

GENERALIZED INTEGRALS WITH RESPECT TO FUNCTIONS OF BOUNDED VARIATION

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Introduction. A considerable literature has grown up around the analysis of the structure of a function in terms of its derivative, and the structure of functions $F(x)$ which are integrals of various kinds. Some of this relates to derivatives and integrals of $F(x)$ with respect to functions of bounded variation $\omega(x)$ (1-6) or, in the case of a paper by Ward (4), with respect to a function of generalized bounded variation in the restricted sense. While functions of bounded variation have at most a denumerable set of discontinuities, yet this set can be everywhere dense, and in the studies to which we refer a consideration of these discontinuities enters, sometimes in a complicated way. In the present paper results are obtained without reference to the values of $F(x)$ or $\omega(x)$ at the points of discontinuity of ω . The results lead to a descriptive definition of a Lebesgue-Stieltjes integral of a function with respect to ω and a descriptive definition of a generalized integral with respect to ω . The latter involves functions $F(x)$ which are generalized absolutely continuous relatively to ω .

Because $\omega(x)$ can be written as the difference of two non-decreasing functions there is no loss of generality in taking $\omega(x)$ to be non-decreasing. We shall consider a function $\omega(x)$ on a closed interval $[a, b]$ with the understanding that $w(x) = \omega(a)$, $x < a$, $\omega(x) = \omega(b)$, $x > b$. With $\omega(x)$ given we shall denote by \mathfrak{U} a class of functions $F(x)$ defined at the points of continuity of $\omega(x)$ on $[a, b]$. Furthermore, if \mathfrak{C} is the set over which ω is continuous, then $F(x)$ is continuous over \mathfrak{C} at points of \mathfrak{C} , and if x_0 is a point of discontinuity of ω then $F(x)$ tends to a limit as x tends to x_0+ and to x_0- , $x \in \mathfrak{C}$. These limits will be denoted by $F(x_0+)$, $F(x_0-)$. Also $F(x) = F(a+)$ for $x < a$ and $F(x) = F(b-)$ for $b > x$. $F(x)$ may, or may not, be defined at points of discontinuity of ω .

1. The ω -measure of a set E on $[a, b]$. Let (a', b') be an open interval on $[a, b]$. The ω -measure of (a', b') is $\omega(b'-) - \omega(a'+)$, and is denoted by $|(a', b')|_\omega$. Let E be any set on $[a, b]$. Let $\alpha_1, \alpha_2, \dots$ be a set of non-overlapping open intervals containing E . The outer ω -measure of the set E is the infimum of $\sum |\alpha_i|_\omega$ for all such sets of open intervals. This outer measure is denoted by $|E|_\omega^0$. Let \tilde{E} be the complement of the set E . If for $\epsilon > 0$ there exists a set of non-overlapping open intervals $\alpha = \alpha_1 + \alpha_2 + \dots$, $\alpha \supset E$ and a similar set $\beta \supset \tilde{E}$ for which $|\alpha\beta|_\omega^0 < \epsilon$ then the set E is said to be ω -measurable. The ω -measure of E , denoted by $|E|_\omega$, is equal to $|E|_\omega^0$.

Received October 21, 1957.

LEMMA 1. If $\alpha = \alpha_1 + \alpha_2 + \dots$ is a set of non-overlapping open intervals $\alpha_1, \alpha_2, \dots$ then α is ω -measurable.

Because $\omega(x)$ is BV it follows that $\Sigma |\alpha_i|_\omega$ converges. Hence for $\epsilon > 0$ there exists an integer n_0 such that for $n > n_0$,

$$\sum_{n+1}^{\infty} |\alpha_i|_\omega < \epsilon.$$

Fix $n > n_0$. Let $\alpha_i = (a_i, b_i)$, $i \leq n$. Take a'_i, b'_i with $a_i < a'_i < b'_i < b_i$ and such that

$$\sum_{i=1}^n \{ |(a_i, a'_i)|_\omega + |(b'_i, b_i)|_\omega \} < \epsilon.$$

Let β be the open set complementary to the finite set of closed intervals $[a'_1, b'_1], \dots, [a'_n, b'_n]$. Then $\beta \supset \bar{E}$ and

$$|\alpha\beta|_\omega^0 = \sum_{n+1}^{\infty} |\alpha_i|_\omega + \sum_{i=1}^n \{ |(a_i, a'_i)|_\omega + |(b'_i, b_i)|_\omega \} < 2\epsilon.$$

Because ϵ is arbitrary it follows that α satisfies the definition of measurability.

