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The Oscillatory Hyper-Hilbert Transform Associated with Plane Curves

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Abstract. In this paper, the bounded properties of oscillatory hyper-Hilbert transform along certain plane curves $\gamma(t)$,

$$T_{\alpha,\beta}f(x,y)=\int_0^1 f(x-t,y-\gamma(t))e^{it^{-\beta}}\frac{\mathrm{d}t}{t^{1+\alpha}}$$

are studied. For general curves, these operators are bounded in $L^2(\mathbb{R}^2)$ if $\beta\geq 3\alpha$. Their boundedness in $L^p(\mathbb{R}^2)$ is also obtained, whenever $\beta>3\alpha$ and $\frac{2\beta}{2\beta-3\alpha}<p<\frac{2\beta}{3\alpha}$.

1 Introduction

Let $\Gamma(t)\colon\mathbb{R}\to\mathbb{R}^n$ be a continuous curve with $\Gamma(0)=0$. For an appropriate function f, the Hilbert transform along curve is defined as

$$Hf(x)=\mathrm{pv}\int_{-\infty}^{+\infty} f(x-\Gamma(t))\frac{\mathrm{d}t}{t},$$

where pv is used to indicate a principal-value integral. A fundamental problem is the study of its boundedness in $L^p(\mathbb{R}^n)$, which has attracted enormous attention. We list some of the literature here: [1,3,4,10,11,14,17,18]. Our view of the problem is to set up the boundedness of singular operator along a general curve Γ. More clearly, we want to know under what conditions the following inequality holds:

$$\|Hf\|_{L^p(\mathbb{R}^n)}\leq C\|f\|_{L^p(\mathbb{R}^n)},\ \text{for } 1<p<\infty. \tag{1.1}$$

Stein and Waigner pointed out in [16] that the curvature of the curve plays a crucial role in this project. In the same paper, they showed that if $\Gamma(t)$ is well curved,[1] then (1.1) holds. In [5], the *well-curved condition* was used for an odd or even convex curve where $\gamma(t)\in C^2(0,\infty)$, $\gamma(0)=0$, and the following doubling condition holds:

(D) There exists $1<\lambda<\infty$ so that $\gamma'(\lambda t)\geq 2\gamma'(t)$ for every $0<t<\infty$.

Let $h(t)=t\gamma'(t)-\gamma(t)$. In [2] condition (D) was replaced by the following infinitesimally doubling condition.

(ID) There exists $0<\varepsilon_0<\infty$ so that $h'(t)\geq\varepsilon_0\dfrac{h(t)}{t}$ for every $0<t<\infty$.

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[1]We refer the reader to the paper [16, p. 1240] for the definition of the well-curved curves.

Later, in [20] (D) and (ID) were extended to the *mixed condition*:

(M) There exist $0 < \varepsilon_0 < \infty$ and $1 < \lambda < \infty$ so that

$$\max\Big\{\frac{h'(t)}{\varepsilon_0}, \frac{1}{2}\Big(\gamma'(\lambda t) - 2\frac{\gamma(t)}{t}\Big)\Big\} \geq \frac{h(t)}{t}.$$

There exists a curve $\gamma(t)$ that satisfies (D) but not (ID) and also a curve $\gamma(t)$ that satisfies (ID) but not (D) (see [20]). Condition (M) is weaker than (D) and (ID). Ziesler in [20] gave an example that satisfies (M), but neither (D) nor (ID).

We now turn to the oscillatory hyper-Hilbert transform associated with plane curves $\gamma(t)$:

$$T_{\alpha,\beta}f(x,y) = \int_0^1 f(x-t, y-\gamma(t))e^{it^{-\beta}}\frac{\mathrm{d}t}{t^{1+\alpha}},$$

with $\alpha \geq 0$, $\beta > 0$. We are interested in determining the boundedness in $L^p(\mathbb{R}^2)$ for some general curves $\Gamma(t) = (t, \gamma(t))$. We recall the classical Hyper-Hilbert transform,

$$T(f)(x) = \int_0^1 f(x-t)e^{it^{-\beta}}\frac{dt}{t^{1+\alpha}}.$$

It is also known as the weakly strongly singular integral operator (see [12]). The interest in this operator is that it does not fall into the category of Calderón–Zygmund operators and is bounded on some, but not all, of the L^p spaces. For its L^p boundedness we refer the reader to [12], and a simple proof can also be found in [13]. This operator also has a close relation with the Bochner–Riesz mean operator.

