Bull. Aust. Math. Soc. **96** (2017), 468–478
doi:10.1017/S0004972717000752

ITERATED LOGARITHM SPEED OF RETURN TIMES

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(Received 13 March 2017; accepted 27 June 2017; first published online 4 October 2017)

Abstract

In a general setting of an ergodic dynamical system, we give a more accurate calculation of the speed of the recurrence of a point to itself (or to a fixed point). Precisely, we show that for a certain ξ depending on the dimension of the space, $\liminf_{n\to+\infty}(n\log\log n)^{\xi} d(T^n x, x) = 0$ almost everywhere and $\liminf_{n\to+\infty}(n\log\log n)^{\xi} d(T^n x, y) = 0$ for almost all x and y. This is done by assuming the exponential decay of correlations and making a weak assumption on the invariant measure.

2010 *Mathematics subject classification*: primary 37A25; secondary 37B20.

Keywords and phrases: Poincaré theorem, recurrence, decay of correlations, shrinking target.

1. Introduction

Let (X, d) be a separable metric space, let $T : X \to X$ be a transformation and let μ be a T-invariant Borel probability measure. The classical Poincaré lemma in such a setting may be restated as

$$\liminf_{n\to\infty} d(T^n x, x) = 0 \quad \text{for } \mu\text{-almost all } x.$$

A natural question follows: how fast is this convergence? A partial answer was given in a pioneering paper by Boshernitzan [2], namely,

$$\liminf_{n\to\infty} n^{1/\beta} d(T^n x, x) < +\infty, \tag{1.1}$$

if the Hausdorff measure of the space satisfies $H_\beta(X) < +\infty$. Moreover, if $H_\beta(X) = 0$, then the lower limit equals zero.

Since then, there have been significant developments in this area, leading to the notion of lower and upper recurrence rates, defined as

$$\underline{R}(x) = \liminf_{r\to 0^+} \frac{\log \tau_{B(x,r)}(x)}{-\log(r)} \quad \text{and} \quad \overline{R}(x) = \limsup_{r\to 0^+} \frac{\log \tau_{B(x,r)}(x)}{-\log(r)},$$

where $\tau_U(x)$ is the first-return time of the point x into U and $B(x, r)$ is the ball with centre x and radius r.

In several cases, those quantities are connected to the lower and upper pointwise dimensions of the invariant measure (actually, for various systems, recurrence rates and pointwise dimensions coincide). For a good introduction to quantitative recurrence, see [1]. Under assumptions similar to the ones in this paper (although weaker with respect to the measure), Urbański [9] proved the equality of recurrence rates and pointwise dimensions.

Calculating (approximating) the limit in (1.1) may be seen as a more exact calculation of the recurrence rate. However, this estimate can only be *one sided*, because taking the upper limit in (1.1) would not make much sense. We shall prove that if the transformation has good mixing properties (exponential decay of correlations) and the measure of a ball is at least its radius to some power β, then

$$\liminf_{n\to+\infty}(n\log\log n)^{1/\beta}d(T^n x, x) = 0.$$

Another way of looking at such a limit is to consider the dimension or measure of so-called *shrinking target sets*, that is,

$$E(x, r) = \{y \in X : d(T^n x, y) < r_n \text{ for infinitely many } n\},$$

where $r = (r_n)_{n=1}^{\infty}$ is a given decreasing sequence. For example, Persson and Rams [6] give the formula for the Hausdorff dimension depending on (r_n).

Our result applied in this setting states that, for any $\varepsilon > 0$ and the sequence $r_n = \varepsilon \cdot (n\log\log n)^{-1/\beta}$, the set $E(x, r)$ has full μ-measure.

2. Basic definitions

Throughout this paper, we will assume that (X, d) is a metric space, $T : X \to X$ is a Borel measurable map and μ is a T-invariant Borel probability measure on X.

Definition 2.1. We say that a dynamical system has an *exponential decay of correlations* in Lipschitz continuous functions (denoted by $\mathcal{L}$), if there exist $\gamma \in (0, 1)$ and $C < +\infty$ such that, for all $g \in \mathcal{L}$, all $f \in L_1(\mu)$ and every $n \in \mathbb{N}$,

$$|\mu(f \circ T^n \cdot g) - \mu(g) \cdot \mu(f)| \le C\gamma^n \|g\|_{\mathcal{L}}\mu(|f|),$$

where $\|\cdot\|_{\mathcal{L}}$ denotes the standard Lipschitz norm.

