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COUNTING RATIONAL POINTS ON CUBIC HYPERSURFACES: CORRIGENDUM

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There is an error in [1] which invalidates the proof of the main theorem from [1] and also the proof of Lemma 11 from [2]. In attempting to apply Proposition 3 in [1, §5], it is claimed that

$$\begin{aligned}\sum_{R_0<b_1\leqslant 2R_0} M_1 &\ll R_0^{-1/2} \sum_{R_0<b_1\leqslant 2R_0} \max_{0<N\ll (HP)^\theta} \gcd(b_1, N)^{1/2}\\ &\ll R_0^{-1/2} \max_{0<N\ll (HP)^\theta} \sum_{R_0<b_1\leqslant 2R_0} \gcd(b_1, N)^{1/2}\\ &\ll R_0^{1/2}(HP)^\varepsilon.\end{aligned}$$

The second line is false and in fact one has $M_1 = 1$ in Proposition 3. The author is very grateful to Professor Hongze Li for drawing his attention to this flaw.

The error can be fixed by introducing an average over b_1 into the statement of Proposition 3. This allows us to recover the main theorem in [1], and also [2, Lemma 11], via the following modification.

PROPOSITION 3. *Let $w \in \mathcal{W}_n$, let $\varepsilon > 0$ and let $g \in \mathbb{Z}[x_1, \ldots, x_n]$ be a cubic polynomial such that g_0 is non-singular and $\|g\|_P \leqslant H$, for some $H \leqslant P$. Let $\tilde{q} = b_2^2 c^2 d$, where*

$$b_2 := \prod_{p^2 \| \tilde{q}} p, \qquad d := \prod_{\substack{p^e \| \tilde{q}\\ e\geqslant 3,\ 2\nmid e}} p,$$

and let $R_0 \geqslant 1/2$. Define

$$V := R_0 \tilde{q} P^{-1} \max\{1, \sqrt{|z|P^3}\}, \tag{4.2}$$

and

$$W := V + (c^2 d)^{1/3}. \tag{4.3}$$

Then there exists a positive number θ such that

$$\sum_{\substack{R_0<b_1\leqslant 2R_0\\ b_1 \text{ square-free}}} |\mathcal{S}_u(b_1\tilde{q}; z)| \ll H^\theta (R_0\tilde{q})^{-n/2+1} P^{n+\varepsilon} \times (W^n \tilde{M}_1 + R_0 \min\{M_2, M_3\}),$$

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where

$$\tilde{M}_1 := \min\left\{R_0,\ \frac{P^{3/4}}{\tilde{q}^{1/2}}\right\}$$

and

$$M_2 := c^n\left(1+\frac{V}{c}\right)^{n-3/2}, \qquad M_3 := V^n\left(1+\frac{c^2d}{V^3}\right)^{n/2}.$$

In order to prove this result we will need a new technical lemma, which allows us to separate variables at a crucial point in our argument.

LEMMA A. *Let* $\mathbf{h}\in\mathbb{R}^n$, *let* $M, N>0$ *and let* $f(m;\mathbf{n})\geqslant 0$ *for every* $m\in\mathbb{N}$ *and* $\mathbf{n}\in\mathbb{Z}^n$. *Then we have*

$$\sum_{M<m\leqslant 2M}\ \sum_{\substack{\mathbf{n}\in\mathbb{Z}^n\\ |\mathbf{n}-m\mathbf{h}|\leqslant N}} f(m;\mathbf{n}) \leqslant \sum_{1\leqslant i\leqslant I}\ \sum_{\substack{\mathbf{n}\in\mathbb{Z}^n\\ |\mathbf{n}-M_i\mathbf{h}|\leqslant 2N}}\ \sum_{M<m\leqslant 2M} f(m;\mathbf{n}),$$

for appropriate $M_i\in(M,2M]$, *where* $I=M\min\{1,|\mathbf{h}|/N\}+1$.

Proof. We break the outer sum into smaller intervals of length $U\geqslant 1$, writing

$$(M,2M]=\bigcup_{1\leqslant i\leqslant M/U+1}(M_i, M_{i+1}],$$

with $M_i=M+(i-1)U$. We will take U to be maximal so that $U\geqslant 1$ and $|\mathbf{h}|U\leqslant N$. Let $m\in(M_i, M_{i+1}]$ and note that

$$N\geqslant|\mathbf{n}-m\mathbf{h}|=|\mathbf{n}-M_i\mathbf{h}+M_i\mathbf{h}-m\mathbf{h}|\geqslant\big||\mathbf{n}-M_i\mathbf{h}|-(m-M_i)|\mathbf{h}|\big|.$$