LEMMA 2. Let E be any set on $[a, b]$. Let each point $x \in E$ be the left hand end point of a set of intervals $(x, x + h_i)$ for which $h_i \rightarrow 0$ as $i \rightarrow \infty$. Let \mathfrak{F} denote the set of intervals thus associated with the set E . Let $\epsilon > 0$ be given.

There exists a finite non-overlapping set Δ_i of the intervals of \mathfrak{F} for which

$$\sum_{i=1}^n |\Delta_i|_\omega < |E|_\omega^0 + \epsilon, \quad \sum_{i=1}^n |\Delta_i E|_\omega^0 > |E|_\omega^0 - \epsilon.$$

Let $\eta > 0$ be given. Put E in a set of open intervals $\alpha_1, \alpha_2, \dots$ in such a way that $\Sigma |\alpha_i|_\omega < |E|_\omega^0 + \eta$. Now let α be a finite set of the intervals $\alpha_1, \alpha_2, \dots, \alpha = \alpha_1 + \alpha_2 + \dots + \alpha_n$ where n is sufficiently great to insure that $|\alpha E|_\omega^0 > |E|_\omega^0 - \eta$. Let $\alpha_i = (a_i, b_i)$. Let η' be so fixed that if b'_i is on (a_i, b_i) , $a_i < b'_i < b_i$ and $b_i - b'_i < \eta'$ then

$$(1) \quad \sum_{i=1}^n |E(b'_i, b_i)|_\omega^0 < \eta.$$

Let E_δ be the points of $E\alpha$ which are such that if x is in E_δ then there is an interval $(x, x + h)$ of \mathfrak{F} with $h > \delta$ and $x, x + h$ on the same interval of the set α . If $\delta_2 < \delta_1$ then

$$E_{\delta_2} \supset E_{\delta_1}.$$

Hence $|E_\delta|_\omega^0 \rightarrow |E\alpha|_\omega^0$ as $\delta \rightarrow 0$. Fix $\delta < \eta'$ and sufficiently near zero to insure that

$$(2) \quad |E_\delta|_\omega^0 > |E\alpha|_\omega^0 - \eta > |E|_\omega^0 - 2\eta.$$

Now consider the interval $\alpha_1 = (a_1, b_1)$. There is a point $x_1' \geq a_1$ which is (a) the first point of E_δ to the right of a_1 or (b) the infimum of points of E_δ on

(a_1, b_1) . For case (a) let $x_1' = x_1$ and let $(x_1, x_1 + h_1)$ be an interval of \mathfrak{F} on (a_1, b_1) with $h_1 > \delta$. For case (b) take x_1 a point of E_δ and such that

$$(3) \quad |E(x_1', x_1)|_\omega^0 < \epsilon_1$$

where ϵ_1 is the first member of a sequence of positive numbers $\epsilon_1, \epsilon_2, \dots$ for which $\Sigma \epsilon_i < \eta/n$, and take $(x_1, x_1 + h_1)$ an interval of \mathfrak{F} with $h_1 > \delta$.

In the foregoing replace a_1 by $x_1 + h_1$ and arrive at $x_2' \geq x_1 + h_1$ satisfying (a) or (b) and an interval $(x_2, x_2 + h_2)$ of \mathfrak{F} on (a_1, b_1) corresponding to $(x_1, x_1 + h_1)$ with $h_2 > \delta$. If case (b) holds choose x_2 so that

$$(4) \quad |E(x_2', x_2)|_\omega^0 < \epsilon_2.$$

Since each $h_i > \delta$, and $\delta < \eta'$ this procedure can be continued to get a finite non-overlapping set of intervals $[x_1, x_1 + h_1], [x_2, x_2 + h_2], \dots, (x_m, x_m + h_m)$ of the set \mathfrak{F} for which no points of $\alpha_1 E_\delta$ are to the right of $x_m + h_m$ or

$$(5) \quad b_1 - (x_m + h_m) < \eta'.$$

Also the intervals $(x_i, x_i + h_i), i = 1, 2, \dots, m$, have been so chosen that

$$(6) \quad \sum_{i=1}^m |E(x_i', x_i)|_\omega < \sum \epsilon_i < \frac{\eta}{n}.$$

This procedure can be repeated for each of the remaining intervals $\alpha_2, \alpha_3, \dots, \alpha_n$ of the set α .