Remark. Usually, this property is proved with respect to Hölder continuous functions or functions of bounded variation. Our result works in both these situations. In fact, the proof of the main theorem only uses the exponential decay for explicitly defined Lipschitz continuous functions.

We shall now state the necessary property of a measure.

Definition 2.2. The measure μ has the *β-property* if there exist measurable functions $D(x) > 0$ and $R(x) > 0$ such that, for μ-almost all $x \in X$ and for all $r < R(x)$, we have $\mu(B(x, r)) \ge D(x) \cdot r^{\beta}$.

Let us denote the packing measure by Π_β and the packing dimension of a measure μ by $\mathrm{PD}(\mu)$. (For definitions, see [7, Ch. 8].)

In Euclidean space, we will prove a useful characterisation of the *β-property*.

Lemma 2.3. *If X is a Borel bounded subset of $\mathbb{R}^n$, then μ (a Borel probability measure on X) has the β-property if and only if there exists a set A of full measure such that the packing measure Π_β is σ-finite on A.*

In particular, if Π_β is σ-finite on X, then the β-property holds for μ (in fact, for every probability measure) and if the measure μ has the β-property, then $\mathrm{PD}(\mu) \le \beta$.

We will use another property of measure; this one is well known.

Definition 2.4. A measure ν is called a *doubling measure*, if there exist constants $\eta > 1$ and $c > 0$ such that $\nu(B(x, \eta r)) \le c\nu(B(x, r))$ for every $x \in X$ and $r > 0$. It is obvious that if a measure is *doubling* for some η, then it is also doubling for any $\eta > 1$.

3. Main result

We shall prove the following theorem.

Theorem 3.1. *Let (X, d) be a metric space and let $(X, \mu, \mathcal{F}, T)$ be a Borel-probability-measure-preserving dynamical system with an exponential decay of correlations whose measure μ has the β-property. In addition, assume that either μ is a doubling measure or that X is a subset of a Euclidean space. Then, for μ-almost all $x \in X$,*

$$\liminf_{n\to\infty} n \log\log n \cdot (d(T^n x, x))^\beta = 0 \tag{3.1}$$

and, for every $y \in \mathrm{supp}(\mu)$ such that $D(y) < +\infty$ and $R(y) > 0$ (see Definition 2.2),

$$\liminf_{n\to\infty} n \log\log(n) \cdot (d(T^n x, y))^\beta = 0. \tag{3.2}$$

Using Lemma 2.3, we can obtain a version which is easier to apply.

Corollary 3.2. *Let X be a bounded Borel subset of $\mathbb{R}^n$ such that the packing measure Π_β is σ-finite on X. Let T be a mapping on X preserving a Borel probability measure μ with an exponential decay of correlations. Then (3.1) and (3.2) hold with respect to the metric used to define the packing measure.*

4. Examples

The main result is general enough to be applicable to many different dynamical systems. We give several classes of examples.

(1) A hyperbolic rational function of degree greater than or equal to two on the complex sphere has an invariant probability measure which is equivalent (up to a constant factor) to some β-Hausdorff measure, where $\beta = \mathrm{HD}(J_f)$ (see [8]). This proves the β-property. The needed decay of correlations is another well-known fact, so we may apply Theorem 3.1.

https://doi.org/10.1017/S0004972717000752 Published online by Cambridge University Press

(2) A piecewise expanding transformation $f\colon [a,b] \to [a,b]$ with $g(x) = 1/|f'(x)|$ having bounded variation admits an absolutely continuous invariant measure whose density is of bounded variation (see [3, Theorem 5.2.1]). The density can be redefined on a countable set to become lower semicontinuous and positive on an open set [3, Theorem 8.1.2] and also bounded away from zero on its support, which gives the β-property. If, in addition, the system is weakly mixing, then we have an exponential decay of correlations in the functions of bounded variation [3, Theorem 8.3.1]. Thus all the assumptions of Theorem 3.1 are satisfied.