Since $m-M_i\leqslant M_{i+1}-M_i=U$, we see that the overall contribution to the left-hand side from such m is at most

$$\sum_{M_i<m\leqslant M_{i+1}}\ \sum_{\substack{\mathbf{n}\in\mathbb{Z}^n\\ |\mathbf{n}-M_i\mathbf{h}|\leqslant 2N}} f(m;\mathbf{n}).$$

We conclude the proof on enlarging the outer sum to all $m\in(M,2M]$ and interchanging it with the sum over $\mathbf{n}$. □

Proof of Proposition 3. We adopt the equation numbering from [1] and write $\mathcal{B}$ for the set of square-free integers $b_1\in(R_0,2R_0]$. For given $b_1\in\mathcal{B}$ we write $q=b_1\tilde{q}$ and $b=b_1b_2^2$. Our chief difficulty in introducing averaging over b_1 will be that we can no longer merely take a maximum over $\mathbf{v}_0\ll HP$ in (4.5) in every case. We begin, using (4.5) and (4.11), by noting that

$$\mathcal{S}_u(q;z)\ll P^{-N}+q^{-n}\int_{\mathbf{x}\ll P}\ \sum_{|\mathbf{v}-qz\nabla g(\mathbf{x})|\leqslant P^{\varepsilon}V}|S_u(q;\mathbf{v})|\,d\mathbf{x},$$

where

$$S_u(q;\mathbf{v})\ll H^{\theta}b^{(n+1)/2+\varepsilon}b_2\gcd(b_1,u,g^*(\mathbf{v}))^{1/2} \times \max_{\bar{b}\in(\mathbb{Z}/c^2d\mathbb{Z})^*}|S_{u\bar{b}^2}(c^2d;\bar{b}\mathbf{v})|. \qquad (*)$$

Let $\mathcal{S}_2(\mathbf{v}_0)$ be the overall contribution obtained by taking $u = 0$ and summing the right-hand side of ($*$) over $|\mathbf{v} - \mathbf{v}_0| \leqslant P^\varepsilon V$ for which $g^*(\mathbf{v}) = 0$. Then

$$\sum_{b_1 \in \mathcal{B}} q^{-n} \int_{\mathbf{x} \ll P} \mathcal{S}_2(qz\nabla g(\mathbf{x}))\, d\mathbf{x} \ll \sum_{b_1 \in \mathcal{B}} q^{-n} P^n \max_{\mathbf{v}_0 \ll HP} \mathcal{S}_2(\mathbf{v}_0).$$

But the treatment of $\mathcal{S}_2(\mathbf{v}_0)$, which is uniform in $\mathbf{v}_0$, is correct and leads via (4.15)–(4.16) to

$$\sum_{b_1 \in \mathcal{B}} q^{-n} \int_{\mathbf{x} \ll P} \mathcal{S}_2(qz\nabla g(\mathbf{x}))\, d\mathbf{x} \ll H^\theta R_0 (R_0 \tilde{q})^{-n/2+1} P^{n+\varepsilon} \min\{M_2, M_3\},$$

the effect of the sum over b_1 being merely to multiply the bound by R_0.

Interchanging the sum over b_1 and the integral over $\mathbf{x}$, we are now led to examine

$$J = \sum_{b_1 \in \mathcal{B}} \mathcal{S}_1(b_1 \tilde{q} z \nabla g(\mathbf{x})),$$

for given $\mathbf{x} \ll P$, where for given $\mathbf{v}_0 \in \mathbb{R}^n$, we denote by $\mathcal{S}_1(\mathbf{v}_0)$ the overall contribution from summing ($*$) over $|\mathbf{v} - \mathbf{v}_0| \leqslant P^\varepsilon V$ for which

$$(u, g^*(\mathbf{v})) \neq (0, 0).$$

We will produce two bounds for J. The first arises from taking

$$\gcd(b_1, u, g^*(\mathbf{v})) \leqslant b_1$$

in the existing argument and summing trivially over b_1. This leads to the estimate

$$J \ll H^\theta R_0 (R_0 \tilde{q})^{n/2+1+\varepsilon} W^n. \qquad (**)$$

To deduce an alternative estimate we first analyze

$$\begin{aligned} J(\mathbf{v}_0) &= \sum_{b_1 \in \mathcal{B}} \mathcal{S}_1(\mathbf{v}_0) \\ &\ll H^\theta R_0^{(n+1)/2+\varepsilon} b_2^{n+2+\varepsilon} \sum_{\substack{|\mathbf{v}-\mathbf{v}_0| \leqslant P^\varepsilon V \\ (u, g^*(\mathbf{v})) \neq (0,0)}} \left\{ \max_{\bar{b} \in (\mathbb{Z}/c^2 d\mathbb{Z})^*} |S_{u\bar{b}^2}(c^2 d; \bar{b}\mathbf{v})| \right\} \\ &\quad \times \sum_{b_1 \in \mathcal{B}} \gcd(b_1, u, g^*(\mathbf{v}))^{1/2}, \end{aligned}$$

