem is a generalization of the result of Isaacs and Passman. The techniques in this paper are largely refinements and extensions of their approach.

2. Proof of Theorem

Write the lower central series of a group G as $G = G_1 \geq G_2 \geq \cdots$. As is well known,

(1) $$G_{k+n} = \left\langle [g, g_1, \ldots, g_n] \;\middle|\; g \in G_k,\ g_1, \ldots, g_n \in G \right\rangle.$$

We will need the following easy commutator lemma.

LEMMA 2.1. *If* $L \subseteq G$ *with* $L' = G'$, *then* $L_n = G_n$ *for* $n \geq 2$.

PROOF. The case $n = 2$ is true by hypothesis. Since $L_n \subseteq G_n$, we need to show $G_n \subseteq L_n$. Note that $G' = L' \subseteq L$ implies that $L \trianglelefteq G$ and so $L_n \trianglelefteq G$. By the inductive hypothesis,

$$G_n = [G_{n-1}, G] = [L_{n-1}, G].$$

We will use the Three Subgroups Lemma to show that $[L_{n-2}, L, G]$ is in L_n. Now

$$[L, G, L_{n-2}] \subseteq \left[G', L_{n-2}\right] = \left[L', L_{n-2}\right] \subseteq L_n$$

and

$$[G, L_{n-2}, L] \subseteq [G, G_{n-2}, L] = [G_{n-1}, L] = [L_{n-1}, L] = L_n.$$

Thus

$$G_n = [L_{n-2}, L, G] \subseteq L_n.$$

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THEOREM 2.2. *Given a prime p and integers a and b with $1 \leq a \leq b \leq 2a-2$, let G be a metabelian p-group such that $p^a \leq \chi(1) \leq p^b$ for all non-linear χ in* $\mathrm{Irr}(G)$. *Then the nilpotence class $c(G)$ of G satisfies*

$$c(G) \leq \big(p^{b-a+1}-1\big)\big(p(b-a+1)-1\big)+3.$$

PROOF. If the theorem is not true, choose a, b, and G so that the theorem is true for any smaller value of $b-a$ and is true with the given $b-a$ for any group whose order is less than $|G|$. Let

$$m = (p^{b-a+1}-1)(p(b-a+1)-1)+4,$$

so our assumption is that $G_m \neq 1$.

Step 1: $Z(G)$ is cyclic.

If not, choose two distinct subgroups U, V of order p in $Z(G)$. Then G/U and G/V are metabelian p-groups with $cd(G/U) \subseteq cd(G)$ and $cd(G/V) \subseteq cd(G)$. Therefore, by the minimality of $|G|$, we see that $G_m \subseteq U \cap V = 1$. This contradicts the choice of G, and so proves that $Z(G)$ is cyclic.

Step 2: There exists an abelian normal subgroup A of G with G/A abelian and $|G:A| = p^b$.

By Step 1, $Z(G)$ contains a unique subgroup of order p, which is contained in the kernel of any non-faithful character of G. Thus G must have a faithful irreducible character χ. Since G' is abelian, we can find $A \subseteq G$ with $G' \subseteq A$ and a linear character α of A such that $\alpha^G = \chi$. Now $G' \subseteq A$ implies $A \trianglelefteq G$ and so every irreducible constituent of χ_A is linear. Thus $A' \subseteq \mathrm{Ker}\chi = 1$, hence A is abelian. Now $|G:A| = \chi(1)$ is an upper bound for the irreducible character degrees. If this were less than p^b, G would satisfy the hypotheses with a smaller value of $b-a$, contrary to our choice. Therefore $|G:A| = p^b$.

Step 3: The quotient G/A has an elementary abelian group of order p^a as a homomorphic image, and so as a subgroup.

If we let $K \trianglelefteq G$ be chosen maximal with respect to G/K being non-abelian, then by the maximality of K any central subgroup of order p in G/K must contain the derived subgroup of G/K. It follows that G/K has derived subgroup of order p and cyclic center Z/K. Then AZ/K is an abelian normal subgroup of G/K. Since $cd(G/K) \subseteq cd(G)$, we must have $|G:AZ| \geq p^a$. Also, since the derived subgroup of G/K is central of order p, G/Z is elementary abelian, hence G/A has an elementary abelian image of order p^a.