The first result on the bounded property of a hyper-Hilbert transform along plane curve is due to Zielinski. For $\Gamma(t) = (t, t^2)$ he showed in [19] that $\|T_{\alpha,\beta}f\|_{L^2(\mathbb{R}^2)} \leq C\|f\|_{L^2(\mathbb{R}^2)}$ if and only if $\beta \geq 3\alpha$. L^p boundedness was then studied by Chandarana in [6]. He showed that along $\Gamma(t) = (t, |t|^k)$ or $\Gamma(t) = (t, |t|^k \operatorname{sgn} t)$, $k \geq 2$,

(a) $\|T_{\alpha,\beta}f\|_{L^2(\mathbb{R}^2)} \leq C\|f\|_{L^2(\mathbb{R}^2)}$ if and only if $\beta \geq 3\alpha$;
(b) $\|T_{\alpha,\beta}f\|_{L^p(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}$, when $\beta > 3\alpha$ and

$$1 + \frac{3\alpha(\beta+1)}{\beta(\beta+1)+(\beta-3\alpha)} < p < 1 + \frac{\beta(\beta+1)+(\beta-3\alpha)}{3\alpha(\beta+1)}.$$

For the higher dimensional case, for $\Gamma(t) = (t^{p_1}, t^{p_2}, \ldots, t^{p_n})$, Chen, Fan, and Zhu [9] proved that $\|T_{\alpha,\beta}f\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)}$ if and only if $\beta \geq (n+1)\alpha$, where $\beta > \alpha \geq 0$ and $p_1 < p_2 < \cdots < p_n$. The corresponding L^p boundedness was set up in [8]. For a more general curve, very recently, Chen, Damtew, and Zhu obtained the following theorem.

Theorem A (see [7]) *Let $\beta > 3\alpha$ when $2\beta/2\beta - 3\alpha < p < 2\beta/3\alpha$ and $\beta \geq 3\alpha$ when $p = 2$. Let $\gamma(t) \in C^2(0,1)$ be a convex curve with $\gamma(0) = \gamma'(0) = 0$. If there exists a positive constant $\delta < \beta + 2$ such that $t^\delta\gamma''(t)$ is an increasing function on $(0,1)$, then*

$$\|T_{\alpha,\beta}f\|_{L^p(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}.$$

In this paper, we consider a curve $\gamma(t) \in C^1(0,1)$ satisfying the following conditions:

$$\lim_{t\to 0^+} \gamma(t) = 0, \quad \lim_{t\to 0^+} \gamma'(t) = 0, \tag{1.2}$$

$$\gamma''(t) > 0, \quad \gamma'''(t) \ge 0 \quad \text{on } (2^{-j}, 2^{-j+1}) \text{ for } j \in \mathbb{Z}^+. \tag{1.3}$$

There exists $0 < C_1 \le C_2 < \infty$ so that

$$C_1 h'(t) \le \frac{h(t)}{t} \le C_2 h'(t) \text{on } (2^{-j}, 2^{-j+1}) \tag{1.4}$$

for each $j \in \mathbb{Z}^+$, where $h(t) = t\gamma'(t) - \gamma(t)$.

We now state our main result.

Theorem 1.1 *Let $\beta > 3\alpha$ when $\frac{2\beta}{2\beta - 3\alpha} < p < \frac{2\beta}{3\alpha}$ and $\beta \ge 3\alpha$ when $p = 2$. If $\gamma(t)$ satisfies (1.2)–(1.4), then*

$$\|T_{\alpha,\beta} f\|_{L^p(\mathbb{R}^2)} \le C\|f\|_{L^p(\mathbb{R}^2)}.$$

Remark 1.2 Condition (1.4) is not very strong. In fact, if $\lim_{t\to 0^+} h(t)/(t^2\gamma''(t)) = C$, where $t \in (2^{-j}, 2^{-j+1})$, $j \in \mathbb{Z}^+$ and $0 < C < \infty$, then there exist constant $0 < C_1 \le C_2 < \infty$ and $0 < C_3 < \infty$ such that $C_1 h'(t) \le \frac{h(t)}{t} \le C_2 h'(t)$ on $(2^{-j}, 2^{-j+1})$ for each $j \ge C_3$ and $j \in \mathbb{Z}^+$. The model example (t, t^k) with $k \ge 2$ satisfies this condition.

Remark 1.3 Condition (1.4) is needed only in the case $\beta = 3\alpha$. For the case $\beta > 3\alpha$, we do not assume this condition. This can be seen in Section 2.

Remark 1.4 There exists a curve $\gamma(t)$ satisfying (1.2)–(1.4) but not the assumptions in Theorem A.