for fixed $\mathbf{v}_0 \in \mathbb{R}^n$. The inner sum over b_1 is $O(R_0 P^\varepsilon)$, by the third displayed equation on page 107 of [1], whence

$$J(\mathbf{v}_0) \ll H^\theta R_0^{(n+3)/2} b_2^{n+2} P^\varepsilon \sum_{|\mathbf{v}-\mathbf{v}_0| \leqslant P^\varepsilon V} \max_{\bar{b} \in (\mathbb{Z}/c^2 d\mathbb{Z})^*} \sum_{\substack{a \bmod c^2 d \\ \gcd(a, c^2 d) = 1}} |T(a, c^2 d; \bar{b}\mathbf{v})|,$$

where $T(a, c^2 d; \bar{b}\mathbf{v})$ is given by (4.6). The path is now clear for the final bound

$$J(\mathbf{v}_0) \ll H^\theta R_0^{1/2} (R_0 \tilde{q})^{n/2+1+\varepsilon} W^n,$$

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which is obtained by combining [3, Lemmas 11, 15 and 16] in the manner indicated at the close of [3, §5]. In particular, this bound is uniform in $\mathbf{v}_0$. Returning to the estimation of J we apply Lemma A with

$$M = R_0, \qquad N = P^{\varepsilon} V, \qquad \mathbf{h} = \tilde{q} z \nabla g(\mathbf{x}),$$

which leads to the bound

$$\begin{aligned} J &\ll \min\left\{R_0, \frac{R_0 \tilde{q}|z|HP^2}{V}\right\} \max_{\mathbf{v}_0 \ll HP} J(\mathbf{v}_0) \\ &\ll H\sqrt{\frac{P^{3/2}}{R_0\tilde{q}}} \max_{\mathbf{v}_0 \ll HP} J(\mathbf{v}_0), \end{aligned}$$

since

$$\frac{R_0\tilde{q}|z|P^2}{V} = \frac{|z|P^3}{\max\{1, \sqrt{|z|P^3}\}} \leqslant \sqrt{|z|P^3} \leqslant \sqrt{\frac{P^{3/2}}{R_0\tilde{q}}},$$

by (3.2). Drawing our argument together with (∗∗), this therefore shows that

$$(R_0\tilde{q})^{-n} \int_{\mathbf{x} \ll P} \sum_{b_1 \in \mathcal{B}} \mathcal{S}_1(qz\nabla g(\mathbf{x}))\, d\mathbf{x} \ll H^{\theta}(R_0\tilde{q})^{-n/2+1} P^{n+\varepsilon} W^n \tilde{M}_1,$$

which concludes our proof of the proposition. □

It remains to show that our modified Proposition 3 suffices to prove [1, Proposition 1] and [2, Lemma 11].

Proof of Proposition 1. Let us adopt the equation and page numbering from [1]. We begin as in §5, with the aim of showing (5.2) for $i = 1, 2$, under the assumption that $n \geqslant 5$ and $s(g_0) = -1$. We supplant Lemma 3 with the modified bound

$$\#\{\tilde{q} = b_2^2 c^2 d : \text{ (5.1) holds}\} \ll R_1 R_2^{1/2} R_3^{1/2}.$$

The estimation of $\Sigma_2(R, \mathbf{R}; t) = \Sigma_2(R, \mathbf{R})$ in §5.1 begins with (5.5), the estimation of $\Sigma_{2,b}$ running through unchanged. On the other hand, we now have

$$\Sigma_{2,a} \ll H^{\theta} P^{n-3+\varepsilon} \mathcal{M} \sum_{\tilde{q}} R_0^{1/2} R^{1-n/2} \max_{|z| \asymp (RQ)^{-1}} W^n,$$

where

$$\mathcal{M} = \min\left\{R_0^{1/2}, \frac{P^{3/4}}{R^{1/2}}\right\}$$

and the summation over $\tilde{q}$ is over all $\tilde{q} = b_2^2 c^2 d$ such that b_2, c, d are constrained to lie in the dyadic ranges (5.1). Hence

$$\Sigma_{2,a} \ll H^{\theta} \frac{P^{n-3+\varepsilon}}{R^{n/2-3/2}} \mathcal{M}(R^{1/2}P^{-1/4} + (R_2^2 R_3)^{1/3})^n. \qquad (***)$$

