Let $G$ be a finite group and let $N$ be a normal subgroup of $G$. If $G/N$ is solvable and $H/N$ is a nilpotent injector of $G/N$, then there exists a canonical basis of the complex space of the class functions of $G$ which vanish off the $G$-conjugates of $H$.

1. INTRODUCTION

Let $G$ be a finite group and let $\text{cf}(G)$ be the space of complex class functions of $G$. Let $\text{Irr}(G)$ be the set of irreducible complex characters of $G$. A subgroup $H$ of $G$ is good if there exists a basis $P(G \mid H)$ of the subspace $\text{cf}(H)^G$ of the class functions of $G$ induced from the class functions of $H$ satisfying:

(I) for each $\eta \in P(G \mid H)$, there is $\gamma \in \text{Irr}(H)$ such that $\eta = \gamma^G$; and

(D) for every $\alpha \in \text{Irr}(H)$, we have $\alpha^G$ is a nonnegative linear combination of $P(G \mid H)$.

This basis (if it exists) is unique and does not depend on $H$ but on its $G$-conjugacy class.

There are examples of good subgroups. For instance, if $H \triangleleft G$, then it easy to show that $H$ is good (although this is already false if $H \lhd G$.) If $G$ is $p$-solvable and $H$ is a $p$-complement of $G$, then $P(G \mid H)$ exists and it is the set of projective indecomposable characters by Fong theory. If $G$ is $\pi$-separable and $H$ is a Hall $\pi$-subgroup of $G$, then $P(G \mid H)$ also exists by Isaacs $\pi$-theory. (It is interesting to notice that, in general, a Sylow $p$-subgroup of a finite group $G$ need not to be good as shown by $A_9$ and $p = 3$.) If $G/N$ is $\pi$-separable and $H/N$ is a Hall $\pi$-subgroup of $G/N$, we have recently proved that $P(G \mid H)$ exists (see [6, Theorem A]). If $H$ is a nilpotent injector of the solvable group $G$, then $P(G \mid H)$ exists (see [3, Theorem (3.1)]).

In this note we find a new class of good subgroups.

**Theorem A.** Let $N \lhd G$ and suppose that $G/N$ is a solvable group. If $H/N$ is a nilpotent injector of $G/N$, then $H$ is good.

As pointed out in [5], good subgroups provide a canonical partition of $\text{Irr}(G)$ which behaves like the Brauer $p$-blocks of $G$. 

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2. PRELIMINARIES: GOOD BASES

In this section we are going to review the notation and terminology of good bases. We refer the reader to [4, Section 2] for more details.

Let $G$ be a finite group and let $\text{cf}(G)$ be the space of complex class functions defined on $G$. If $H \subseteq G$, let $\text{cf}(H)^G = \{ \gamma^G \mid \gamma \in \text{cf}(H) \}$.

**Definition 2.1:** Let $H$ be a subgroup of $G$. We say that $H$ is **good** if there exists a basis $P(G \mid H)$ of $\text{cf}(H)^G$ satisfying:

(I) for each $\eta \in P(G \mid H)$, there is $\gamma \in \text{Irr}(H)$ such that $\eta = \gamma^G$; and

(D) for every $\alpha \in \text{Irr}(H)$, we have

$$\alpha^G = \sum_{\eta \in P(G \mid H)} a_\eta \eta$$

for nonnegative integers $a_\eta$.

In this case, we say that $P(G \mid H)$ is the **good basis** of the space $\text{cf}(H)^G$. (Good bases are unique by [3, Theorem (2.2)].)

Now, let $N$ be a normal subgroup of $G$ contained in $H$. If $\theta \in \text{Irr}(N)$, then we write $\text{Irr}(G \mid \theta)$ for the set of irreducible constituents of $\theta^G$. Also, $\text{cf}(G \mid \theta)$ is the $\mathbb{C}$-span of the set $\text{Irr}(G \mid \theta)$. We denote by

$$\text{vcf}(G \mid H, \theta) = \text{cf}(H)^G \cap \text{cf}(G \mid \theta).$$

Let $\Theta$ be a complete set of representatives of the orbits of the action of $G$ on $\text{Irr}(N)$. We know that

$$\text{cf}(H)^G = \bigoplus_{\theta \in \Theta} \text{vcf}(G \mid H, \theta)$$

by [4, Lemma (2.2)].

We define good bases "over" an irreducible character of a certain normal subgroup.

**Definition 2.2:** Let $N \trianglelefteq G$, let $\theta \in \text{Irr}(N)$ and let $N \subseteq H \subseteq G$. A basis $B$ of $\text{vcf}(G \mid H, \theta)$ is **good** if satisfies the following conditions:

(I) If $\eta \in B$, then there exists $\alpha \in \text{Irr}(H \mid \theta)$ such that $\alpha^G = \eta$.

(D) If $\gamma \in \text{Irr}(H \mid \theta)$, then $\gamma^G = \sum_{\eta \in B} a_\eta \eta$ for uniquely determined integers $a_\eta$.

Good bases "over" irreducible characters are necessarily unique (by the same argument as in [3, Theorem (2.2)]). We shall denote by $P(G \mid H, \theta)$ the unique good basis (if it exists) of $\text{vcf}(G \mid H, \theta)$.