(3) One can also apply the result to some conformal graph directed Markov systems (or just to conformal iterated function systems). Definitions and necessary results cited here may be found in [5, Ch. 4]. The systems have an appropriate conformal measure [5, Theorem 3.2.3; check also Lemma 4.2.2] and exhibit the exponential decay of correlations [5, Theorem 2.4.6]. The only assumption we need to check is the β-property. All finite conformal systems have this property [5, Theorem 4.2.11], as do infinite systems with finite packing measure of the limit set. We may also extend the result to some finite parabolic iterated function systems [5, Ch. 8], again if the packing measure is finite. Additionally, [4, Theorem 1.6] shows that if the limit set has dimension less than or equal to one, then the packing measure is always finite.

5. Proofs

Proof of lemma 2.3. We shall use the following volume lemma of Frostman type (see [7, Theorem 8.6.2]). Suppose A is a bounded subset of $\mathbb{R}^n$ and $0 < D < +\infty$.

(a) If, for all $x \in A$,

$$\liminf_{r\to 0} \frac{\mu(B(x,r))}{r^\beta} \le D,$$

then $\mu(E) \le b(n) D \Pi_\beta(E)$ for every Borel subset $E \subset A$, where $b(n)$ is a constant depending only on the dimension.

(b) If, for all $x \in A$,

$$\liminf_{r\to 0} \frac{\mu(B(x,r))}{r^\beta} \ge D,$$

then $\mu(E) \ge D \Pi_\beta(E)$ for every Borel subset $E \subset A$.

We first assume the β-property holds on a set A_β (of full measure) and we construct a set A of σ-finite Π_β measure.

Fix $\lambda > 0$. There exists a set $A_\lambda \subset A_\beta$ and a constant $D_\lambda > 0$ such that $\mu(A_\lambda) \ge 1 - \lambda$ and $D(x) \ge D_\lambda$ for all $x \in A_\lambda$, where $D(x)$ comes from the definition of the β-property. Take a Borel subset $E_\lambda \subset A_\lambda$ of measure $\mu(E_\lambda) = 1 - \lambda$ (μ is regular). By part (b) of the volume lemma, $\Pi_\beta(E_\lambda) \le D_\lambda^{-1}\mu(E_\lambda) < +\infty$. Take $A = \bigcup_{k=1}^{+\infty} E_{1/k}$. Observing that $\Pi_\beta(E_{1/k}) < +\infty$ for all k and $\mu(A) = 1$ ends this part of the proof.

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To prove the converse, assume that the β-property does not hold. This means that there exists a set H with $\mu(H) > 0$ such that, for every $x \in H$,

$$\liminf_{r\to 0} \frac{\mu(B(x,r))}{r^\beta} = 0.$$

The assumption means that we may write $A = \bigcup_{n=1}^{+\infty} A_n$, where $\Pi_\beta(A_n) < +\infty$. Define sets $H_n = H \cap A_n$. Since $\mu(H) > 0$, there exists n such that $\mu(H_n) > 0$. Fix a Borel subset $E \subset H_n$ of positive μ measure. For any $D > 0$, part (a) of the volume lemma shows that $\Pi_\beta(E) \geq (Db(n))^{-1}\mu(E)$ and so $\Pi_\beta(E) = +\infty$, which is a contradiction. □

Proof of theorem 3.1. The proof of the second part of the theorem (limit with fixed y) is contained in the first part. We will give the necessary changes in the proof.

Fix $\varepsilon > 0$. We will show that there exists an infinite sequence n_k, such that

$$(n_k \log\log n_k)^{1/\beta} d(T^{n_k}x, x) < \varepsilon$$

for all k. In the proof, we will write $A' = X \setminus A$.