Step 1 The construction of $\{a_m\}$ and $\{b_m\}$.
Set

$$\sum_{m\ge 2} \frac{1}{m^2} + \frac{1}{(m - \frac{1}{2})^2} := \Delta.$$

We define the series $\{a_m\}$ by

$$a_1 = 3 + \Delta \quad \text{and} \quad a_{m-1} - a_m = \frac{1}{m^2} + \frac{1}{(m - \frac{1}{2})^2} \text{ for each } m > 1.$$

We define series $\{b_m\}$ by

$$b_m - a_m = \frac{1}{m^2} \text{ for each } m \ge 1.$$

Then we have

$$\lim_{m\to\infty} a_m = \lim_{m\to\infty} b_m = 3$$

and

$$3 < a_m < b_m < a_{m-1} < b_{m-1} \text{ for each } m > 1.$$

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Step 2 The construction of an oscillatory function $\gamma''(t)$.

Let

$$\gamma''(t) = \begin{cases} t^{a_m}, & t \in (\frac{1}{2^{2m+1}}, \frac{1}{2^{2m}}), \\ t^{b_m}, & t \in (\frac{1}{2^{2m}}, \frac{1}{2^{2m-1}}), \end{cases}$$

where $m \geq 1$. From $a_m < b_m < a_{m-1} < b_{m-1}$ for each $m > 1$, we have that $\gamma''(t)$ is an oscillatory function, so there does not exists a positive constant $\delta < \beta + 2$ and constant $0 < \Theta < 1$ such that $t^\delta \gamma''(t)$ is an increasing function on $(0, \Theta)$.

Step 3 From the definition of $\gamma''(t)$, we have $\gamma''(t) > 0$ on $(2^{-j}, 2^{-j+1})$ for each $j \geq 2$ and $j \in \mathbb{Z}^+$, and

$$\gamma'''(t) = \begin{cases} a_m t^{a_m - 1}, & t \in (\frac{1}{2^{2m+1}}, \frac{1}{2^{2m}}), \\ b_m t^{b_m - 1}, & t \in (\frac{1}{2^{2m}}, \frac{1}{2^{2m-1}}). \end{cases}$$

Thus, $\gamma'''(t) > 0$ for each $j \geq 2$ and $j \in \mathbb{Z}^+$, and

$$\frac{\gamma''(t)}{2\gamma''(t) + t\gamma'''(t)} = \begin{cases} \frac{1}{2+a_m}, & t \in (\frac{1}{2^{2m+1}}, \frac{1}{2^{2m}}), \\ \frac{1}{2+b_m}, & t \in (\frac{1}{2^{2m}}, \frac{1}{2^{2m-1}}). \end{cases}$$

We also have

$$\gamma(t) = \int_0^t \int_0^\tau \gamma''(s) \mathrm{d}s \mathrm{d}\tau + t\gamma'_+(0) + \gamma_+(0),$$
$$\gamma'(t) = \int_0^t \gamma''(s) \mathrm{d}s + \gamma'_+(0),$$

where $\gamma'_+(0) = \lim_{t \to 0^+} \gamma'(t)$, $\gamma_+(0) = \lim_{t \to 0^+} \gamma(t)$. The conditions $\gamma'_+(0) = 0$ and $\gamma_+(0) = 0$ will be checked in Step 5. Noticing that $\lim_{m\to\infty} a_m = \lim_{m\to\infty} b_m = 3$, we have

$$\lim_{t\to 0^+} \frac{h(t)}{t^2\gamma''(t)} = \lim_{t\to 0^+} \frac{t\int_0^t \gamma''(s)\mathrm{d}s - \int_0^t \int_0^\tau \gamma''(s)\mathrm{d}s\mathrm{d}\tau}{t^2\gamma''(t)}$$
$$= \lim_{t\to 0^+} \frac{\gamma''(t)}{2\gamma''(t) + t\gamma'''(t)} = \frac{1}{5},$$

where $t \in (2^{-j}, 2^{-j+1})$, $j \in \mathbb{Z}^+$. From Remark 1.2, there exists constants $0 < C_1 \leq C_2 < \infty$ and $0 < C_3 < \infty$ such that $C_1 h'(t) \leq \frac{h(t)}{t} \leq C_2 h'(t)$ on $(2^{-j}, 2^{-j+1})$ for each $j \geq C_3$ and $j \in \mathbb{Z}^+$.

