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A simple model for a weak system of arithmetic

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The natural first order version of Peano's axioms (the theory T with 0, the successor function and an induction schema) is shown to possess the following nonstandard model: the natural numbers together with a collection of 'infinite' elements isomorphic to the integers. In fact, a complete list of the models of this theory is obtained by showing that T is equivalent to the apparently weaker theory with the induction axiom replaced by axioms stating that there are no finite cycles under the successor function and that 0 is the only non-successor.

We work in the predicate calculus with equality, one constant symbol 0 and one unary function symbol $'$ (successor). We consider the following axioms:

(i) $(\forall x)(\forall y)(x' = y' \to x = y)$,

(ii) $(\forall x)(x' \neq 0)$,

(iii) $(\forall x_1) \ldots (\forall x_n)\{[\phi(x_1, \ldots, x_n, 0) \mathbin{\&} (\forall y)(\phi(x_1, \ldots, x_n, y) \to \phi(x_1, \ldots, x_n, y'))] \to (\forall y)\phi(x_1, \ldots, x_n, y)\}$

where ϕ is any formula of the language with free variables among $x_1, \ldots, x_n, y$,

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(iv) $(\forall x \neq 0)(\exists y)(x = y')$,

(v_n) $(\forall x)(x \neq x'^{\cdots\,'})$ (where $'$ occurs n times).

T is the theory with axioms (i), (ii), (iii); we denote by T^* the theory with axioms (i), (ii), (iv), (v_n) for $n = 1, 2, \ldots$.

N denotes the set of natural numbers $\{0, 1, 2, \ldots\}$, Z the integers. The following lemma is easily proved.

LEMMA. *The class of all models of* T^* *is just the class of all structures* $N \cup (Z \times A)$ *, where* A *is an arbitrary (possibly empty) index set,* 0 *is interpreted as* $0 \in N$ *and the successor function is defined thus:*

for $n \in N$, $n' = n + 1$ *;*

for $(m, a) \in Z \times A$, $(m, a)' = (m+1, a)$.

We call elements of $Z \times A$ *infinite* elements of the structure.

THEOREM. T *and* T^* *have the same theorems.*

Proof. Every axiom of T^* is easily seen to be a theorem of T . However T^* has only infinite models and all models of T^* of cardinality $\aleph_1$ are isomorphic. Thus T^* is complete, by the Łoś-Vaught test ([*1*], p. 179), and the theorems of T^* form a maximal consistent set.

COROLLARY 1. *Every instance of the induction schema* (iii) *can be proved from a finite sub-collection of the axioms* (i), (ii), (iv), (v_n).

COROLLARY 2. *The class of all models of* T *is the class of all structures of the form* $N \cup (Z \times A)$.

COROLLARY 3. *Addition cannot be defined in* T .

Proof. The structure $M = N \cup (Z \times \{0\})$ is a model of T , and there is no way of defining the sum of two 'infinite' elements of M in such a way that the cancellation law holds.

Reference

[1] J.L. Bell and A.B. Slomson, *Models and ultraproducts: an introduction* (North-Holland, Amsterdam, London, 1969).

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