Set

$$r_n = \frac{\varepsilon}{(n \log\log n)^{1/\beta}}$$

and observe that this sequence is decreasing. Define sets $C_n = \{x \in X : d(T^n x, x) < r_n\}$. It suffices to prove that almost all $x \in X$ belong to an infinite number of sets C_n. This is equivalent to proving that

$$\forall_{k>1} \quad \mu\Big(\bigcup_{n=k}^{\infty} C_n\Big) = 1.$$

Taking the complements of the sets C_n and because the sequence of sets is monotonic, we only need to show that, for some large W,

$$\forall_{k>W} \quad \mu\Big(\bigcap_{n=k}^{\infty} C'_n\Big) = 0.$$

Summing up, we want to show that

$$\forall_{\varepsilon>0}\ \forall_{k>W}\ \forall_{\delta>0}\ \exists_N \quad \mu\Big(\bigcap_{n=k}^{N} C'_n\Big) < \delta. \tag{5.1}$$

It should be emphasised that we will only find one (a few) such N. Certainly, it follows that for all bigger N the statement remains true, but the method fails. The correct N will be calculated at the end of the proof.

First, let us work with the β-property. For fixed $\lambda > 0$, there exists a measurable set F_λ and finite positive values D_λ and R_λ such that $\mu(X \setminus F_\lambda) \leq \lambda$ and $D(x) \geq D_\lambda$ for all $x \in F_\lambda$ and $r \leq R_\lambda$. (For brevity, we shall write F, D, R instead of F_λ, D_λ, R_λ.) We may assume that F is bounded and Borel. We denote by A'_n the *good* part of C'_n, that is, $A'_n := C'_n \cap F$.

We will now prove (5.1) with A'_n instead of C'_n, which will show that almost every point $x \in F$ has the desired recurrence speed (3.1). As λ can be made arbitrary small (so that the measure of F can be made arbitrarily close to one), this will be sufficient.

Fix some N (presumably quite large) and cover the set F with balls of radius $\frac{1}{2}r_N$ centred in the set F. We need this covering to have the property $\sum_i \mu(B(x_i, r_N)) \leq \alpha$, where α does not depend on N.

Lemma 5.1. *If X is Euclidean space or if the measure is doubling, then, for any Borel set $F \subset X$, there exists a covering of F by balls $B(x, r)$ of equal radius, centred at F such that $\sum \mu(B(x, 2r)) \leq \alpha$, where α does not depend on r.*

Proof. If the measure is doubling, then we take any covering of F by $B(x_i, \frac{1}{5}r)$, where $x_i \in \mathcal{A} \subset F$ for a set $\mathcal{A}$. Using the Vitali $5r$-lemma, there is a set $\mathcal{A}' \subset \mathcal{A}$ such that $\{B(x_i, r)\}_{i\in\mathcal{A}'}$ is a covering while the balls $\{B(x_i, \frac{1}{5}r)\}_{i\in\mathcal{A}'}$ are disjoint. Set $\alpha = c^{[\log_\eta 10]}$, where c and η come from the definition of the doubling measure. We easily estimate

$$\sum_{i\in\mathcal{A}'} \mu(B(x_i, 2r)) \leq \alpha \sum_{i\in\mathcal{A}'} \mu\Big(B\Big(x_i, \frac{1}{5}r\Big)\Big) \leq \alpha\mu(X) \leq \alpha.$$

If $X = \mathbb{R}^d$, we take a maximal r-separated set $E \subset F$. Maximality means that $\bigcup_{i\in E} B(x, r)$ is a covering. Also, there exists a constant α (depending only on d) such that E can be decomposed as a union $E = \bigcup_{k=1}^{\alpha} E_k$, where each set E_k is a $2r$-separated set. This can be seen by dividing the space into boxes of side $2r$ and dividing each of those boxes into 4^d boxes of sides $\frac{1}{2}r$. Take any ordering of the small boxes (from 1 to 4^d) in each big box. Put all points from E that belong to the first of the small boxes (in any big box) into E_1, those in the second small box into E_2, and so on. As every small box has at most one of the points from E and the distance between boxes with the same number is equal to $2r$, this is enough. So for $i, j \in E_k$, the sets $B(x_i, 2r)$ and $B(x_j, 2r)$ are disjoint, giving the estimate

$$\sum_{i\in E} \mu(B(x_i, 2r)) = \sum_{k=1}^{\alpha} \sum_{i\in E_k} \mu(B(x_i, 2r)) \leq \sum_{k=1}^{\alpha} \mu(X) \leq \alpha. \qquad \square$$

This covering will be called $\mathcal{B}$. By definition, $\mathcal{B} = \{B(x_i, \frac{1}{2}r_N)\}_{i=1}^{M}$ for some $x_i \in F$, where M depends on ε, N and λ. Put $B_i = B(x_i, \frac{1}{2}r_N)$.