Step 4 To show that $\gamma(t) \in C^1(0,1)$, we define $\gamma'(t)$ as

$$\gamma'(t) = \begin{cases} \frac{1}{a_m+1} t^{a_m+1} + a'_m, & t \in (\frac{1}{2^{2m+1}}, \frac{1}{2^{2m}}), \\ \frac{1}{b_m+1} t^{b_m+1} + b'_m, & t \in (\frac{1}{2^{2m}}, \frac{1}{2^{2m-1}}). \end{cases}$$

To keep the continuity, the series $\{a'_m\}$ and $\{b'_m\}$ will be chosen such that

$$\lim_{n\to+\infty} a'_m = \lim_{n\to+\infty} b'_m = 0,$$
$$\frac{1}{a_m+1}\frac{1}{2^{2m(a_m+1)}} + a'_m = \frac{1}{b_m+1}\frac{1}{2^{2m(b_m+1)}} + b'_m,$$
$$\frac{1}{b_m+1}\frac{1}{2^{(2m-1)(b_m+1)}} + b'_m = \frac{1}{a_{m-1}+1}\frac{1}{2^{(2m-1)(a_{m-1}+1)}} + a'_{m-1}.$$

In fact, a'_m satisfies

$$a'_m - a'_{m-1} = \frac{1}{b_m+1}\frac{1}{2^{2m(b_m+1)}} - \frac{1}{a_m+1}\frac{1}{2^{2m(a_m+1)}} + \frac{1}{a_{m-1}+1}\frac{1}{2^{(2m-1)(a_{m-1}+1)}} - \frac{1}{b_m+1}\frac{1}{2^{(2m-1)(b_m+1)}}.$$

for each $m > 1$. Noticing that $3 \le a_m < b_m < a_{m-1} < b_{m-1}$ for each $m > 1$, we have that $\{a'_m\}$ is a decreasing series and there exists $-\infty < \nabla < 0$ such that

$$\nabla = \sum_2^\infty \frac{1}{b_m+1}\frac{1}{2^{2m(b_m+1)}} - \sum_2^\infty \frac{1}{a_m+1}\frac{1}{2^{2m(a_m+1)}} + \sum_2^\infty \frac{1}{a_{m-1}+1}\frac{1}{2^{(2m-1)(a_{m-1}+1)}} - \sum_2^\infty \frac{1}{b_m+1}\frac{1}{2^{(2m-1)(b_m+1)}}.$$

We now choose $a'_1 = -\nabla$ and take b'_m such that

$$b'_m = \frac{1}{a_m+1}\frac{1}{2^{2m(a_m+1)}} + a'_m - \frac{1}{b_m+1}\frac{1}{2^{2m(b_m+1)}}$$

for each $m \ge 1$.

Step 5 From the definition of series $\{a'_m\}$ and $\{b'_m\}$, we have

$$0 < a'_m < b'_m < a'_{m-1} < b'_{m-1} \text{ for each } m > 1,$$
$$\lim_{n\to+\infty} a'_m = \lim_{n\to+\infty} b'_m = 0.$$

And from the definition of $\gamma'(t)$, noticing that

$$3 \le a_m < b_m < a_{m-1} < b_{m-1} \text{ for each } m > 1,$$

we have

$$0 < \gamma'(t) \le \frac{1}{4}t^4 + b'_m \text{ for } t \in \Big(\frac{1}{2^{2m+1}}, \frac{1}{2^{2m-1}}\Big).$$

Thus, $\gamma'_+(0) = \lim_{t\to 0^+} \gamma'(t) = 0$ and $\gamma(t) = \int_0^t \gamma'(s)\mathrm{d}s + \gamma'_+(0)$ for every $0 < t < \frac{1}{2}$. Then $\lim_{t\to 0^+} \gamma(t) = 0$.

We finish this section by recalling Van der Corput's well known lemma, which is a key tool for setting up the L^2 boundedness.

Lemma 1.5 (Van der Corput's lemma [15, p. 334]) *Let ψ and ϕ be two real-valued smooth functions on the interval (a,b) and $k \in \mathbb{N}$. If ψ satisfies $|\psi^{(k)}(t)| \ge 1$ for all $t \in (a,b)$ and either $k = 1$, $\psi'(t)$ is monotone on (a,b) or $k \ge 2$, then*

$$\Big|\int_a^b e^{i\lambda\psi(t)}\phi(t)dt\Big| \le C_k\lambda^{-\frac{1}{k}}\Big(|\phi(b)| + \int_a^b |\phi'(t)|dt\Big).$$

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2 The Proofs

2.1 The L^2 Bounds

For $f \in L^2(\mathbb{R}^2)$, we have

$$T_{\alpha,\beta} f(x,y) = \sum_{j=1}^{\infty} \int_{2^{-j}}^{2^{1-j}} f(x-t, y-\gamma(t)) e^{it^{-\beta}} \frac{dt}{t^{1+\alpha}} \triangleq \sum_{j=1}^{\infty} T_j f(x,y),$$

$$\widehat{(T_j f)}(\xi_1,\xi_2) = m_j(\xi_1,\xi_2)\widehat{f}(\xi_1,\xi_2),$$

where $\widehat{f}$ denotes the Fourier transform of f, and $m_j(\xi_1,\xi_2)$ is the multiplier given by

$$m_j(\xi_1,\xi_2) = \int_{2^{-j}}^{2^{1-j}} e^{i[t^{-\beta}-\xi_1 t-\xi_2\gamma(t)]} \frac{dt}{t^{1+\alpha}}.$$