Step 4: If $L \trianglelefteq G$ with $|G:L| < p^a$, then $L' = G'$ and so $L_n = G_n$ for $n \geq 2$.

The last statement follows from Lemma 2.1, so we need only show $L' = G'$. Suppose $L' < G'$. Then G/L' is a non-abelian factor group of G and so it has some

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irreducible character of degree $\geq p^a$. However, L/L' is a normal abelian subgroup of index $< p^a$, contrary to this situation. Thus we must have $L' = G'$.

Step 5: The exponent of G_3 divides p^{b-a+1}.

Let $B = G/A$ and note that $\mathbb{Z}B$, the group ring of B over the integers, acts on A via conjugation extended linearly (since A is abelian). With the corresponding change in notation, if $k \geq 2$ then (1) becomes

$$(2) \qquad G_{k+n} = \left\langle g^{(b_1-1)\cdots(b_n-1)} \;\middle|\; g \in G_k,\ b_1, \ldots, b_n \in B \right\rangle$$

If S is any subset of B, we write $\widehat{S}$ to denote the sum of the elements of S in $\mathbb{Z}B$. Note that if D is any subgroup of B with $|B : D| < p^a$, then Step 4 and (2) imply that

$$G_3 = \left\langle x^{y-1} \;\middle|\; x \in G',\ y \in D \right\rangle.$$

Thus $\widehat{D}$ annihilates G_3.

By Step 3, we may choose $W \subseteq B$ an elementary abelian subgroup of order p^{b-a+2} (since $b - a + 2 \leq a$). Let $k = b - a + 1$. Then W has $p^{k+1} - 1$ elements of order p, each of which is contained in precisely $p^{k-1} + p^{k-2} + \ldots + p + 1$ subgroups of order p^k. Therefore in $\mathbb{Z}B$ we have

$$\left(\sum_D \widehat{D}\right) - \left(p^{k-1} + \ldots + p + 1\right)\widehat{W} = p^k \cdot 1$$

where the sum is over all subgroups D of order p^k in W. But for any such D we have

$$|B : D| = p^b / p^{b-a+1} = p^{a-1} < p^a$$

and so, by above, $\widehat{D}$ and $\widehat{W}$ annihilate G_3. Thus $g^{p^k} = 1$ for all $g \in G_3$.

Step 6: Contradiction. We will show that $G_m = 1$ contrary to our choice of G.

With B as in the proof of Step 5, choose $D \subseteq B$ with D elementary abelian of order p^{b-a+1}. Then, by Step 4 and (2), we have

$$G_{n+3} = \left\langle x^{(y_1-1)\cdots(y_n-1)} \;\middle|\; x \in G_3,\ y_1, \ldots, y_n \in D \right\rangle.$$

Now, for any $y \in D$,

$$(y-1)^p \equiv 0 \pmod{p\,\mathbb{Z}B}$$

and so

$$(y-1)^{p(b-a+1)} \equiv 0 \pmod{p^{b-a+1}\,\mathbb{Z}B}.$$

Since D is abelian and of order p^{b-a+1}, if $n > (p^{b-a+1} - 1)(p(b-a+1) - 1)$ then each product $(y_1 - 1) \cdots (y_n - 1)$ with $y_1, \ldots, y_n \in D$ must have either a factor $(1-1)$

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or a factor $(y-1)^{p(b-a+1)}$. In view of Step 5, for such n it follows that each generator $x^{(y_1-1)\cdots(y_n-1)}$ of G_{n+3} is 1, which implies that $G_{n+3}=1$.

In particular, since $m=(p^{b-a+1}-1)(p(b-a+1)-1)+4$, we have $G_m=1$. This final contradiction shows that no counterexample exists and so the theorem is established.

References

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