We will need the inclusion

$$A'_n \cap B_i \subset B_i \cap T^{-n}B'(x_i, \tfrac{1}{2}r_n). \tag{5.2}$$

We prove this by the following sequence of simple implications (where we use the triangle inequality, the monotonicity of r_n and the fact that $n \leq N$).

$$\begin{aligned}
y \in A'_n \cap B(x_i, \tfrac{1}{2}r_N) &\implies d(T^n y, y) > r_n \quad \text{and} \quad d(x_i, y) < \tfrac{1}{2}r_N \\
d(x_i, T^n y) + d(x_i, y) \geq d(y, T^n y) &\implies d(x_i, T^n y) \geq d(y, T^n y) - d(x_i, y) \\
&\implies d(x_i, T^n y) \geq r_n - \tfrac{1}{2}r_N \geq \tfrac{1}{2}r_n,
\end{aligned}$$

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and the last inequality proves (5.2). Summing over the covering,

$$A'_n = \bigcup_{i=1}^{M} B_i \cap A'_n \subset \bigcup_{i=1}^{M} B_i \cap T^{-n} B'\Big(x_i, \frac{1}{2} r_n\Big),$$

giving an estimate for the measure of the set in (5.1) with C'_n replaced by A'_n, namely,

$$\mu\Big(\bigcap_{n=k}^{N} A'_n\Big) \le \mu\Big(\bigcup_{i=1}^{M}\Big(B_i \cap \bigcap_{n=k}^{N} A'_n\Big)\Big) \le \sum_{i=1}^{M} \mu\Big(B_i \cap \bigcap_{n=k}^{N} T^{-n} B'\Big(x_i, \frac{1}{2} r_n\Big)\Big). \tag{5.3}$$

At this point, we note that the proof of the second part of the theorem is similar (with return to a fixed point y), but with the following changes.

(i) Instead of the whole covering with B_i, we just take sets $B(y, \frac{1}{2} r_n)$, that is, the only centre is $x_1 = y$.

(ii) Instead of D_λ, R_λ, we take $D(y)$ and $R(y)$.

Although the proof follows that of the first part, it can, in fact, be simplified, for the following reasons.

(a) We do not need the set F_λ, so that part of the proof may be removed.

(b) The covering $\mathcal{B}$ consists only of one set so we do not really need it at all.

(c) This means that we can skip the subsequent part of the proof involving the bound on N.

(d) This, in turn, simplifies the estimates on the sum at the end of the proof (Lemma 5.2) because the series is infinite instead of finite.

Now let us define two classes of Lipschitz functions: $g_{n,i}(z)$ approximating the characteristic function of the ball $B(x_i, r_n)$ and $h_{n,i}(z)$ approximating the characteristic function of the complement $B'(x_i, r_n)$. These functions depend on a parameter $\kappa > 1$, radius $r > 0$ and $x \in X$. First, we define the auxiliary functions

$$\phi_{r,\kappa}(t) = \begin{cases} 1 & \text{for } 0 \le t \le r, \\ r^{-\kappa}(r + r^\kappa - t) & \text{for } r \le t \le r + r^\kappa, \\ 0 & \text{for } t \ge r + r^\kappa, \end{cases}$$

$$\psi_{r,\kappa}(t) = \begin{cases} 0 & \text{for } 0 \le t \le r - r^\kappa, \\ r^{-\kappa}(t - r + r^\kappa) & \text{for } r - r^\kappa \le t \le r, \\ 1 & \text{for } t \ge r. \end{cases}$$

The approximating functions are $g_{n,i}(z) = \phi_{(1/2)r_n,\kappa}(d(z, x_i))$, $h_{n,i}(z) = \psi_{(1/2)r_n,\kappa}(d(z, x_i))$. These functions have a Lipschitz constant equal to

$$\Big(\frac{1}{2} r_n\Big)^{-\kappa} = \Big(\frac{2}{\varepsilon}\Big)^{\kappa} (n \log\log n)^{\kappa/\beta}$$

and Lipschitz norm $\|g\|_{\mathcal{L}} = \|h\|_{\mathcal{L}} = 1 + 2^\kappa (r_n)^{-\kappa} \le 3^\kappa (r_n)^{-\kappa}$ (for r_n small enough).