From the Plancherel theorem, the oscillatory hyper-Hilbert transform $T_{\alpha,\beta}$ is bounded on $L^2(\mathbb{R}^2)$ if and only if $m(\xi_1,\xi_2) = \sum_{j=1}^{\infty} m_j(\xi_1,\xi_2)$ is a bounded function. For $t \in (2^{-j}, 2^{-j+1})$ where $j \in \mathbb{Z}^+$, let $\xi = (\xi_1,\xi_2)$ and $\psi_\xi(t) = t^{-\beta} - \xi_1 t - \xi_2\gamma(t)$. Then

$$\begin{aligned}
\psi_\xi'(t) &= -\beta t^{-\beta-1} - \xi_1 - \xi_2\gamma'(t),\\
\psi_\xi''(t) &= \beta(\beta+1)t^{-\beta-2} - \xi_2\gamma''(t) = t^{-\beta-2}\Big(\beta(\beta+1) - \xi_2 t^{\beta+2}\gamma''(t)\Big),\\
\psi_\xi'''(t) &= -\beta(\beta+1)(\beta+2)t^{-\beta-3} - \xi_2\gamma'''(t).
\end{aligned}$$

Case 1 $\xi_2 \le 0$.

For $t \in (2^{-j}, 2^{-j+1})$ with $j \in \mathbb{Z}^+$ and noticing $\gamma''(t) > 0$, we then have

$$\psi_\xi''(t) \ge \beta(\beta+1)t^{-\beta-2} \ge 2^{-\beta-2}\beta(\beta+1)2^{j(\beta+2)}.$$

From Lemma 1.5, we have

$$|m_j(\xi_1,\xi_2)| \le C2^{j(\alpha-\frac{\beta}{2})},$$

for each $j \in \mathbb{Z}^+$. Then, noticing $\beta \ge 3\alpha$, we have

$$|m(\xi_1,\xi_2)| \le \sum_{j=1}^{\infty} |m_j(\xi_1,\xi_2)| \le \sum_{j=1}^{\infty} C2^{j(\alpha-\frac{\beta}{2})} < C.$$

Case 2 $\xi_2 > 0$.

For $t \in (2^{-j}, 2^{-j+1})$ with $j \in \mathbb{Z}^+$, from $C_1 h'(t) \le \frac{h(t)}{t} \le C_2 h'(t)$, we have

$$\frac{1}{C_2}\frac{h(t)}{t^2} \le \gamma''(t) \le \frac{1}{C_1}\frac{h(t)}{t^2}.$$

Then

$$\psi_\xi''(t) \le t^{-\beta-2}\Big(\beta(\beta+1) - \xi_2\frac{1}{C_2}t^\beta h(t)\Big), \tag{2.1}$$

$$\psi_\xi''(t) \ge t^{-\beta-2}\Big(\beta(\beta+1) - \xi_2\frac{1}{C_1}t^\beta h(t)\Big). \tag{2.2}$$

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Noticing that $h'(t) = t\gamma''(t) > 0$ on $(2^{-j}, 2^{-j+1})$ for each $j \in \mathbb{Z}^+$ and $\gamma(t) \in C^1(0,1)$, for any $\xi_2 > 0$, we have that $h(t)$ and $\xi_2 \frac{1}{C_1} t^\beta h(t)$ are increasing functions about $t \in (0,1)$. Then the equation

$$\beta(\beta+1) = \xi_2 \frac{1}{C_1} t^\beta h(t) \tag{2.3}$$

can have at most one solution on $(0, \frac{1}{2})$.

Case 2.1 Equation (2.3) has no solution on $(0, \frac{1}{2})$.