From (1), (5), (6) and the relations similar to (5) and (6) for the intervals $\alpha_2, \dots, \alpha_n$ it follows that the total set $\Delta_i = (x_i, x_i + h_i)$ obtained by this process are on α , are non-overlapping and

$$\sum |E_\omega \Delta_i| > |\alpha E_\delta| - 2\eta.$$

This, with (2) and the fact that E_δ is on α and $\Delta_i E_\delta \subset \Delta_i E$, gives $\Sigma \Delta_i E > |E|_\omega^0 - 4\eta$. Because Δ_i is on α and because η is arbitrary the lemma follows.

Definition 1. A function $F(x)$ defined on $[a, b]$ and in class \mathfrak{U} is *absolutely continuous relative to ω* , $AC - \omega$ if for $\epsilon > 0$ there exists $\delta > 0$ such that for any set of non-overlapping intervals (x_i, x_i') on $[a, b]$ with $\Sigma \{\alpha(x_i' +) - \alpha(x_i -)\} < \delta$ the relation $\Sigma |F(x_i' +) - F(x_i -)| < \epsilon$ is satisfied.

2. The derivatives of $F(x)$ with respect to ω . Let $F(x)$ be a function in class \mathfrak{U} . Define the function $\psi(x, h)$ by the relation

$$\psi(x, h) = \begin{cases} \frac{F(x+h) - F(x-)}{\omega(x+h) - \omega(x-)}, & h > 0, \omega(x+h) - \omega(x-) \neq 0, \\ \frac{F(x+h) - F(x+)}{\omega(x+h) - \omega(x+)}, & h < 0, \omega(x+h) - \omega(x+) \neq 0, \\ 0, & \omega(x+h) - \omega(x\pm) = 0, \end{cases}$$

for points of continuity $x+h$ of ω . If $\psi(x, h)$ tends to a limit as $h \rightarrow 0$ this limit is the derivative of $F(x)$ with respect to $\omega(x)$, $D_\omega F$. The upper and lower limits of this ratio are the upper and lower derived numbers of $F(x)$

with respect to ω . It is to be noted that $D_\omega F$ exists at points of discontinuity of ω even when $F(x)$ is not defined at such a point.

Definition 2. Let $f(x)$ be defined on an ω -measurable set E on $[a, b]$. Let e_a be the part of E for which $f(x) < a$. If e_a is ω -measurable for every real number a then f is ω -measurable on E .

Definition 3. Let $f(x)$ be ω -measurable on an ω -measurable set E . Let (l_{i-1}, l_i) be a subdivision of the range of f on E . Let e_i be the points of E for which $l_{i-1} \leq f < l_i$. If $\sum l_{i-1}|e_i|_\omega$ tends to a limit as the supremum of $l_i - l_{i-1} \rightarrow 0$ then this limit is the Lebesgue-Stieltjes integral of $f(x)$ over the set E .

If x is a point on $[a, b]$ and $f(x)$ is ω -measurable on $[a, b]$ then

$$F(x) = \int_a^x f(x) d\omega$$

is absolutely continuous relatively to ω . Also $F(x)$ is in class \mathfrak{U} and $D_\omega F = f$ except for a set of ω -measure zero. This definition of a Lebesgue-Stieltjes integral is the usual one and the stated properties may be proved in the usual way.

3. Properties of functions in class \mathfrak{U} .

Definition 4. Let $F(x)$ be a function in class \mathfrak{U} . The set $[x, F(x)]$ is the union of the graphs $[x, F(x+)]$ and $[x, F(x-)]$. For any interval of continuity $[x, F(x)]$ is thus the graph of $F(x)$ in the usual sense.