From this point on, we will assume that

$$3^\kappa C r_N^{-\kappa} \gamma^k \le D r_N^{\beta}, \tag{5.4}$$

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which, after transformation, gives a bound on N by

$$r_N \geq \left(3^\kappa \frac{C}{D}\gamma^k\right)^{1/(\kappa+\beta)}. \tag{5.5}$$

Take N to be the largest integer which satisfies the inequality (5.5). Setting $E := B_i$ and $F := \bigcap_{n=0}^{N-k} T^{-n}B'(x_i, \frac{1}{2}r_{n+k})$ leads to the key estimate

$$\mu\Big(B_i \cap \bigcap_{n=k}^{N} T^{-n}B'\Big(x_i, \frac{1}{2}r_n\Big)\Big) = \mu(\mathbb{1}_E \cdot \mathbb{1}_F \circ T^k) \leq \mu(g_{N,i} \cdot \mathbb{1}_F \circ T^k). \tag{5.6}$$

Using the exponential decay of correlations and then assumption (5.4), the above is

$$\leq \mu(g_{N,i})\mu(F) + \mu(F) \cdot C\gamma^k \cdot 3^\kappa r_N^{-\kappa} \leq \mu(F)(\mu(B(x_i, r_N)) + Dr_N^\beta).$$

Then, from the β-property and $D(x_i) \geq D$, we arrive at the estimate that the above quantity is

$$\leq \mu\Big(\bigcap_{n=0}^{N-k} T^{-n}B'\Big(x_i, \frac{1}{2}r_{n+k}\Big)\Big) \cdot 2\mu(B(x_i, r_N)).$$

Applying this to (5.3),

$$\mu\Big(\bigcap_{n=k}^{N} A'_n\Big) \leq \sum_{i=1}^{M} 2\mu(B(x_i, r_N)) \cdot \mu\Big(\bigcap_{n=0}^{N-k} T^{-n}B'\Big(x_i, \frac{1}{2}r_{n+k}\Big)\Big).$$

Because of the way in which we defined our covering (Lemma 5.1), we can estimate the measure further by

$$\mu\Big(\bigcap_{n=k}^{N} A'_n\Big) \leq 2\alpha \sup_{1\leq i\leq M} \mu\Big(\bigcap_{n=0}^{N-k} T^{-n}B'\Big(x_i, \frac{1}{2}r_{n+k}\Big)\Big).$$

Thus it suffices to show (with the condition set on N) that

$$\mu\Big(\bigcap_{n=0}^{N-k} T^{-n}B'\Big(x_i, \frac{1}{2}r_{n+k}\Big)\Big) < \frac{\delta}{2\alpha} \quad \text{for } i = 1, \ldots, M. \tag{5.7}$$

Fix any i. We will show that, for $N = N(k)$, which is the largest integer satisfying the inequality (5.5), the measure of the set in equation (5.7) tends to zero as $k \to \infty$. This will complete the proof.

Now, define a function $l(p)$ as the smallest number satisfying the inequality

$$\gamma^{l(p)}C \cdot 3^\kappa r_p^{-\kappa} \leq \frac{D}{2 \cdot 4^\beta} r_p^\beta. \tag{5.8}$$

Transforming this inequality gives the formula for $l(p)$,

$$l(p) = \frac{(\kappa+\beta)}{\log(\gamma)} \log(r_p) - \log_\gamma\Big(\frac{4^\beta 3^\kappa \cdot 2C}{D}\Big). \tag{5.9}$$