Next is a key definition in order to find good bases for the subspace $\text{cf}(H)^G$.

**Definition 2.3:** Suppose that $N \trianglelefteq G$ is contained in $H \subseteq G$. Let $\theta \in \text{Irr}(N)$ and and write $T = I_G(\theta)$ for the stabiliser of $\theta$ in $G$. We say that $\theta$ is $H$-**good** (with respect to $G$), if for every $g \in G$, we have that $H^g \cap T$ is contained in some $T$-conjugate of $H \cap T$. 

[156 L. Sanus [2] 0.016468687674744786]
3. PROOF OF THEOREM A

Let $G$ be a solvable group. We recall that a nilpotent injector is a maximal nilpotent subgroup of $G$ containing $F(G)$. Any two of them are $G$-conjugate [2].

We need the following new property of the nilpotent injectors.

**Theorem 3.1.** Let $G$ be a solvable group and let $T$ be a subgroup of $G$. Then there exists a nilpotent injector $H$ of $G$ such that for every $g \in G$

$$H^g \cap T \subseteq (H \cap T)^t,$$

for some $t \in T$.

**Proof:** We argue by induction on $|G|$. Write $F = F(G)$. We claim that we may assume that $FT = G$. Otherwise, we have that there exists a nilpotent injector $J$ of $FT$ such that for every $x \in FT$

$$J^x \cap T \subseteq (J \cap T)^t,$$

for some $t \in T$. Now, by [2, Theorem 2(b)], we know that there exists a nilpotent injector $H$ of $G$ such that

$$H \cap FT = J.$$

Let $g \in G$. We have that $H^g \cap T = H^g \cap FT \cap T$. Now, by [2, Theorem 2(c)], there exists $y \in FT$ such that $H^y \cap FT \subseteq J^y$. Then, we have that there exists an element $t \in T$ such that

$$H^g \cap T = H^g \cap FT \cap T \subseteq J^y \cap T \subseteq (J \cap T)^t.$$

Now, since $H \cap FT = J$, it follows that $$H^g \cap T \subseteq (J \cap T)^t = (H \cap FT \cap T)^t = (H \cap T)^t,$$

as claimed.

Therefore, we assume that $G = FT$. Let $H$ be a nilpotent injector of $G$. For every $g \in G$, we write $g = ft$ with $f \in F$ and $t \in T$. We have that

$$H^g \cap T = H^t \cap T = (H \cap T)^t$$

and the proof of the theorem is complete. 

**Corollary 3.2.** Let $N$ be a normal subgroup of a solvable group $G$. Suppose that $\theta \in \text{Irr}(N)$ and let $T = I_G(\theta)$ be the stabiliser of $\theta$ in $G$. Then there exists a nilpotent injector $H/N$ of $G/N$ such that $\theta$ is $H$-good.

**Proof:** The proof easily follows from Theorem (3.1). 

Now, let $N \triangleleft G$ and let $\theta \in \text{Irr}(G)$ be invariant in $G$. Under these hypotheses we say that $(G, N, \theta)$ is a character triple. For the definition and main properties of isomorphisms of character triples we refer the reader to [1, Chapter 11].
We are now ready to prove Theorem A.

**Theorem 3.3.** Let $G/N$ be a solvable group. If $H/N$ is a nilpotent injector of $G/N$, then $H$ is good.

**Proof.** Given $\theta \in \text{Irr}(N)$, by Corollary (3.2) we know that there exists $x \in G$ such that $\theta$ is $H^x$-good. It follows that $\theta^{x^{-1}}$ is $H$-good. Hence, we may find a complete set $\Theta$ of representatives of the orbits of the action of $G$ on $\text{Irr}(N)$ such that each $\theta \in \Theta$ is $H$-good. Now, we are going to prove that there exists a good basis of $\text{vcf}(G \mid H, \theta)$ for every $\theta \in \Theta$.

We fix $\theta \in \Theta$. Since there is a "Clifford correspondence" for good bases over $\theta \in \Theta$ [4, Lemma (2.10)], we may assume that $\theta$ is $G$-invariant. Hence $(G, N, \theta)$ is a character triple. By [1, Theorem (11.28)], there exists an isomorphic character triple $(G^*, N^*, \theta^*)$ such that $N^*$ is a central subgroup of $G^*$. Since $N^* \subseteq Z(G^*)$, we have that $F(G^*/N^*) = F(G^*)/N^*$. Now, since $H^*/N^*$ is nilpotent if and only if $H^*$ is nilpotent, it easily follows that $H^*/N^*$ is a nilpotent injector of $G^*/N^*$ if and only if $H^*$ is a nilpotent injector of $G^*$. We know that $P(G^* \mid H^*, \theta^*)$, the good basis of $\text{vcf}(G^* \mid H^*, \theta^*)$ exists by [3, Theorem (3.1)] and [6, Theorem (2.4)]. Now, by [6, Lemma (3.4)], it follows that $P(G \mid H, \theta)$ is the good basis of $\text{vcf}(G \mid H, \theta)$. We conclude that $\bigcup_{\theta \in \Theta} P(G \mid H, \theta)$ is $P(G \mid H)$ by [4, Lemma (2.9)].

**References**


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