For $t \in (0, \frac{1}{4})$ we have

$$\xi_2 \frac{1}{C_1} t^\beta h(t) \le \xi_2 \frac{1}{C_1} \Big(\frac{1}{2}\Big)^\beta h\Big(\frac{1}{2}\Big) \le \beta(\beta+1);$$

then

$$\begin{aligned} \beta(\beta+1) - \xi_2 \frac{1}{C_1} t^\beta h(t) &\ge \beta(\beta+1) - \xi_2 \frac{1}{C_1} \Big(\frac{t}{\frac{1}{2}}\Big)^\beta \Big(\frac{1}{2}\Big)^\beta h\Big(\frac{1}{2}\Big) \\ &\ge \beta(\beta+1) - \beta(\beta+1)\Big(\frac{1}{2}\Big)^\beta \\ &= \beta(\beta+1)\Big(1 - \Big(\frac{1}{2}\Big)^\beta\Big). \end{aligned}$$

From (2.2) and Lemma 1.5, we have

$$|m_j(\xi_1, \xi_2)| \le C 2^{j(\alpha - \frac{\beta}{2})}$$

for each $j \in \mathbb{Z}^+ \setminus \{1,2\}$. Then noticing $\beta \ge 3\alpha$, we have

$$|m(\xi_1, \xi_2)| \le \Sigma_{j=1}^\infty |m_j(\xi_1, \xi_2)| \le \Sigma_{j=1}^\infty C 2^{j(\alpha - \frac{\beta}{2})} < C.$$

Case 2.2 Equation (2.3) has a solution $t_\xi \in (0, \frac{1}{2})$.

We have

$$\beta(\beta+1) = \xi_2 \frac{1}{C_1} (t_\xi)^\beta h(t_\xi). \tag{2.4}$$

Case 2.2.1 $t \le \frac{t_\xi}{2}$.

From (2.4) we have

$$\begin{aligned} \beta(\beta+1) - \xi_2 \frac{1}{C_1} t^\beta h(t) &\ge \beta(\beta+1) - \xi_2 \frac{1}{C_1} \Big(\frac{t_\xi}{2}\Big)^\beta h(t_\xi) \\ &\ge \beta(\beta+1) - \beta(\beta+1)\Big(\frac{1}{2}\Big)^\beta \\ &= \beta(\beta+1)\Big(1 - \Big(\frac{1}{2}\Big)^\beta\Big). \end{aligned}$$

From (2.2) and Lemma 1.5, we have

$$|m_j(\xi_1, \xi_2)| \le C 2^{j(\alpha - \frac{\beta}{2})}, \tag{2.5}$$

for each $j \in \mathbb{Z}^+$ and $2^{1-j} \le \frac{t_\xi}{2}$.

Case 2.2.2 $t \ge C_3 t_\xi$, where $C_3 = (\frac{2C_2}{C_1})^{\frac{1}{\beta}} > 1$.

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From (2.4) we have

$$\begin{aligned}\beta(\beta+1)-\xi_2\frac{1}{C_2}t^{\beta}h(t)&\leq\beta(\beta+1)-\xi_2\frac{1}{C_2}(C_3t_\xi)^{\beta}h(t_\xi)\\&\leq\beta(\beta+1)-\frac{C_1}{C_2}(C_3)^{\beta}\beta(\beta+1)\\&=-\beta(\beta+1).\end{aligned}$$

Then from (2.1) and Lemma 1.5, we have

$$|m_j(\xi_1,\xi_2)|\leq C2^{j(\alpha-\frac{\beta}{2})}, \tag{2.6}$$

for each $j\in\mathbb{Z}^+$ and $2^{-j}\geq C_3t_\xi$.

Case 2.2.3 $t\in(\frac{t_\xi}{2},C_3t_\xi)$.

In this case, we only care about those $j\in\mathbb{Z}^+$ that satisfy $\frac{t_\xi}{2}\leq 2^{1-j}$ or $2^{-j}\leq C_3t_\xi$. Then we have

$$\frac{1}{C_3}2^{-j}\leq t_\xi\leq 2^{2-j}.$$

For $t\in(2^{-j},2^{-j+1})$, we have

$$\psi_\xi'''(t)\leq-\beta(\beta+1)(\beta+2)t^{-\beta-3}\leq-\beta(\beta+1)(\beta+2)2^{-\beta-3}2^{j(\beta+3)}.$$

From Lemma 1.5, we have

$$|m_j(\xi_1,\xi_2)|\leq C2^{j(\alpha-\frac{\beta}{3})}. \tag{2.7}$$

Thus, from (2.5), (2.6), (2.7), noticing $\beta\geq 3\alpha$, we have

$$\begin{aligned}&|m(\xi_1,\xi_2)|\\&\quad\leq\sum_{j=1}^{\infty}|m_j(\xi_1,\xi_2)|\\&\quad\leq\sum_{2^{1-j}\leq\frac{t_\xi}{2}}|m_j(\xi_1,\xi_2)|+\sum_{2^{-j}\geq C_3t_\xi}|m_j(\xi_1,\xi_2)|+\sum_{\frac{1}{C_3}2^{-j}\leq t_\xi\leq 2^{2-j}}|m_j(\xi_1,\xi_2)|\\&\quad\leq\sum_{2^{1-j}\leq\frac{t_\xi}{2}}C2^{j(\alpha-\frac{\beta}{2})}+\sum_{j:2^{-j}\geq C_3t_\xi}C2^{j(\alpha-\frac{\beta}{2})}+\sum_{\frac{1}{C_3}2^{-j}\leq t_\xi\leq 2^{2-j}}C2^{j(\alpha-\frac{\beta}{3})}\\&\quad\leq C.\end{aligned}$$

We obtain the L^2 boundedness.