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Next, define a *finite* sequence a_j by the recurrence scheme

$$a_0 = k,$$
$$a_{j+1} = a_j + l(a_j).$$

The sequence ends at Ω (depending on N) such that $a_{\Omega+1} > N \geq a_\Omega$. Clearly,

$$\mu\Big(\bigcap_{n=k}^{N} T^{-n} B'\Big(x_i, \frac{1}{2} r_n\Big)\Big) \leq \mu\Big(\bigcap_{j=0}^{\Omega} T^{-a_j} B'\Big(x_i, \frac{1}{2} r_{a_j}\Big)\Big).$$

We proceed to deal with the set on the right-hand side as in the sequence of inequalities starting with (5.6) (the only difference is that we take the function h instead of g). For brevity, set $B' = B'(x_i, \frac{1}{2} r_{a_0})$. Then

$$\mu\Big(\bigcap_{j=0}^{\Omega} T^{-a_j} B'\Big) = \mu\Big(T^{-a_0} B' \cap \bigcap_{j=1}^{\Omega} T^{-a_j} B'\Big)$$
$$= \mu\Big(B' \cap \bigcap_{j=1}^{\Omega} T^{-a_j + a_0} B'\Big) = \mu(B' \cap T^{-l(a_0)} F_1), \tag{5.10}$$

where $F_1 = \bigcap_{j=1}^{\Omega} T^{-a_j + a_1} B'$. Using again the decay of correlations and then the definition of l in (5.8),

$$\begin{aligned} &\mu(B' \cap T^{-l(a_0)} F_1) \\ &\quad = \mu(\mathbb{1}_{B'} \cdot \mathbb{1}_{F_1} \circ T^{l(a_0)}) \leq \mu(h_{a_0,i} \cdot \mathbb{1}_{F_1} \circ T^{l(a_0)}) \\ &\quad \leq \mu(F_1) \cdot (C\gamma^{l(a_0)} \|h_{a_0,i}\|_{\mathcal{L}} + \mu(h_{a_0,i})) \\ &\quad \leq \mu(F_1) \cdot (C\gamma^{l(a_0)} 3^\kappa r_{a_0}^{-\kappa} + \mu(h_{a_0,i})) \leq \mu(F_1) \cdot \Big(\frac{D}{2 \cdot 4^\beta} r_{a_0}^\beta + \mu(h_{a_0,i})\Big). \end{aligned} \tag{5.11}$$

Now we repeat the calculation (5.10), but this time for F_1, giving

$$\mu(F_1) = \mu\Big(\bigcap_{j=1}^{\Omega} T^{-a_j + a_1} B'\Big) = \mu\Big(B' \cap \bigcap_{j=2}^{\Omega} T^{-a_j + a_1} B'\Big) = \mu(B' \cap T^{-l(a_1)} F_2),$$

where $F_2 = \bigcap_{j=2}^{\Omega} T^{-a_j + a_2} B'$. Repeating (5.11) yields

$$\mu(B' \cap T^{-l(a_1)} F_2) \leq \mu(F_2) \cdot \Big(\frac{D}{2 \cdot 4^\beta} r_{a_1}^\beta + \mu(h_{a_1,i})\Big).$$

Repeating those two calculations for $a_1, a_2 \ldots a_\Omega$,

$$\begin{aligned} \mu\Big(\bigcap_{j=0}^{\Omega} T^{-a_j} B'\Big(x_i, \frac{1}{2} r_{a_j}\Big)\Big) &\leq \prod_{j=0}^{\Omega} \Big(\mu(h_{a_j,i}) + \frac{D}{2 \cdot 4^\beta} r_{a_j}^\beta\Big) \\ &\leq \prod_{j=0}^{\Omega} \Big(\mu\Big(B'\Big(x_i, \frac{1}{4} r_{a_j}\Big)\Big) + \frac{D}{2 \cdot 4^\beta} r_{a_j}^\beta\Big) \\ &\leq \prod_{j=0}^{\Omega} \Big(1 - D\Big(\frac{r_{a_j}}{4}\Big)^\beta + \frac{D}{2 \cdot 4^\beta} r_{a_j}^\beta\Big) \leq \prod_{j=0}^{\Omega} \Big(1 - \frac{D}{2 \cdot 4^\beta} r_{a_j}^\beta\Big). \end{aligned}$$