THEOREM 1. Let $F(x)$ and $G(x)$ be two functions in class \mathfrak{U} . If F and G are $AC - \omega$ and if $D_\omega F = D_\omega G$, except at a set of ω -measure zero, then the sets $[x, F(x)]$, $[x, G(x)]$ are identical or one is a translation parallel to the y -axis of the other.

Set $H(x) = F(x) - G(x)$. Then H is $AC - \omega$ and $D_\omega H = 0$ except for a set of ω -measure zero. At a point of discontinuity x_0 of $\omega(x)$, $H(x_0-) = H(x_0+)$. For otherwise

$$\lim_{h \rightarrow 0} \frac{H(x_0+) - H(x_0 \pm)}{\omega(x_0 + h) - \omega(x_0 \pm)} \neq 0$$

and it follows that $D_\omega H \neq 0$ on a set of ω -measure greater than zero, which is a contradiction. If at points of discontinuity of ω we set $H(x) = H(x+) = H(x-)$ then $H(x)$ is continuous on $[a, b]$. We now prove that $H(x)$ is constant on $[a, b]$.

Let E be the set on $[a, b]$ at which $D_\omega H = 0$. Then $|E|_\omega = |[a, b]|_\omega$. If x , is a point of E and $\epsilon > 0$ is given there is a sequence of intervals $[x, x + h_i]$, $h_i > 0$, $h_i \rightarrow 0$ as $i \rightarrow \infty$ such that

$$(1) \quad \left| \frac{H(x + h_i) - H(x-)}{\omega(x + h_i) - \omega(x-)} \right| < \epsilon.$$

By Lemma 2 there is a finite set Δ_i of these intervals associated with the set E such that

$$(2) \quad |\Sigma|\Delta_i|_\omega - |[a, b]|\omega| < \epsilon, \quad \Sigma|\bar{\Delta}_j|_\omega < \epsilon,$$

where $\bar{\Delta}_j$ is the finite set of intervals complementary to the set Δ_i . It follows from the second member of (2) and the fact that $H(x)$ is $AC - \omega$ that if η is given then ϵ can be so fixed that if $[x_j, x'_j]$ are the intervals of $\bar{\Delta}_j$ then

$$\Sigma|H(x'_j) - H(x_j)| < \eta.$$

It follows from (1) that for the intervals Δ_i

$$\Sigma|H(x'_i) - H(x_i)| \leq \epsilon M$$

where M depends on the function of bounded variation $\omega(x)$. It then follows that $|H(b) - H(a)| < \epsilon M + \eta$. Because ϵ and η are arbitrary, ϵ fixed after η , it follows that $F(b) = F(a)$. If $a < x < b$ it can be shown in the same way that $H(x) = H(a)$. Hence $H(x)$ is constant on $[a, b]$.

It now follows that $F(x) - G(x) = C$, a constant, at points where both functions are continuous, that is, at the points of continuity of ω . Furthermore, at points x_0 of discontinuity of ω ,

$$\lim_{x \rightarrow x_0 \pm 0} [F(x) - G(x)] = C,$$

x a point of continuity of ω , from which it follows that at points of discontinuity of ω the jumps, if any, of F and G are the same and in the same direction. It then follows that the sets $[x, F(x)]$, $[x, G(x)]$ are identical or one is a translation parallel to the y -axis of the other.

THEOREM 2. *If $F(x)$ is in class \mathfrak{U} and is $AC - \omega$ and if $D_\omega F = f(x)$, except for a set of ω -measure zero, then if a and x are points of continuity of ω*

$$F(x) - F(a) = \int_a^x f(x) d\omega.$$

Let

$$G(x) = \int_a^x f(x) d\omega.$$

Then $G(x)$ is in class \mathfrak{U} , is $AC - \omega$, and $D_\omega G = f(x)$ except for a set of x of ω -measure zero. Hence G and F satisfy the conditions of Theorem 1, and, except at points of discontinuity of ω

$$F(x) - G(x) = C.$$

Since $G(a) = 0$ it follows that $C = F(a)$ and

$$F(x) - F(a) = G(x) = \int_a^x f(x) d\omega.$$

This theorem gives point to the following definition:

Definition 5. Let $f(x)$ be defined on $[a, b]$ and be measurable relative to ω . Let $F(x)$ be a function in class \mathfrak{U} which is $AC - \omega$ and such that $D_\omega F = f(x)$ except possibly for a set of ω -measure zero. If the series $\Sigma |f(x_i)\{\omega(x_i+) - \omega(x_i-)\}|$, where $x_i, i = 1, 2, \dots$, are the points of discontinuity of ω , converges and if $F(x_i+) - F(x_i-) = f(x_i)\{\omega(x_i+) - \omega(x_i-)\}$ then $F(x)$ is the Lebesgue-Stieltjes integral of $f(x)$:

$$F(x) - F(a) = \int_a^x f(t) d\omega(t).$$

Definition 6. A function $F(x)$ is generalized absolutely continuous with respect to ω , $ACG - \omega$, over $[a, b]$ if this interval is the sum of a denumerable sequence of closed sets over each of which $F(x)$ is $AC - \omega$.

THEOREM 3. Let $F(x)$ and $G(x)$ be two functions in class \mathfrak{U} each of which is $ACG - \omega$ on $[a, b]$, and such that $D_\omega F = D_\omega G$ except for a set of ω -measure zero. Then the sets $[x, F(x)]$, $[x, G(x)]$ are identical or one is a translation parallel to the y -axis of the other.

As in Theorem 1, let $H(x) = F(x) - G(x)$ at points of continuity of $\omega(x)$. Then $D_\omega H = 0$ except for set of ω -measure zero, and it follows as before that $H(x_0-) = H(x_0+)$ at points of discontinuity of ω . Hence if $H(x_0)$ is equal to this common value then $H(x)$ is continuous on $[a, b]$. We now have $H(x)$ continuous and $ACG - \omega$ on $[a, b]$ and $D_\omega H = 0$ except for a set of ω -measure zero. We show that $H(x)$ is constant on $[a, b]$.

Because F and G are both $ACG - \omega$ on $[a, b]$ it follows that

$$[a, b] = \sum E_i^1, \quad [a, b] = \sum E_j^2, \quad \text{where } E_i^1, E_j^2 \text{ are closed,}$$

where F is $AC - \omega$ over each E_i^1 and G is $AC - \omega$ over each E_j^2 . Hence $H(x)$ is $AC - \omega$ over each of the sets $E_i^1 E_j^2, i, j = 1, 2, \dots$. This is a denumerable sequence $\mathfrak{E}_1, \mathfrak{E}_2, \dots$ of closed sets which cover $[a, b]$. Let E be the set on $[a, b]$ which is such that if $x \in E$ then in every interval ω with x as an interior point $H(x)$ fails to be constant. The set E is closed. It then follows from Baire's theorem that there is an integer n and an interval $[l, m]$ such that $E[l, m]$ is not empty, $E[l, m] = \mathfrak{E}_n[l, m] = e$. The set e is closed and H is $AC - \omega$ over e . Hence, if ϵ is given, there exists δ such that if $(x_i x_i')$ is a set of non-overlapping intervals on $[l, m]$ with x_i, x_i' points of e and with $\Sigma |\omega(x_i, x_i')|_\omega < \delta$ then $\Sigma |H(x_i') - H(x_i)| < \epsilon$. Now let (x_j, x_j') be any set of non-overlapping intervals on $[l, m]$ with $\Sigma |\omega(x_j, x_j')|_\omega < \delta$. If there are points of e on $[x_j, x_j']$ let \bar{x}_j be the first point of e to the right of x_j , $\bar{x}_j = x_j$ if $x_j \in e$. Let \bar{x}_j' be the first point of e to the left of x_j' , $\bar{x}_j' = x_j'$ if $x_j' \in e$. Because $H(x)$ is continuous, and constant on intervals of $[l, m]$ which are complementary to the set $e = \mathfrak{E}_n[l, m]$, it follows that $H(x)$ is constant on $[x_j, \bar{x}_j]$ and on $[\bar{x}_j', x_j']$. We now have $H(\bar{x}_j) - H(x_j) = H(x_j') - H(\bar{x}_j') = 0$ and

$$\begin{aligned} \sum |H(x'_j) - H(x_j)| &\leq \sum |H(\bar{x}_j) - H(x_j)| + \\ &\quad \sum |H(\bar{x}'_j) - H(\bar{x}_j)| + \sum |H(x'_j) - H(\bar{x}'_j)| \\ &\leq \sum |H(\bar{x}'_j) - H(\bar{x}_j)| < \epsilon. \end{aligned}$$

