ON A CLASS OF SEMIGROUPS

D. Ž. Djoković

Let $S_{n,r}$ be the semigroup generated by two elements a and b with two defining relations

$$(1) \qquad aba = b^n, \quad bab = a^r .$$

By symmetry, we may assume that $n \leq r$. In [1] and [2] it was shown that the semigroups

$$\left.\begin{array}{ll} S_{1,r}, & r \geq 1 \\ S_{2,r}, & r = 2, 3, 4 \end{array}\right\} \quad \text{are finite,}$$

and

$$\left.\begin{array}{ll} S_{2,5} & \\ S_{n,r} & r \geq n \geq 3) \end{array}\right\} \quad \text{are infinite.}$$

We shall prove that the semigroups $S_{2,r}$, $r \geq 6$, are also infinite and that there exist decision algorithms for $S_{2,5}$ and $S_{3,3}$.

THEOREM 1. _The semigroups_ $S_{2,r}$, $r \geq 6$, _are infinite_.

Proof. Every word in $S_{2,r}$ which is equal to a given word w can be obtained from w by a finite number of the following four types of elementary operations:

(i) replace a subword aba by b^2,

(ii) replace a subword b^2 by aba,

(iii) replace a subword bab by a^r,

(iv) replace a subword a^r by bab.

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Let $u_0 = ab^2a$, $u_1 = a^2ba^2$. For any function $f : \{1, 2, \ldots, k\} \to \{0, 1\}$ we define the word

$$w(f) = u_{f(1)} u_{f(2)} \cdots u_{f(k)} .$$

This set of 2^k words $w(f)$ has the following properties:

1^o the elementary operations which can be performed on $w(f)$ are necessarily of type (i) or (ii).

2^o if an elementary operation is performed on a word of this set then the resulting word also belongs to this set.

The property 2^o implies that the words $(ab^2a)^k$, $k = 1, 2, 3, \ldots,$ are all distinct. Hence $S_{2,r}$, $r \geq 6$, are infinite.

The fact that $S_{2,5}$ and $S_{3,3}$ are infinite is also a consequence of the following

LEMMA. $S_{2,5}$ <u>and</u> $S_{3,3}$ <u>have the following property: the number of words which are equal to a given word is finite.</u>

<u>Proof</u>. Let us assign the weights to the letters a, b:

$$a \to 1, \quad b \to 2 \qquad \text{in } S_{2,5},$$

$$a \to 1, \quad b \to 1 \qquad \text{in } S_{3,3}.$$

The weight of a word is the sum of the weights of all its letters. The elementary operations determined by (1) preserve the weight. Hence any two equal words must have the same weight. Since there are only finitely many words of given weight the lemma follows.

THEOREM 2. <u>There exists a decision algorithm for</u> $S_{2,5}$ and $S_{3,3}$.

<u>Proof</u>. Since the number of words equal to a given word w is finite we can easily construct an algorithm which will give us a list of all words equal to w. The algorithm has to perform elementary operations on w and on the words which are already in the list. If a new word occurs it is added to the list. After a finite number of operations all words equal to w will be listed.

REFERENCES

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University of Waterloo

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