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The penultimate inequality follows from the β-property (definition of D). The only thing remaining to prove is that

$$\sum_{i=0}^{\Omega} \frac{D}{2 \cdot 4^{\beta}} r_{a_j}^{\beta} \to +\infty \quad \text{as } k \to \infty.$$

We state this below as a separate technical lemma. It follows that the measure of the intersection may be arbitrary small, which ends the proof of Theorem 3.1. □

Lemma 5.2. *For any $D > 0$ and $\varepsilon > 0$ (note that Ω and the sequence (a_j) depend on k),*

$$\lim_{k\to+\infty} \sum_{i=0}^{\Omega} \frac{D}{2 \cdot 4^{\beta}} r_{a_j}^{\beta} = \lim_{k\to+\infty} \frac{D\varepsilon^{\beta}}{2 \cdot 4^{\beta}} \sum_{i=0}^{\Omega} \frac{1}{a_j \log\log a_j} = +\infty.$$

Proof. We begin by simplifying the conditions on N (5.5) and $l(p)$ (5.9). Write $a = \gamma^{-\beta/(\kappa+\beta)} > 1$. After using the definition of r_n and collecting all the constants into one number Z, (5.5) becomes $N \log\log N) \geq Za^k$, which, by simple estimates, gives

$$N(k) \geq \widetilde{Z} \frac{a^k}{\log(k)}. \tag{5.12}$$

We may take $N(k)$ equal to the right-hand side above (after rounding to an integer). Doing the same for (5.9) yields (with another constant called S)

$$l(p) = \log_a(p) + \log_a(\log\log p) - S.$$

For clarity, let us assume that every logarithm is to base a and written just as log (this just changes the constants). The elements of the sequence a_j are contained in the interval $[k, N(k)]$. Divide this interval into subintervals $I_n = [a^n k, a^{n+1} k]$, where n takes values $0, 1, \ldots, M$. It is easy to estimate M, that is, the number of the last subinterval, because $a^M k \leq N \leq a^{M+1} k$, which, after applying the definition of N (5.12), becomes

$$M \approx k - \log k - \log\log k + \widetilde{S}. \tag{5.13}$$

On each subinterval I_n, we can estimate the function $l(p)$ from above by

$$l(p) \leq l(a^{n+1}k) = n + \log k + \log\log(n + \log k) + \widehat{S}.$$

This means that the number of elements a_j in I_n is not less than

$$\frac{\text{length of interval } I_n}{\text{increment of } a_j \text{ on } I_n} \geq \frac{a^n k}{n + \log k + \log\log(n + \log k) + \widehat{S}}.$$

Now we can estimate the sum from below by estimating on each I_n separately.

$$\begin{aligned}
\sum_{i=0}^{\Omega} \frac{1}{a_j \log\log a_j} &\geq \sum_{n=0}^{M} (\text{number of } a_j \text{ in interval } I_n) \cdot (\text{min of } (a_j \log\log a_j)^{-1} \text{ in } I_n) \\
&\geq \sum_{n=0}^{M} \frac{a^n k}{n + \log k + \log\log(n + \log k) + \widehat{S}} \cdot \frac{1}{a^{n+1} k \log(n + \log k)} \\
&\geq \frac{1}{a} \sum_{n=\lceil \log k \rceil}^{M + \lfloor \log k \rfloor} \frac{1}{n + \log\log n + \widehat{S}} \frac{1}{\log n}.
\end{aligned}$$

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Dropping the negligible constants a and $\widehat{S}$, using the integral approximation and ignoring the floor and ceiling notation (which changes the value by at most ± 1) shows that the sum is

$$\geq \log\log(M+\log k) - \log\log\log(M+\log k) - \log\log(\log k) + \log\log\log(\log k).$$

Finally, by inserting the estimate for M (5.13), we see that the sum is

$$\geq \log\log(k-\log\log k) - \log\log\log(k-\log\log k) - \log\log\log k + \log\log\log\log k,$$

and this expression goes to $+\infty$ as $k \to +\infty$. □

Acknowledgement

I am indebted to Anna Zdunik for showing me this problem and for her valuable comments and discussions on the topic.

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https://doi.org/10.1017/S0004972717000752 Published online by Cambridge University Press