The last relation follows because \bar{x}_j, \bar{x}'_j are points of \mathfrak{G}_n and $\Sigma |(\bar{x}_j, \bar{x}'_j)|_\omega \leq \Sigma |(x_j, x'_j)|_\omega < \delta$. Then, because (x_j, x'_j) is any set of intervals on $[l, m]$ it follows that $H(x)$ is $AC - \omega$ on $[l, m]$. Because $D_\omega H = 0$ except for a set of ω -measure zero it now follows from Theorem 1 that $H(x)$ is constant on $[l, m]$. Consequently, there are no points of E on $[l, m]$. But this contradicts the fact that E is not empty. Hence the set E is empty. It now follows that every point of $[a, b]$ is interior to an interval on which $H(x)$ is constant. By the Heine-Borel theorem there is a finite set of intervals covering $[a, b]$ on each of which $H(x)$ is constant. Then, because $H(x)$ is continuous it easily follows that $H(x)$ is constant on $[a, b]$. The proof of Theorem 3 may now be completed as in Theorem 1.

THEOREM 3. *Let $F(x)$ be a function in class \mathfrak{U} which is $ACG - \omega$ on $[a, b]$. Let $f(x)$ be a ω -measurable function on (a, b) and let $D_\omega F = f$ except for at most a set of ω -measure zero. Then $F(b+) - F(a-)$ can be determined in at most a denumerable set of operations.*

LEMMA 2. *Let E be a closed set on $[a, b]$. If $F(x)$ satisfies the conditions of Theorem 3 there is an interval $[l, m]$ on $[a, b]$ such that $E[l, m]$ is not empty, such that $D_\omega F$ is Lebesgue-Stieltjes integrable with respect to ω over $E[l, m]$, and such that if $[\alpha_i, \beta_i]$ are the intervals on $[l, m]$ contiguous to $E[l, m]$, then $\Sigma |F(\beta_i+) - F(\alpha_i-)|$ converges.*

Let $[a, b] = \mathfrak{G}_1 + \mathfrak{G}_2 + \dots$ where each \mathfrak{G}_n is closed and F is $AC - \omega$ over \supset_n . There is then an integer n and an interval $[l, m]$ such that $E[l, m] = \mathfrak{G}_n[l, m] = e$. We now turn our attention to the function $F(x)$ on $[l, m]$. F is $AC - \omega$ over e . It then easily follows that there exists a positive number M such that for any set of non-overlapping intervals (x_i, x'_i) on $[a, b]$ with x_i, x'_i points of e the relation $\Sigma |F(x'_i+) - F(x_i-)| < M$ holds. We use this fact in showing that $|D_\omega F| = |f|$ is Lebesgue-Stieltjes integrable with respect to ω over e . Let (l_{i-1}, l_i) , $i = 1, 2, \dots$, be a subdivision of the range $[0, \infty]$. Let $e_i = E(l_{i-1} < |f| \leq l_i, x \in e)$, $i > 1$, $e_0 = E(l_0 \leq f \leq l_1)$. Suppose that $\Sigma l_{i-1}|e_i|_\omega$ diverges. Fix n so that

$$\sum_{i=1}^n l_{i-1}|e_i|_\omega > 2M.$$

If $x \in e_i$ there is a sequence of intervals $(x, x + h_i)$, $h_i > 0$, $h_i \rightarrow 0$ such that

$$(3) \quad \frac{|F(x + h_i) - F(x-)|}{|\alpha(x + h_i) - \alpha(x-)|} > l_{i-1}.$$

Let $\epsilon_0, \epsilon_1, \epsilon_2, \dots$ be a sequence of positive numbers with $\epsilon_n \rightarrow 0$. By Lemma 2 there is a finite set Δ_i^0 of non-overlapping intervals of the set associated with the set e_0 by means of (3) for which

$$(4) \quad \sum |\Delta_i^0|_\omega > |e_0|_\omega - \epsilon_0, \quad \sum |\tilde{e}_0 \Delta_i^0|_\omega < \epsilon_0.$$