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This is the same bound for $\Sigma_{2,a}$ that features in the middle of page 107, except that we have an additional factor $\mathcal{M}$. The term involving $R^{1/2}P^{-1/4}$ is now found to contribute

$$\ll H^\theta P^{3n/4-3+3/4+\varepsilon} R \ll H^\theta P^{3n/4-3/4+\varepsilon},$$

since $R \leqslant P^{3/2}$, whereas the term involving $(R_2^2 R_3)^{1/3}$ contributes

$$\ll H^\theta \frac{P^{n-3+3/4+\varepsilon}(R_2^2 R_3)^{n/3}}{R^{n/2-1}} \ll H^\theta P^{n-3+3/4+\varepsilon} R^{1-n/6}.$$

since $R_2^2 R_3 \ll R$. Both of these are satisfactory, concluding the proof of (5.7).

We now turn to the treatment of $\Sigma_1(R, \mathbf{R}; t)$ in §5.2, with the estimation of $\Sigma_{1,b}$ running through unchanged. On the other hand, we now have

$$\Sigma_{1,a} \ll H^\theta P^{n+\varepsilon} t \mathcal{M}\left(\frac{R^{3/2-n/2}(V + (R_2^2 R_3)^{1/3})^n}{R_2^{1/2}}\right),$$

where V has order (5.10) and the difference between this and the existing bound for $\Sigma_{1,a}$ is the additional factor $\mathcal{M}$. Following the argument in §5.2, we need to check that this does not alter the truth of (5.9). Thus, when $t \geqslant P^{-3}$, we take $\mathcal{M} \leqslant R^{-1/2}P^{3/4}$ and find that the term involving V makes the contribution

$$\ll H^\theta P^{3n/2+3/4+\varepsilon} t^{1+n/2} R^{1+n/2} \ll H^\theta P^{3n/4-3/4+\varepsilon},$$

since $t \leqslant (RP^{3/2})^{-1}$. This is satisfactory for $n \geqslant 5$. Likewise, when $t < P^{-3}$, one obtains a satisfactory contribution. Turning to the contribution from the term involving $(R_2^2 R_3)^{1/3}$, we suppose first that $t < P^{-3}$. Taking $R_2 \geqslant (R_2^2 R_3)^{1/3}$, the contribution from this case is found to be

$$\begin{aligned} &\ll H^\theta P^{n+\varepsilon} \mathcal{M} \frac{R^{3/2-n/2} t (R_2^2 R_3)^{n/3}}{R_2^{1/2}} \\ &\ll H^\theta P^{n-3+\varepsilon} \mathcal{M} R^{3/2-n/2} (R_2^2 R_3)^{n/3-1/6}. \end{aligned}$$

Taking $\mathcal{M} \leqslant R^{-1/2}P^{3/4}$ gives $O(H^\theta P^{n-3+3/4+\varepsilon} R^{(5-n)/6})$, which is satisfactory since $n \geqslant 5$. Next, assuming that $t \geqslant P^{-3}$ and adjoining Proposition 2, it remains to analyze the contribution

$$\ll H^\theta P^{n+\varepsilon} \min\left\{\mathcal{M} R^{3/2-n/2} t (R_2^2 R_3)^{n/3-1/6}, \frac{R^{2-n/8} t^{1-n/8}}{(R_2^2 R_3)^{2/3} P^{3n/8}}\right\}. \qquad (****)$$

For $n \geqslant 6$ we apply the inequality $\min\{A, B\} \leqslant A^{1/3}B^{2/3}$, to get the overall contribution $O(H^\theta P^{n-2+\varepsilon} \mathcal{M}^{1/3} E_n)$, with E_n given at the bottom of page 109. When $n \geqslant 13$ we take $t \geqslant P^{-3}$, getting

$$\mathcal{M}^{1/3} E_n \ll P^{-3/4} R^{7/6-5n/36} \ll 1.$$

When $6 \leqslant n \leqslant 12$ we take $t \leqslant (RP^{3/2})^{-1}$ to deduce that

$$\mathcal{M}^{1/3} E_n \ll P^{3/4-n/8} R^{1/6-n/18} \ll 1.$$

Finally we dispatch the case $n = 5$, for which we return to (∗∗∗∗) and take $t \leqslant (RP^{3/2})^{-1}$. This leads to the contribution

$$\begin{aligned} &\ll H^\theta P^{3+\varepsilon} \\ &\quad \times \min\left\{ P^{1/2} R^{-2} R_0^{1/2} (R_2^2 R_3)^{3/2},\ P^{5/4} R^{-5/2} (R_2^2 R_3)^{3/2},\ \frac{P^{-7/16} R}{(R_2^2 R_3)^{2/3}} \right\} \\ &\ll H^\theta P^{3+\varepsilon} \min\left\{ P^{1/2} R^{-3/2} R_2^2 R_3,\ P^{5/4} R^{-5/2} (R_2^2 R_3)^{3/2},\ \frac{P^{-7/16} R}{(R_2^2 R_3)^{2/3}} \right\}. \end{aligned}$$