2.2 The L^p bounds

From

$$T_{\alpha,\beta}f(x,y)=\sum_{j=1}^{\infty}\int_{2^{-j}}^{2^{1-j}}f\big(x-t,y-\gamma(t)\big)\,e^{it^{-\beta}}\frac{\mathrm{d}t}{t^{1+\alpha}}\triangleq\sum_{j=1}^{\infty}T_jf(x,y),$$

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for any j large enough, we have

$$\|T_j f\|_{L^1(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \Big| \int_{2^{-j}}^{2^{1-j}} f(x-t, y-\gamma(t)) e^{it^{-\beta}} \frac{dt}{t^{1+\alpha}} \Big| dx dy$$
$$\leq \int_{2^{-j}}^{2^{1-j}} \int_{\mathbb{R}^2} |f(x-t, y-\gamma(t)) e^{it^{-\beta}}| dx dy \frac{dt}{t^{1+\alpha}}$$
$$\leq C 2^{j\alpha} \|f\|_{L^1(\mathbb{R}^2)}.$$

From Subsection 2.1, we have

$$|m_j(\xi_1, \xi_2)| \leq C 2^{j(\alpha - \frac{\beta}{3})},$$

for each $j \in \mathbb{Z}^+$. Then

$$\|T_j f\|_{L^2(\mathbb{R}^2)} \leq C 2^{j(\alpha - \frac{\beta}{3})} \|f\|_{L^2(\mathbb{R}^2)}.$$

By interpolation, for $1 < p < 2$ we have

$$\|T_j f\|_{L^p(\mathbb{R}^2)} \leq C 2^{j\left(\vartheta\alpha + (\alpha - \frac{\beta}{3})(1-\vartheta)\right)} \|f\|_{L^p(\mathbb{R}^2)},$$

where $\frac{1}{p} = \vartheta + \frac{1-\vartheta}{2}$ and $0 \leq \vartheta \leq 1$.

Since $\beta > 3\alpha$, for $\frac{2\beta}{2\beta - 3\alpha} < p \leq 2$, we have $\vartheta\alpha + (\alpha - \frac{\beta}{3})(1-\vartheta) < 0$. Thus,

$$\|T_{\alpha,\beta} f\|_{L^p(\mathbb{R}^2)} \leq \sum_{j=1}^{\infty} \|T_j f(x,y)\|_{L^p(\mathbb{R}^2)} \leq \sum_{j=1}^{\infty} C 2^{j\left(\vartheta\alpha + (\alpha - \frac{\beta}{3})(1-\vartheta)\right)} \|f\|_{L^p(\mathbb{R}^2)}$$
$$\leq C \|f\|_{L^p(\mathbb{R}^2)}.$$

When $2 < p < \frac{2\beta}{3\alpha}$, we have the trivial estimate

$$\|T_j f\|_{L^\infty(\mathbb{R}^2)} \leq C 2^{j\alpha} \|f\|_{L^\infty(\mathbb{R}^2)}.$$

An interpolation with the L^2 boundedness give

$$\|T_j f\|_{L^p(\mathbb{R}^2)} \leq C 2^{j\left(\vartheta\alpha + (\alpha - \frac{\beta}{3})(1-\vartheta)\right)} \|f\|_{L^p(\mathbb{R}^2)},$$

with $\frac{1}{p} = \frac{1-\vartheta}{2}$ and $0 \leq \vartheta \leq 1$. Then summation according to $j \in \mathbb{Z}^+$ with $2 < p < \frac{2\beta}{3\alpha}$ gives the inequality

$$\|T_{\alpha,\beta} f\|_{L^p(\mathbb{R}^2)} \leq C \|f\|_{L^p(\mathbb{R}^2)}.$$

This finishes our proof. ■

References

[1] N. Bez, *L^p-boundedness for the Hilbert transform and maximal operator along a class of nonconvex curves.* Proc. Amer. Math. Soc. **135**(2007), no. 1, 151–161. http://dx.doi.org/10.1090/S0002-9939-06-08603-5