Because of the second relation of (4) there exists a finite set Δ_i^1 of the intervals associated with e_1 by means of (3) such that

$$\sum |\Delta_i^1|_\omega > |e_1|_\omega - \epsilon_0 - \epsilon_1, \quad \sum |\tilde{e}_1(\Delta_i^0 + \Delta_i^1)|_\omega < \epsilon_0 + \epsilon_1,$$

and $\Sigma \Delta_i^1$ does not overlap $\Sigma \Delta_i^0$. If this process is continued there is obtained a set of intervals Δ_i^k , which do not overlap the set $\Sigma \Delta_i^0 + \Sigma \Delta_i^1 + \dots + \Sigma \Delta_i^{k-1}$, such that

$$\sum |\Delta_i^k|_\omega > |e_k|_\omega - \epsilon_0 - \epsilon_1 - \dots - \epsilon_k,$$

and

$$\sum |\tilde{e}_k(\Delta_i^0 + \Delta_i^1 + \dots + \Delta_i^k)|_\omega < \epsilon_0 + \epsilon_1 + \dots + \epsilon_k.$$

Also, because of (3), it is true that if (x_i, x_i') are the intervals of the set Δ_i^k then

$$\frac{F(x_i' +) - F(x_i -)}{\alpha(x_i' +) - \alpha(x_i -)} > l_{k-1},$$

and

$$\sum \{F(x_i' +) - F(x_i -)\} > l_{k-1}(|e_k|_\omega - \epsilon_0 - \epsilon_1 - \dots - \epsilon_k).$$

Combining all the sets Δ_i^k into a single set $\Delta_i = (x_i, x_i')$ and summing over this set we get

$$\sum |F(x_i' +) - F(x_i -)| > \sum_{k=1}^n l_{k-1}|e_k|_\omega - nl_0\epsilon_0 - (n-1)l_1\epsilon_1 - \dots - l_{n-1}\epsilon_n.$$

The first sum on the right is greater than $2M$. The numbers $\epsilon_0, \epsilon_1, \dots, \epsilon_n$ are independent of n , and independent of the numbers l_0, l_1, \dots, l_{n-1} . Hence the left side is not less than $2M$. But x_i, x_i' are points of e which compels the left side to be less than M . Thus there is a contradiction and we may conclude that $\Sigma l_{i-1}|e_i|_\omega$ converges. It then follows that $|D_\omega F|$ is Lebesgue-Stieltjes integrable with respect to ω over e .

Let (α_i, β_i) be the intervals on $[l, m]$ complementary to e . Let $\epsilon > 0$ be given. Fix n_0 with

$$\sum_{n_0+1}^{\infty} |F(\beta_i +) - F(\alpha_i -)| < \epsilon.$$

Take (l_{i-1}, l_i) a subdivision of $(-\infty, \infty)$. Let e_i, e_i' be respectively the parts of e for which $l_{i-1} < D_\omega F \leq l_i$, $l_{i-1} \leq D_\omega F < l_i$. Let the subdivision (l_{i-1}, l_i) and the integer n_1 be such that

$$\sum_{-n_1}^{n_1} l_{i-1}|e_i|_\omega, \quad \sum_{-n_1}^{n_1} l_i|e_i'|_\omega$$

differ from the Lebesgue-Stieltjes interval of $D_\omega F$ over e by not more than ϵ . By working as before with the sets e_i we can get a set of intervals Δ_i with end points x_i, x_i' belonging to e and such that the intervals Δ_i do not overlap the intervals (α_i, β_i) , $i = 1, 2, \dots, n_0$, such that

$$\sum_{i=1}^k F(x_i' +) - F(x_i -) > \sum_{i=1}^k l_{i-1}|e_i|_\omega - \epsilon > \int_e D_\omega F d\omega - 2\epsilon,$$

and such that the finite set of intervals $\bar{\Delta}_j$ complementary to the set Δ_i and the set (α_i, β_i) , $i = 1, 2, \dots, n_0$, satisfy $\Sigma |\bar{\Delta}_j|_\omega < \delta$, from which it follows that $\Sigma |F(x_j') - F(x_j)| < \epsilon$. Hence

$$\begin{aligned} F(m+) - F(l-) &= \sum_1^{n_0} \{F(\beta_i) - F(\alpha_i)\} + \sum \{F(x_i' +) - F(x_i -)\} \\ &\quad + \sum \{F(x_j') - F(x_j)\} \\ &> \sum_1^\infty \{F(\beta_i) - F(\alpha_i)\} + \int_e D_\omega F d\omega - 3\epsilon. \end{aligned}$$