Taking $\min\{A, B, C\} \leqslant A^{17/75} B^{2/15} C^{16/25}$ leads to the contribution $O(H^\theta P^{3+\varepsilon} R^{-1/30})$. This is satisfactory and so concludes the proof of Proposition 1 in [1]. □

Proof of Lemma 11. We now adopt the equation and page numbering from [2]. The treatments of $\Sigma_{1,b}$ and $\Sigma_{2,b}$ go through unchanged, leaving us the task of showing that

$$\Sigma_{i,a} \ll H^\theta P^{n-5/2+\varepsilon},$$

for $i = 1, 2$ and $n \geqslant 8$. Beginning with $i = 2$, it follows from (∗∗∗) that our estimate at the top of page 866 gets replaced by

$$\Sigma_{2,a} \ll H^\theta P^{3n/4-3/4+\varepsilon} + H^\theta \frac{P^{n-3+\varepsilon} (R_2^2 R_3)^{n/3}}{R^{n/2-3/2}} \min\left\{ R_0^{1/2}, \frac{P^{3/4}}{R^{1/2}} \right\}.$$

The first term is satisfactory. We take $\min\{\cdot, \cdot\} \leqslant R_0^{1/2}$ in the second term and note that $R_0^{1/2} (R_2^2 R_3)^{n/3} \ll R^{n/3}$. Thus the second term is

$$\ll H^\theta P^{n-3+\varepsilon} R^{-n/6+3/2},$$

which is satisfactory for $n \geqslant 8$, since $R \leqslant P^{3/2}$.

Turning to $i = 1$, our analogue of the third displayed equation on page 866 is

$$\Sigma_{1,a} \ll H^\theta P^\varepsilon (P^{3n/4-3/4} + P^{n-3} \mathcal{M} R^{3/2-n/2} (R_2^2 R_3)^{n/3-1/6} + \mathcal{E}),$$

where, in view of (∗∗∗∗),

$$\mathcal{E} = P^n \min\left\{ \mathcal{M} R^{3/2-n/2} t (R_2^2 R_3)^{n/3-1/6}, \frac{R^{2-n/8} t^{1-n/8}}{(R_2^2 R_3)^{2/3} P^{3n/8}} \right\}.$$

In our bound for $\Sigma_{1,a}$ the second and third terms correspond to the contribution from the term involving $(R_2^2 R_3)^{1/3}$, with the second dealing with the case $t < P^{-3}$ and the third dealing with the case $t \geqslant P^{-3}$. The first term is satisfactory. Taking $\mathcal{M} \leqslant R_0^{1/2}$ shows that the second term makes the satisfactory contribution

$$\ll H^\theta P^{n-3+\varepsilon} R^{3/2-n/2} R_0^{1/2} (R_2^2 R_3)^{n/3-1/6} \ll H^\theta P^{n-3+\varepsilon} R^{4/3-n/6}.$$

We handle $\mathcal{E}$ as in [1, §5.2] by applying the inequality $\min\{A, B\} \leqslant A^{1/3} B^{2/3}$, to get the overall contribution $O(H^\theta P^{n-2+\varepsilon} \mathcal{M}^{1/3} E_n)$, with

$$E_n = P^{2-n/4} t^{1-n/12} R^{11/6-n/4} (R_2^2 R_3)^{n/9-1/2}.$$

We need to check that $P^{1/2} \mathcal{M}^{1/3} E_n \ll 1$ for $n \geqslant 8$. When $n \geqslant 13$ we take $t \geqslant P^{-3}$, getting

$$P^{1/2} \mathcal{M}^{1/3} E_n \ll P^{-1/4} R^{7/6-5n/36} \ll 1.$$

When $8 \leqslant n \leqslant 12$ we take $t \leqslant (RP^{3/2})^{-1}$ to deduce that

$$P^{1/2} \mathcal{M}^{1/3} E_n \ll R_0^{1/6} P^{1-n/8} R^{5/6-n/6} (R_2^2 R_3)^{n/9-1/2} \ll P^{1-n/8} R^{1/3-n/18} \ll 1.$$

This is satisfactory and so concludes the proof of Lemma 11 in [2]. □

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