[2] A. Carbery, M. Christ, J. Vance, S. Wainger, and D. Watson, *Operators associated to flat plane curves: L^p estimates via dilation methods.* Duke Math. J. **59**(1989), no. 3, 675–700. http://dx.doi.org/10.1215/S0012-7094-89-05930-9

[3] A. Carbery, J. Vance, S. Wainger, and D. Watson, *The Hilbert transform and maximal function along flat curves, dilations, and differential equations.* Amer. J. Math. **116**(1994), no. 5, 1203–1239. http://dx.doi.org/10.2307/2374944

[4] A. Carbery and S. Ziesler, *Hilbert transforms and maximal functions along rough flat curves.* Rev. Mat. Iberoamericana **10**(1994), no. 2, 379–393. http://dx.doi.org/10.4171/RMI/156

[5] H. Carlsson, M. Christ, A. Cordoba, J. Duoandikoetxea, J. L. Rubio de Francia, J. Vance, S. Wainger, and D. Weinberg, *L^p estimates for maximal functions and Hilbert transforms along flat convex curves in* $\mathbb{R}^2$. Bull. Amer. Math. Soc. (N.S.) **14**(1986), no. 2, 263–267. http://dx.doi.org/10.1090/S0273-0979-1986-15433-9
[6] S. Chandarana, *L^p-bounds for hypersingular integral operators along curves.* Pacific J. Math. **175**(1996), no. 2, 389–416. http://dx.doi.org/10.2140/pjm.1996.175.389
[7] J. Chen, B. Damtew, and X. Zhu, *Oscillatory hyper Hilbert transforms along general curves.* Front. Math. China **12**(2017), no. 2, 281–299. http://dx.doi.org/10.1007/s11464-016-0574-3
[8] J. Chen, D. Fan, M. Wang, and X. Zhu, *L^p bounds for oscillatory hyper-Hilbert transform along curves.* Proc. Amer. Math. Soc. **136**(2008), no. 9, 3145–3153. http://dx.doi.org/10.1090/S0002-9939-08-09325-8
[9] J. Chen, D. Fan, and X. Zhu, *Sharp L^2 boundedness of the oscillatory hyper-Hilbert transform along curves.* Acta Math. Sin. (Engl. Ser.) **26**(2010), no. 4, 653–658. http://dx.doi.org/10.1007/s10114-010-7396-0
[10] M. Christ, *Hilbert transforms along curves. II. A flat case.* Duke Math. J. **52**(1985), no. 4, 887–894. http://dx.doi.org/10.1215/S0012-7094-85-05246-9
[11] A. Cordoba and J. L. Rubio de Francia, *Estimates for Wainger's singular integrals along curves.* Rev. Mat. Iberoamericana **2**(1986), no. 1–2, 105–117. http://dx.doi.org/10.4171/RMI/29
[12] C. Fefferman. *Inequality for strongly singular convolutions operators.* Acta Math. **124**(1970), 9–36. http://dx.doi.org/10.1007/BF02394567
[13] J. Li and S. Lu, *Applications of the scale changing method to boundedness of certain commutators.* Internat. J. Anal. Math. Sci. **1**(2004), 1–12.
[14] A. Nagel and S. Wainger, *Hilbert transforms associated with plane curves.* Trans. Amer. Math. Soc. **223**(1976), 235–252. http://dx.doi.org/10.1090/S0002-9947-1976-0423010-8
[15] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals.* Princeton Mathematical Series, 43, Princeton University Press, Princeton, NJ, 1993.
[16] E. M. Stein and S. Wainger, *Problems in harmonic analysis related to curvature.* Bull. Amer. Math. Soc. **84**(1978), no. 6, 1239–1295. http://dx.doi.org/10.1090/S0002-9904-1978-14554-6
[17] J. Vance, S. Wainger, and J. Wright, *The Hilbert transform and maximal function along nonconvex curves in the plane.* Rev. Mat. Iberoamericana **10**(1994), no. 1, 93–121. http://dx.doi.org/10.4171/RMI/146
[18] J. Wright, *L^p estimates for operators associated to oscillating plane curves.* Duke Math. J. **67**(1992), no. 1, 101–157. http://dx.doi.org/10.1215/S0012-7094-92-06705-6
[19] M. Zielinski, *Highly oscillatory singular integrals along curves.* Ph.D. Thesis, The University of Wisconsin - Madison, 1985.
[20] S. Ziesler, *L^p-boundedness of the Hilbert transform and maximal function associated to flat plane curves.* Proc. Amer. Math. Soc. **122**(1994), no. 4, 1035–1043. http://dx.doi.org/10.2307/2161171

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