If a similar procedure is used with the sets e_i' it may be shown that

$$F(m+) - F(l-) < \sum_1^\infty \{F(\beta_i +) - F(\alpha_i -)\} + \int_e D_\omega F d\omega + 3\epsilon.$$

Because ϵ is arbitrary we conclude that

$$F(m+) - F(l-) = \int_e D_\omega F d\omega + \sum \{F(\beta_i -) - F(\alpha_i +)\}.$$

We can now state that if E is any closed set on $[a, b]$ there is an interval $[l, m]$ containing points of E such that $D_\omega F$ is summable over $E[l, m]$ and such that if (α_i, β_i) are the intervals on $[l, m]$ complementary to the set $E[l, m]$ then

$$F(m+) - F(l-) = \int_{E[l, m]} D_\omega F d\omega + \sum \{F(\beta_i +) - F(\alpha_i -)\}.$$

Let E_1 be the points of non-summability of $D_\omega F$ over $[a, b]$, (α_i, β_i) the intervals complementary to E_1 . If (α', β') is an interval such that $\alpha_i < \alpha' < \beta' < \beta_i$ then $D_\omega F$ is Lebesgue-Stieltjes integrable over $[\alpha', \beta']$ and

$$F(\beta' +) - F(\alpha' -) = \int_{\alpha'}^{\beta'} D_\omega F d\omega.$$

Because of the continuity properties of F it follows that as $\alpha' \rightarrow \alpha_i$, $\beta' \rightarrow \beta_i$

$$F(\beta' +) - F(\alpha' -) \rightarrow F(\beta_i -) - F(\alpha_i +),$$

and

$$\begin{aligned} F(\beta_i +) - F(\alpha_i -) &= F(\beta_i -) - F(\alpha_i +) \\ &\quad + \int_{\beta_i} D_\omega F d\omega - \int_{\alpha_i} D_\omega F d\omega. \end{aligned}$$

Thus $F(\beta_i+) - F(\alpha_i-)$ is determined for all intervals (α_i, β_i) contiguous to the set E_1 . Now let E_2 be the points of E_1 which are such that if $x \in E_2$ there is no interval $[l, m]$ containing x with $D_\omega F$ summable over $E_1[l, m]$ and $\Sigma\{F(\beta_i+) - F(\alpha_i-)\}$ converging where (α_i, β_i) are the intervals on $[l, m]$ contiguous to the set $E_1[l, m]$. The set E_2 is closed and, by Lemma 2, non-dense on E_1 . If (α_i, β_i) are the intervals complementary to E_2 the procedure used for the intervals complementary to E_1 can now be used to obtain $F(\beta_i+) - F(\alpha_i-)$ for these intervals (α_i, β_i) complementary to E_2 . This process can be continued by transfinite inductions to arrive at $F(b+) - F(a-)$ in a denumerable number of steps.

A consideration of Theorem 3 leads to the following definition.

Definition 7. Let ω be a non-decreasing function on $[a, b]$ and let $f(x)$ be defined on $[a, b]$ and be measurable relatively to ω . If there exists a function $F(x)$ in class \mathfrak{U} which is $ACG - \omega$ on $[a, b]$ and is such that $D_\omega F = f$ except for a set of ω -measure zero, and for which the relations of Definition 5 are satisfied at the points of discontinuity of ω , then $F(x)$ is an *indefinite Lebesgue-Stieltjes integral of f with respect to ω* .

The descriptive definition of an integral with respect to a non-decreasing function ω given here appears to be equivalent to the constructive definition given in (3, p. 666). This requires proof. Another problem for investigation is that of extending the methods of the present paper to the case in which the base function ω is *VBG*.

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